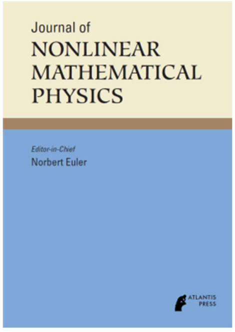

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Cusped solitary wave with algebraic decay governed by the equation for surface waves of moderate amplitude

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The existence of a new type of cusped solitary wave, which decays algebraically at infinity, for a nonlinear equation modeling the free surface evolution of moderate amplitude waves in shallow water is established by employing qualitative analysis for differential equations. Furthermore, the exact parametric representation as well as its planar graph for such type of wave is also given.

Keywords: Cusped solitary wave; Algebraic decay; Free surface; Shallow water; Moderate amplitude.

2000 Mathematics Subject Classification: 35Q35, 37K40, 76B15

1. Introduction

The nonlinear evolution equation

$$u_t + u_x + 6uu_x - 6u^2u_x + 12u^3u_x + u_{xxx} - u_{xxt} + 14uu_{xxx} + 28u_xu_{xx} = 0 \tag{1.1}$$

was established in [2] as a model for the propagation of surface waves of moderate amplitude in shallow water regime. Here the dependent variable u represents the free surface elevation, the independent variables t and x are non-dimensional time and space coordinates. It is worthwhile to mention that Eq. (1.1) originates from the earlier equation in [14] and arises as an approximation of the Euler equations in the context of homogeneous, inviscid gravity water waves propagating over a flat bed.

As shown in [2], Eq. (1.1) approximates the governing equation to the same order as the Camassa-Holm (CH) equation, which models the horizontal fluid velocity at a certain depth beneath the fluid [8]. The great interest in the equations describing the moderate amplitude waves (e.g., the CH equation), lies in the fact that they exhibit a wider range of nonlinear phenomena, such as wave breaking and solitary waves with singularities, which the model equations derived within the small amplitude shallow water regime (e.g., the KdV equation) do not have, despite the fact that the governing equations for irrotational waves do admit peaked traveling waves (periodic, as well as solitary), namely the celebrated Stokes waves of greatest height, see [4, 5, 25] for example. It is shown in [3] that unlike the KdV and CH equations, Eq. (1.1) does not have a bi-Hamiltonian integrable structure. The local well-posedness results in Sobolev space for the initial value problem associated to Eq. (1.1) on the line and on the unit circle were reported in [17, 20, 27] and in [21],

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respectively. Further work on the well-posedness of Eq. (1.1) has been done in Besov space [24]. Moreover, some results on its wave breaking, global conservative solutions, low regularity solutions and continuity and persistence properties of strong solutions can be found in [2,19,21,22,26,27,29]. Eq. (1.1) has been shown to admit various kinds of traveling wave solutions, including smooth solitary wave, compacted solitary wave, cusped solitary wave, and smooth, peaked and cusped periodic wave solutions [9,10,12,28]. Further, it is proved that the smooth solitary waves are orbitally stable in [23] and that all symmetric waves are traveling waves in [11] for this equation.

In the present paper, we will use qualitative analysis method for differential equations, which is proposed by Lenells [15,16], to solve Eq. (1.1). We prove that Eq. (1.1) admits a type of cusped solitary wave featured by decaying to zero algebraically at infinity. Such type of cusped solitary wave is different from those with exponential decay appeared in the former literature and thus is new for Eq. (1.1). Our work may help people to understand deeply the described physical process and possible applications of Eq. (1.1).

The remainder of paper is organized as follows. In Sec. 2, we prove the existence of cusped solitary wave with algebraic decay to Eq. (1.1) based on a weak formulation of Eq. (1.1). In Sec. 3, we give the exact parametric representation of such type of cusped solitary wave as well as its planar graph.

2. Existence of cusped solitary wave with algebraic decay

For a traveling wave solution $u(t,x) = \phi(x-ct)$, with c representing the constant wave velocity, Eq. (1.1) takes the form

$$(1-c)\phi_x + 6\phi\phi_x - 6\phi^2\phi_x + 12\phi^3\phi_x + (c+1)\phi_{xxx} + 14(\phi\phi_{xxx} + 2\phi_x\phi_{xx}) = 0. \tag{2.1}$$

Now we give the definition of solitary waves to Eq. (1.1).

Definition 2.1. A solitary wave to Eq. (1.1) is a nontrivial traveling wave solution to Eq. (1.1) of the form $\phi(x-ct) \in H^1(\mathbb{R})$ with $c \in \mathbb{R}$ and ϕ vanishing at infinity along with the first and second derivatives of ϕ.

Taking account of $\phi(x)$, $\phi_x(x)$ and $\phi_{xx}(x) \to 0$ as $|x| \to \infty$, integration of Eq. (2.1) over $(-\infty, x]$ leads to

$$(\phi-\rho)\phi_{xx} + \frac{1}{2}\phi_x^2 + \frac{1}{14}\phi(3\phi^3 - 2\phi^2 + 3\phi + 1 - c) = 0, \tag{2.2}$$

with $\rho = -(c+1)/14$. Notice that Eq. (2.2) can be written in the form

$$((\phi-\rho)^2)_{xx} = \phi_x^2 - \frac{1}{7}\phi(3\phi^3 - 2\phi^2 + 3\phi + 1 - c). \tag{2.3}$$

To deal with the regularity of the solitary waves, we give the following lemma, which is inspired by the study of traveling waves of Camassa-Holm equation [15].

Lemma 2.1. *Assume that ϕ is a solitary wave to Eq.* (1.1). *Then we have*

$$\phi^k \in C^j(\mathbb{R}) \ \ \textit{for} \ \ k \geq 2^j, \ \ j \geq 1. \tag{2.4}$$

Therefore

$$\phi \in C^\infty(\mathbb{R} \setminus \phi^{-1}(\rho)). \tag{2.5}$$

Proof. Let $\psi = \phi - \rho$ and denote

$$p(\psi) = -\frac{1}{7}(\psi+\rho)[3(\psi+\rho)^3 - 2(\psi+\rho)^2 + 3(\psi+\rho) + 1 - c].$$

Thus $p(\psi)$ is a polynomial in ψ and then Eq. (2.3) can be written as

$$(\psi^2)_{xx} = \psi_x^2 + p(\psi). \tag{2.6}$$

From the assumption, it follows that $(\psi^2)_{xx} \in L^1_{loc}(\mathbb{R})$. Therefore $(\psi^2)_x$ is absolutely continuous and $\psi^2 \in C^1(\mathbb{R})$. Note that $\psi + \rho \in H^1(\mathbb{R}) \subset C(\mathbb{R})$. Moreover,

$$\begin{aligned}(\psi^k)_{xx} &= (k\psi^{k-1}\psi_x)_x = \frac{k}{2}\left(\psi^{k-2}(\psi^2)_x\right)_x \\ &= k(k-2)\psi^{k-2}\psi_x^2 + \frac{k}{2}\psi^{k-2}(\psi^2)_{xx} \\ &= k(k-2)\psi^{k-2}\psi_x^2 + \frac{k}{2}\psi^{k-2}[\psi_x^2 + p(\psi)] \\ &= k\left(k - \frac{3}{2}\right)\psi^{k-2}\psi_x^2 + \frac{k}{2}\psi^{k-2}p(\psi). \end{aligned} \tag{2.7}$$

For $k \geq 3$ the right-hand side of (2.7) is in $L^1_{\text{loc}}(\mathbb{R})$. Therefore

$$\psi^k \in C^1(\mathbb{R}) \ \text{ for } \ k \geq 2. \tag{2.8}$$

Thus (2.4) holds for $j = 1$. Next, we assume that

$$\psi^k \in C^{j-1}(\mathbb{R}) \ \text{ for } \ k \geq 2^{j-1} \ \text{ and } \ j \geq 2.$$

Then for $k \geq 2^j$ we have

$$\begin{aligned}\psi^{k-2}\psi_x^2 &= \frac{1}{2^{j-1}}\left(2^{j-1}\psi^{2^{j-1}-1}\psi_x\right)\frac{1}{k-2^{j-1}}\left[\left(k-2^{j-1}\right)\psi^{k-2^{j-1}-1}\psi_x\right] \\ &= \frac{1}{2^{j-1}(k-2^{j-1})}\left(\psi^{2^{j-1}}\right)_x\left(\psi^{k-2^{j-1}}\right)_x \in C^{j-2}(\mathbb{R}).\end{aligned}$$

Also we have $\psi^{k-2}p(\psi) \in C^{j-1}(\mathbb{R})$. Therefore the right-hand side of (2.7) is in $C^{j-2}(\mathbb{R})$. Hence, in view of the relation $\psi = \phi - \rho$, by induction on j, we know (2.4) holds.

Furthermore, it follows from (2.8) that

$$k\psi^{k-1}\psi_x = (\psi^k)_x \in C(\mathbb{R}).$$

This implies that $\psi_x \in C(\mathbb{R} \setminus \psi^{-1}(0))$ and thus $\psi \in C^1(\mathbb{R} \setminus \psi^{-1}(0))$. Now, we assume that $\psi \in C^j(\mathbb{R} \setminus \psi^{-1}(0))$ for $j \geq 1$. Then for $k \geq 2^{j+1}$, we have $\psi^k \in C^{j+1}(\mathbb{R})$. Thus

$$k\psi^{k-1}\psi_x = (\psi^k)_x \in C^j(\mathbb{R}),$$

which shows that $\psi_x \in C^j(\mathbb{R} \setminus \psi^{-1}(0))$. Hence, $\psi \in C^{j+1}(\mathbb{R} \setminus \psi^{-1}(0))$. Thus, due to the relation $\psi = \phi - \rho$, by induction on j, we know (2.5) holds. □

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Setting $x_0 = \min\{x : \phi(x) = \rho\}$, then we have $x_0 \leq +\infty$. In view of Lemma 2.1, it follows that a solitary wave ϕ is smooth on $(-\infty, x_0)$ and hence Eq. (2.3) holds pointwise on $(-\infty, x_0)$. Therefore we may multiply both sides of Eq. (2.3) by $2\phi_x$ and integrate on $(-\infty, x_0)$ for $x < x_0$ to get

$$\phi_x^2 = \frac{\phi^2[6\phi^3 - 5\phi^2 + 10\phi - 5(c-1)]}{70(\rho - \phi)} := F(\phi). \tag{2.9}$$

We notice that $F(\phi) \geq 0$ if ϕ is a solution to Eq. (2.9).

Remark 2.1. As has been already pointed out in [15], a continuous function ϕ is said to have a cusp at x_0 if ϕ is smooth locally on both sides of x_0 and $\lim_{x\uparrow x_0} \phi_x(x) = -\lim_{x\downarrow x_0} \phi_x(x) = \pm\infty$. A solitary wave to Eq. (1.1) with a cusp on its crest or trough is called a cusped solitary wave. In addition, it should be pointed out that cusped waves are also relevant for the governing equations for water waves, since the limiting form of the Gerstner waves is a cycloidal profile with upward cusps, see [3, 13] for the case of gravity water waves and [6, 7, 18] for equatorial waves.

To determine the cusped solitary waves to Eq. (1.1), we also need the following lemma.

Lemma 2.2. *The solution to Eq.* (2.9) *has the following asymptotic properties:*

(i) *If $F(\phi)$ has a simple pole at ρ, where $\phi(x_0) = \rho$, then*

$$\phi(x) - \rho = \alpha|x - x_0|^{2/3} + O((x - x_0)^{4/3}) \ \ \text{as} \ \ x \to x_0, \tag{2.10}$$

$$\phi_x(x) = \begin{cases} \frac{2}{3}\alpha|x - x_0|^{-1/3} + O((x - x_0)^{1/3}) \ \ \text{as} \ \ x \downarrow x_0, \\ -\frac{2}{3}\alpha|x - x_0|^{-1/3} + O((x - x_0)^{1/3}) \ \ \text{as} \ \ x \uparrow x_0, \end{cases} \tag{2.11}$$

for some constant α, thus ϕ has a cusp.

(ii) *If ϕ approaches a triple zero m of $F(\phi)$ so that $F(m) = F'(m) = F''(m) = 0$, $F'''(m) \neq 0$, then*

$$\phi - m \sim \beta(\sqrt{|F'''(m)|/24}|x|)^{-2}, \ \ \text{as} \ \ x \to \infty, \tag{2.12}$$

for some constant β. Thus $\phi \to m$ algebraically as $x \to \infty$.

Proof. Since the proof of (i) can be found in [15], then here we only consider the proof of (ii). Since m is a triple zero of $F(\phi)$, then it follows from (2.9) that

$$\phi_x^2 = \frac{F'''(m)}{3!}(\phi - m)^3 + O((\phi - m)^4) \ \ \text{as} \ \ \phi \to m.$$

Furthermore, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\phi} = \pm\frac{\sqrt{3!}}{\sqrt{F'''(m)(\phi - m)^3 + O((\phi - m)^4)}}.$$

Since

$$\sqrt{F'''(m)(\phi - m)^3 + O((\phi - m)^4)} = |\phi - m|^{\frac{3}{2}}(\sqrt{|F'''(m)|} + O(|\phi - m|))$$

and

$$\frac{1}{\sqrt{|F'''(m)|+O(|\phi-m|)}}=\frac{1}{\sqrt{|F'''(m)|}}+O(|\phi-m|),$$

then we have

$$\pm \mathrm{d}x=\left[\frac{\sqrt{3!}}{\sqrt{|F'''(m)|}}|\phi-m|^{-\frac{3}{2}}+O((|\phi-m|)^{-\frac{1}{2}})\right]\mathrm{d}\phi.$$

Integration gives

$$|x|=\frac{2\sqrt{6}}{\sqrt{|F'''(m)|}}|\phi-m|^{-\frac{1}{2}}+O(|\phi-m|^{\frac{1}{2}}),$$

from which we get (2.12) and therefore we know ϕ decays algebraically to m at infinity. □

Based on the above derivation, now we give the following theorem on existence of cusped solitary wave with algebraic decay to Eq. (1.1).

Theorem 2.1. *If* $c=1$, *then Eq.* (1.1) *admits an anti-cusped solitary wave* $\phi<0$ *with* $\min_{x\in\mathbb{R}}\phi(x)=-1/7$ *and an algebraic decay to zero at infinity*

$$\phi(x)=O\left(\left(\frac{1}{2}|x|\right)^{-2}\right)\quad as\quad |x|\to\infty. \tag{2.13}$$

Proof. If $c=1$, then Eq. (2.9) becomes

$$\phi_x^2=\frac{3}{35}\cdot\frac{-\phi^3\left(\phi^2-\frac{5}{6}\phi+\frac{5}{3}\right)}{\phi+\frac{1}{7}}:=F_1(\phi). \tag{2.14}$$

Hence we know that $\phi(x)<0$ near $-\infty$. Since $\phi(x)\to 0$ as $x\to-\infty$, there exists some $\bar{x}$ sufficiently large negative so that $\phi(\bar{x})=-\varepsilon<0$, with ε sufficiently small, and $\phi_x(\bar{x})<0$. By the standard ODE theory, we can establish a unique solution $\phi(x)$ on $[\bar{x}-L,\bar{x}+L]$ for some $L>0$.

It is easy to see that $\phi^2-\frac{5}{6}\phi+\frac{5}{3}$ is decreasing when $\phi<0$. Furthermore,

$$\left[\frac{-\phi^3}{\phi+\frac{1}{7}}\right]'=\frac{\phi^2\left(-\frac{3}{7}-2\phi\right)}{\left(\phi+\frac{1}{7}\right)^2}<0\ \text{ for }\ -\varepsilon<\phi<0.$$

Thus $F_1(\phi)$ decreases for $\phi\in(-\varepsilon,0)$. Since $\phi_x(\bar{x})<0$, then ϕ decreases near $\bar{x}$. So $F_1(\phi)$ increases near $\bar{x}$. Therefore from (2.14), ϕ_x decreases near $\bar{x}$, and then both ϕ and ϕ_x decrease on $[\bar{x}-L,\bar{x}+L]$. Since $\sqrt{F_1(\phi)}$ is locally Lipschitz in ϕ for $-1/7<\phi\le 0$, we can easily continue the local solution to $(-\infty,\bar{x}-L]$ with $\phi(x)\to 0$ as $x\to-\infty$. On $[\bar{x}+L,+\infty)$, we can solve the initial value problem

$$\begin{cases}\varphi_x=-\sqrt{F_1(\varphi)},\\ \varphi(\bar{x}+L)=\phi(\bar{x}+L)\end{cases}$$

all the way until $\varphi=-1/7$, which is a simple pole of $F_1(\varphi)$. In view of (2.10) and (2.11), we can construct an anti-cusped solitary wave solution with a cusp singularity at $\phi=-1/7$.

Moreover, since $\phi=0$ is the triple zero of $F_1(\phi)$ and $F_1'''(0)=-6$, then we know from (ii) of Lemma 2.2 that (2.13) holds. □

3. Expression of cusped solitary wave with algebraic decay

In this section we turn our focus to finding the parametric presentation of anti-cusped solitary wave for $c = 1$, whose existence is guaranteed by Theorem 2.1. We will use some symbols on the elliptic functions and elliptic integrals, see [1]. $\mathrm{sn}(\cdot,k)$, $\mathrm{cn}(\cdot,k)$ and $\mathrm{dn}(\cdot,k)$ are Jacobian elliptic functions with the modulus k. $\mathrm{cn}^{-1}(u,k)$ is the inverse function of $\mathrm{cn}(u,k)$. $E(\cdot,k)$ is the Legendre's incomplete elliptic integral of the second kind.

Since ϕ is negative, even with respect to $\bar{x}$ and increasing on $(\bar{x},+\infty)$, then for $x > \bar{x}$ it follows from Eq. (2.14) that

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \sqrt{\frac{3}{35}} \cdot \frac{-\phi}{\phi+1/7}\sqrt{(0-\phi)(\phi+1/7)\left[(\phi-5/12)^2+(\sqrt{215}/12)^2\right]}. \tag{3.1}$$

The substitution of

$$\mathrm{d}x = \sqrt{\frac{35}{3}} \cdot \frac{\phi+1/7}{-\phi}\,\mathrm{d}\tau \tag{3.2}$$

into Eq. (3.1) and integration with the initial condition $\phi(\tau)|_{\tau=0} = -1/7$ leads to

$$\begin{aligned}\tau = \int_0^\tau \mathrm{d}t &= \int_{-1/7}^{\phi} \frac{1}{\sqrt{(0-t)(t+1/7)\left[(t-5/12)^2+(\sqrt{215}/12)^2\right]}}\,\mathrm{d}t \\ &= \frac{1}{\omega}\mathrm{cn}^{-1}\left(\frac{7(\lambda+1)\phi+1}{7(\lambda-1)\phi-1},k\right),\end{aligned}$$

where $\omega = \frac{\sqrt[4]{590}}{\sqrt{14}}$, $\lambda = \frac{3\sqrt{590}}{70}$, $k = \sqrt{\frac{1}{2}-\frac{29\sqrt{590}}{1416}}$, then it follows that

$$\phi = \frac{1+\mathrm{cn}(\omega\tau,k)}{7[(\lambda-1)\mathrm{cn}(\omega\tau,k)-(\lambda+1)]}. \tag{3.3}$$

Inserting (3.3) into (3.2) and solving the resulting equation with the initial value $x(\tau)|_{\tau=0} = \bar{x}$ yields

$$\begin{aligned}x-\bar{x} = \int_{\bar{x}}^{x} \mathrm{d}t &= 2\sqrt{\frac{35}{3}}\frac{\lambda}{\omega}\int_0^\tau\left(\frac{1}{1+\mathrm{cn}(\omega t,k)}-\frac{1}{2}\right)\mathrm{d}\omega t \\ &= 2\sqrt{\frac{35}{3}}\frac{\lambda}{\omega}\left[\frac{1}{2}\omega\tau+\frac{\mathrm{sn}(\omega\tau,k)\mathrm{dn}(\omega\tau,k)}{1+\mathrm{cn}(\omega\tau,k)}-E(\omega\tau,k)\right].\end{aligned}$$

Thus we obtain the exact anti-cusped solitary wave of parametric form with algebraic decay for $c = 1$ to Eq. (1.1) as follows:

$$\begin{cases}\phi(\tau) = \dfrac{1+\mathrm{cn}(\omega\tau,k)}{7[(\lambda-1)\mathrm{cn}(\omega\tau,k)-(\lambda+1)]}, \\ x(\tau) = \bar{x}\pm 2\sqrt{\dfrac{35}{3}}\dfrac{\lambda}{\omega}\left[\dfrac{1}{2}\omega\tau+\dfrac{\mathrm{sn}(\omega\tau,k)\mathrm{dn}(\omega\tau,k)}{1+\mathrm{cn}(\omega\tau,k)}-E(\omega\tau,k)\right].\end{cases} \tag{3.4}$$

The profile of (3.4) with $\bar{x} = 0$ is shown in Fig. 1.

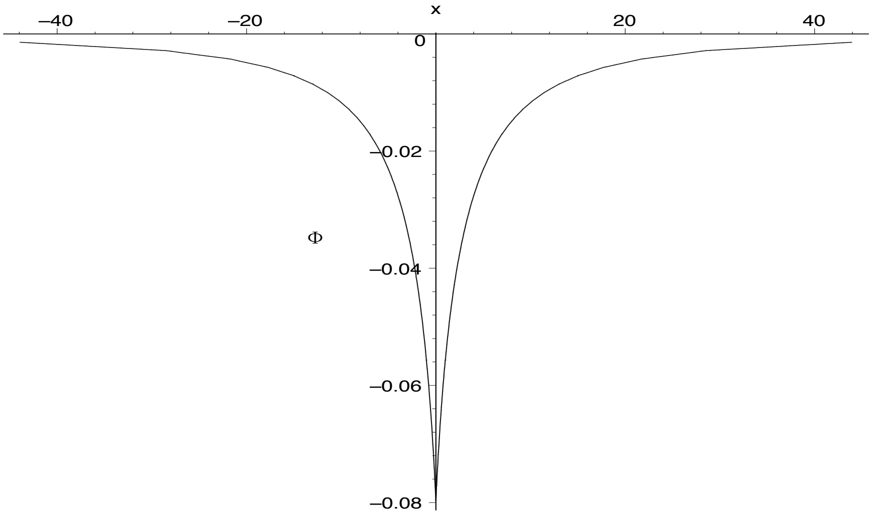

Fig. 1. The planar graph of anti-cusped solitary wave.

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