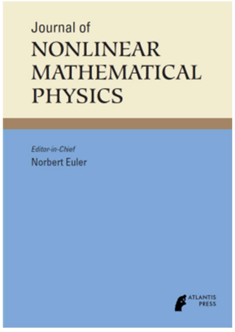

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Askold M. Perelomov

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LETTER TO THE EDITOR

Euler's triangle and the decomposition of tensor powers of the adjoint $\mathfrak{sl}(2)$-module

Askold M. Perelomov
Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia
aperelomo@gmail.com

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By considering a relation between Euler's trinomial problem and the problem of decomposing tensor powers of the adjoint $\mathfrak{sl}(2)$-module I derive some new results for both problems, as announced in arXiv:1902.08065.

1. Introduction

In 1765, Euler [1] investigated the coefficients of trinomial

$$(1+x+x^2)^n = \sum_{k=-n}^{n} a_n^{(k)} x^{n+k}. \tag{1.1}$$

For central trinomial coefficients $a_n^{(0)}$ he found the generating function and a two-term recurrence relation. For a discussion of properties of the $a_n^{(k)}$, see [3].

Let us change variable x by $\exp(i\theta)$ and rewrite the left-hand side of (1.1) as

$$(1+x+x^2)^n = x^n X^n, \text{ where } X = 1+2\cos\theta.$$

Note that X is the character χ_1 of the adjoint $\mathfrak{sl}(2)$-module. In what follows, X^n denotes both the representation with character X^n, and the corresponding module.

So, Euler's problem is equivalent to the **problem of multiplicities of weights in the representation with character** X^n. I also consider, related to the above, the problem of decomposing X^n into irreducible $\mathfrak{sl}(2)$-modules.

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2. Euler's triangle

It is evident that $a_n^{(-k)} = a_n^{(k)}$. So, it suffices to consider only quantities $a_n^{(k)}$ for $k \geq 0$. It is convenient to arrange these coefficients in a triangle. I give here the table of these numbers till $n = 10$:

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	**1**										
1	**1**	1									
2	**3**	2	1								
3	**7**	6	3	1							
4	**19**	16	10	4	1						
5	**51**	45	30	15	5	1					
6	**141**	126	90	50	21	6	1				
7	**393**	357	266	161	77	28	7	1			
8	**1107**	1016	784	504	266	112	36	8	1		
9	**3139**	2907	2304	1554	882	414	156	45	9	1	
10	**8953**	8350	6765	4740	2850	1452	615	210	55	10	1

(2.1)

Eq. (1.1) immediately implies the three-term recurrence relation

$$a_{n+1}^{(k)} = a_n^{(k-1)} + a_n^{(k)} + a_n^{(k+1)}. \tag{2.2}$$

Introduce the generating function $F(t)$ for the central trinomial coefficients:

$$F(t) = \sum_{n=0}^{\infty} a_n t^n, \text{ where } a_n = a_n^{(0)}.$$

Theorem 2.1 (Euler 1765). *The following statements hold.*

1) *The generating function $F(t)$ has the form*

$$F(t) = \frac{1}{\sqrt{(1-2t-3t^2)}}. \tag{2.3}$$

2) *For the a_n, the following two-term recurrence relation takes place*

$$n\, a_n = (2n-1)\, a_{n-1} + 3(n-1)\, a_{n-2}. \tag{2.4}$$

We give here a very short proof of item 1); it is different from Euler's.

Proof. Note that

$$a_n = \frac{1}{\pi}\int_0^{\pi} X^n \, d\theta, \text{ where } X = 1 + 2\cos\theta.$$

So,

$$F(t) = \frac{1}{\pi}\int_0^{\pi} \frac{d\theta}{1 - t - 2t\cos\theta}.$$

Evaluating this integral we obtain formula (2.3). □

Item 2) is a special subcase of the following more general statement.

Theorem 2.2. *For the $a_n^{(k)}$, there is the following two-term recurrence relation*

$$(n^2-k^2)\,a_n^{(k)} = n(2n-1)\,a_{n-1}^{(k)} + 3n(n-1)\,a_{n-2}^{(k)}. \tag{2.5}$$

Proof. We have

$$a_n^{(k)} = \frac{1}{\pi}\int_0^{\pi} X^n \cos k\theta \, d\theta,$$

and

$$\int_0^{\pi} X^n \left[\left(\frac{d^2}{d\theta^2}+k^2\right)\cos k\theta\right] d\theta = 0 = \int_0^{\pi} \cos k\theta \left[\left(\frac{d^2}{d\theta^2}+k^2\right) X^n\right] d\theta.$$

But,

$$\frac{d^2X^n}{d\theta^2} = -n^2X^n + n(2n-1)\,X^{n-1} + 3n(n-1)\,X^{n-2}.$$

This implies formula (2.5). □

Theorem 2.3. *For the $a_n^{(k)}$, there are the following two-term recurrence relations:*

$$k\,a_{n+1}^{(k)} = (n+1)\left(a_n^{(k-1)} - a_n^{(k+1)}\right), \tag{2.6}$$

$$(n-k+1)\,a_n^{(k-1)} = ka_n^{(k)} + (n+k+1)\,a_n^{(k+1)}, \tag{2.7}$$

$$(n-k+1)\,a_{n+1}^{(k)} = (n+1)(a_n^{(k)} + 2a_n^{(k+1)}), \tag{2.8}$$

$$(n+k+1)\,a_{n+1}^{(k)} = (n+1)\,(a_n^{(k)} + 2a_n^{(k-1)}). \tag{2.9}$$

Proof. From the identity

$$\int_0^{\pi} \left[\frac{d}{d\theta}\,(X^n \sin k\theta)\right] d\theta = 0,$$

we obtain relation (2.6). Combining this relation with (2.2), we obtain relations (2.7)–(2.9). □

Note that eq. (2.2) implies

$$a_n^{(1)} = \tfrac{1}{2}\,(a_{n+1}-a_n), \qquad a_n^{(2)} = \tfrac{1}{2}\,(a_{n+2}-2a_{n+1}-a_n),$$

$$a_n^{(3)} = \tfrac{1}{2}\,(a_{n+3}-3a_{n+2}+2a_n), \qquad a_n^{(4)} = \tfrac{1}{2}\,(a_{n+4}-4a_{n+3}+2a_{n+2}+4a_{n+1}-a_n)\,.$$

Corollary 2.1. *Explicit expressions for quantities $a_n^{(n-k)}$ for k small can be obtained from eqs.* (2.5) *and* (2.7) *and we have*

$$a_n^{(n-k)} = \frac{1}{k!}Q_k(n),$$

where $Q_k(n)$ is a degree k polynomial in n.

The recurrence relation for these polynomials follows from eq. (2.7):

$$Q_{k+1}(n) = (n-k)Q_k(n) + k(2n-k+1)Q_{k-1}(n).$$

Here are the explicit expressions for the first ten polynomials.

$$\begin{aligned}
&Q_0 = 1; \quad Q_1 = n; \quad Q_2 = n(n+1); \quad Q_3 = (n-1)n(n+4);\\
&Q_4 = (n-1)n(n^2+7n-6);\\
&Q_5 = (n-2)(n-1)n(n+1)(n+12);\\
&Q_6 = (n-2)(n-1)n\,(n^3+18n^2+17n-120);\\
&Q_7 = (n-3)(n-2)(n-1)n\,(n^3+27n^2+116n-120);\\
&Q_8 = (n-3)(n-2)(n-1)n\,(n+1)(n+10)(n^2+23n-84);\\
&Q_9 = n(n-1)(n-2)(n-3)(n-4)\,(n^4+46n^3+467n^2+86n-3360);\\
&Q_{10} = n(n-1)(n-2)(n-3)(n-4)\,(n^5+55n^4+665n^3-895n^2-16626n+15120).
\end{aligned}$$

3. Decomposition of X^n into irreducible representations

This problem is equivalent to expanding X^n in terms of characters of $\mathfrak{sl}(2)$-modules:

$$X^n = \sum_{k=0}^{n} b_n^{(k)} \chi_k(\theta)\,.$$

These characters are well known (see, for example, [4]):

$$\chi_k = 1 + 2\cos(\theta) + 2\cos(2\theta) + \cdots + 2\cos(k\theta)\,.$$

They are orthogonal

$$\frac{1}{\pi}\int_0^{\pi} \chi_k(\theta)\,\chi_l(\theta)\,(1-\cos(\theta))\,d\theta = \delta_{k,l}\,,$$

and we have

$$b_n^{(k)} = \frac{1}{\pi}\int_0^{\pi} X^n f_k(\theta) d\theta\,, \ \text{ where } f_k(\theta) = \cos(k\theta) - \cos((k+1)\theta)\,.$$

This implies the basic relation

$$b_n^{(k)} = a_n^{(k)} - a_n^{(k+1)}\,,$$

and a three-term recurrence relation similar to relation (2.2)

$$b_{n+1}^{(k)} = b_n^{(k-1)} + b_n^{(k)} + b_n^{(k+1)} \quad \text{for}\, n \geq 2,\ k \geq 1\,,$$

as well as the following relations

$$\begin{aligned}
&b_n = b_n^{(0)} = \tfrac{1}{2}(3a_n - a_{n+1}), \qquad b_n^{(1)} = b_{n+1},\ \ b_n^{(2)} = b_{n+2} - b_{n+1} - b_n,\\
&b_n^{(3)} = b_{n+3} - 2b_{n+2} - b_{n+1} + b_n,\ b_n^{(4)} = b_{n+4} - 3b_{n+3} + 3b_{n-1}\,.
\end{aligned}$$

The triangle for the numbers $b_n^{(k)}$ analogous to the triangle (2.1) is as follows.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	**1**										
1	**0**	1									
2	**1**	1	1								
3	**1**	3	2	1							
4	**3**	6	6	3	1						
5	**6**	15	15	10	4	1					
6	**15**	36	40	29	15	5	1				
7	**36**	91	105	84	49	21	6	1			
8	**91**	232	280	238	154	76	28	7	1		
9	**232**	603	750	672	468	258	111	36	8	1	
10	**603**	1585	2025	1890	1398	837	405	155	45	9	1

(3.1)

Theorem 3.1. *The generating function $G(t)=\sum_{n=0}^{\infty} b_n t^n$ is of the form*

$$G(t)=\frac{1}{2t}\left(1-\frac{\sqrt{1-3t}}{\sqrt{(1+t)}}\right).$$

Proof. Taking into account the identity

$$\frac{1-\cos(\theta)}{1-t-2t\cos(\theta)}=\frac{1}{2t}\left(1-\frac{1-3t}{1-t-2t\cos(\theta)}\right)$$

we reduce the proof to the proof for $F(t)$. We also have the recurrence relation

$$(n+1)b_n=(n-1)(2b_{n-1}+3b_{n-2})$$

which follows from eq. (2.4) and the equality $b_n=a_n-a_n^{(1)}$. □

Theorem 3.2. *There is a four-term recurrence relation*

$$A_{n,k}b_n^{(k)}+B_{n,k}b_{n-1}^{(k)}+C_{n,k}b_{n-2}^{(k)}+D_{n,k}b_{n-3}^{(k)}+E_{n,k}b_{n-4}^{(k)}=0,$$

where

$$\begin{aligned}
A_{n,k}&=(n^2-(k+1)^2)(n^2-k^2);\\
B_{n,k}&=-2n(2n-1)(n+k)(n-k-1);\\
C_{n,k}&=-2n(n-1)(n^2-2n+3-3k(k+1));\\
D_{n,k}&=6n(n-1)(n-2)(2n-3);\\
E_{n,k}&=9n(n-1)(n-2)(n-3).
\end{aligned} \tag{3.2}$$

Proof. We have

$$b_n^{(k)}=\frac{1}{\pi}\int_0^{\pi}X^n f_k(\theta)d\theta, \tag{3.3}$$

where

$$X=1+2\cos(\theta),\quad f_k(\theta)=\cos(k\theta)-\cos((k+1)\theta),$$

and

$$A_k f_k(\theta) = 0, \quad \text{where } A_k = \left(\frac{d^2}{d\theta^2} + k^2\right)\left(\frac{d^2}{d\theta^2} + (k+1)^2\right).$$

Integrating by parts in (3.3) we get (3.2) and

$$\frac{1}{\pi}\int_0^{\pi} f_k(\theta)(A_k X^n)d\theta = 0.$$

□

Theorem 3.3. *There is the following three-term recurrence relation*

$$(k+1)(n+1-k)b_n^{(k-1)} = (k(k+1)-n-1)b_n^{(k)} + k(n+k+2)b_n^{(k+1)}.$$

Proof. This follows from eq. (2.7) and the relation $b_n^{(k)} = a_n^{(k)} - a_n^{(k+1)}$. □

Acknowledgments

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