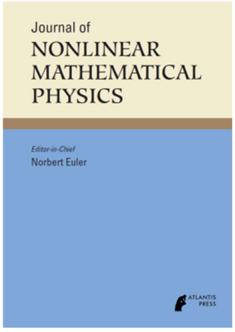

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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Sofiane Bouarroudj, Semyon E. Konstein

To cite this article: Sofiane Bouarroudj, Semyon E. Konstein (2020) SO(4)-symmetry of mechanical systems with 3 degrees of freedom, Journal of Nonlinear Mathematical Physics 27:1, 162–169, DOI: https://doi.org/10.1080/14029251.2020.1683997

To link to this article: https://doi.org/10.1080/14029251.2020.1683997

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 27, No. 1 (2020) 162–169

SO(4)-symmetry of mechanical systems with 3 degrees of freedom

Sofiane Bouarroudj
New York University Abu Dhabi, Division of Science and Mathematics,
P.O. Box 129188, United Arab Emirates
sofiane.bouarroudj@nyu.edu

Semyon E. Konstein *
I.E.Tamm department of Theoretical Physics,
P.N. Lebedev Physical Institute of the Russian Academy of Sciences,
Leninskij prosp. 53, RU-119991 Moscow, Russia
konstein@lpi.ru

Received 30 May 2019

Accepted 20 July 2019

We answered an old question: does there exist a mechanical system with 3 degrees of freedom, except for the Coulomb system, which has 6 first integrals generating the Lie algebra $\mathfrak{o}(4)$ by means of the Poisson brackets? A system which is not centrally symmetric, but has 6 first integrals generating Lie algebra $\mathfrak{o}(4)$, is presented. It is shown also that not every mechanical system with 3 degrees of freedom has first integrals generating $\mathfrak{o}(4)$.

1. Introduction

It is well-known (see, e.g., [8]) that in the Coulomb field, i.e., in the mechanical system with 3 degrees of freedom (3d mechanical system) with the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2} - \frac{1}{r}, \quad \text{where} \quad \mathbf{p}^2 := \sum_{i=1,2,3} p_i^2, \ r := \left(\sum_{i=1,2,3} q_i^2 \right)^{1/2}, \tag{1.1}$$

the symmetry group of canonical transformations[a] keeping Hamiltonian (1.1) invariant has a subgroup isomorphic to SO(4) acting in the domain $H < 0$. This fact, found by V. Fock [6], helps to explain the structure of the spectrum of the hydrogen atom. Sometimes this symmetry is called *hidden*.

An important property of this SO(4) is that the Casimirs of its Lie algebra $\mathfrak{o}(4)$ restore the Hamiltonian. The Hamiltonian in Eq. (1.1) describes, for example, the motion of two particles interacting via gravity, and the motion of two charged particles with the charges of opposite sign. The number of works investigating this Hamiltonian is huge.[b] It is therefore astonishing that the literature does not give (at least, we could not find it) the definite answer to a natural question: "does there exist a mechanical system with 3 degrees of freedom, except for the Coulomb system, which has 6 first integrals generating the Lie algebra $\mathfrak{o}(4)$?" posed, e.g., in [10, 12]. Two different answers to the question were given fifty years ago:

*Corresponding author
[a]Recall, that the transformations of the phase space that preserve the Hamiltonian form of the Hamilton equations, whatever the Hamiltonian function is, are called *canonical*.
[b]See, for example, [3, 5, 11] and references therein.

1) Mukunda [10] claimed that every mechanical system with 3 degrees of freedom has 6 first integrals, generating Lie algebra $\mathfrak{o}(4)$ by means of Poisson brackets.
2) Szymacha and Werle [12] claimed that there are no other mechanical systems with the same property, assuming that $\mathfrak{o}(4)$ contains the Lie algebra of spatial rotations of $\mathbb{R}^3$.

In this note, we showed that some $3d$ systems have 6 first integrals generating Lie algebra $\mathfrak{o}(4)$, and some have not.

To prove that not for every system with 3 degrees of freedom its the first integrals generate $\mathfrak{o}(4)$, we offer a simple necessary condition for existence of $\mathfrak{o}(4)$ symmetry, see Section 4, and in Section 5 we give an example for which this condition is violated.

In Section 6 we consider the Hamiltonian of a charged particle in an homogeneous electric field. For this Hamiltonian, there exists a family of sextuples of first integrals such that every sextuple generates (by means of the Poisson bracket) the Lie algebra $\mathfrak{o}(4)$.

For each element of this $\mathfrak{o}(4)$, we consider the corresponding hamiltonian flow (for details, see the next section) and show that the set of these flows does not constitute the Lie group SO(4) of canonical transformations.

To avoid misunderstanding, note that we consider the *symmetry algebra* (consisting of some first integrals) of the system, not the Lie algebra of *dynamical symmetry group* introduced in [4], which is also called *non-invariance group*, see [9].

2. Generalities (following [1])

Recall the definition of the symmetry group of canonical transformations keeping the Hamiltonian invariant and the Lie algebra of this group. Let $H(q_i, p_i)$, where $i = 1, 2, 3$, be a Hamiltonian of some mechanical system. We will also denote the whole set of the q_i and p_i for $i = 1, 2, 3$ by z_α, where $\alpha = 1, \dots, 6$. Let the first integral F of this system be a real function on some domain $U_F \subset \mathbb{R}^6$. Let $(q, p) \in U_F$; the case $F = H$ is not excluded. Then F generates a 1-dimensional Lie group $\mathscr{L}_F$ of canonical transformations $(q, p) \mapsto (q^F(\tau \mid q, p), p^F(\tau \mid q, p))$ leaving the Hamiltonian H and the domain U_F invariant if

$$(q^F(\tau \mid q, p), p^F(\tau \mid q, p)) \in U_F \quad \text{for any} \quad \tau \in \mathbb{R}.$$

The transformations are defined by the relations

$$\frac{dq_i^F}{d\tau} = \{q_i^F, F\} = \frac{\partial F(q^F, p^F)}{\partial p_i^F}, \tag{2.1}$$

$$\frac{dp_i^F}{d\tau} = \{p_i^F, F\} = -\frac{\partial F(q^F, p^F)}{\partial q_i^F}, \tag{2.2}$$

$$q_i^F(0 \mid q, p) = q_i, \quad p_i^F(0 \mid q, p) = p_i, \tag{2.3}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket[c] in $\mathbb{R}^6$:

$$\{F, G\} := \sum_{i=1,2,3} \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) = \sum_{\alpha,\beta=1,\dots,6} \frac{\partial F}{\partial z_\alpha} \omega_{\alpha\beta} \frac{\partial G}{\partial z_\beta}. \tag{2.4}$$

[c]The definition (2.4) has the opposite sign as compared with the one given in [8], but coincides with the definition of the Poisson bracket given in [1, 2, 7, 13].

Here the symplectic form ω is of shape $\omega = \begin{pmatrix} 0_3 & 1_3 \\ -1_3 & 0_3 \end{pmatrix}$, where 1_3 and 0_3 are 3×3 matrices. We call the transformations Eq. (2.1)–(2.3) the *Hamiltonian flow*, generated by the Hamiltonian F, and denote it $\mathscr{L}_F$. If a certain finite set of first integrals $\mathscr{F} = \{F_\alpha \mid \alpha = 1,2,\ldots\}$ has the same domain U invariant under the action of all Hamiltonian flows $\mathscr{L}_{F_\alpha}$, then these flows generate a Lie group. The Lie algebra of this group coincides with the Lie algebra generated by the set $\mathscr{F}$ by means of the bracket (2.4).

3. The case of the Coulomb field (following [8])

Here we briefly consider the mechanical system (1.1) with the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2} - \frac{1}{r}, \quad \text{where} \quad \mathbf{p}^2 := \sum_{i=1,2,3} p_i^2, \; r := \left(\sum_{i=1,2,3} q_i \right)^{1/2}. \tag{3.1}$$

This Hamiltonian has two well-known triples of first integrals: one consists of the coordinates L_i of the angular momentum vector, the other one consists of the coordinates of the Runge-Lenz vector R_i, defined in the domain

$$U = \{z \in \mathbb{R}^6 \mid H(z) < 0\},$$

or in any of the domains $E_{\min} < H < E_{\max} < 0$, for any pair of numbers $E_{\min} < E_{\max} < 0$ by the formulas

$$L_i := \sum_{j,k=1,2,3} \varepsilon_{ijk} q_j p_k, \tag{3.2}$$

$$R_i := (-2H)^{-1/2} \left(\sum_{j,k=1,2,3} \varepsilon_{ijk} L_j p_k + \frac{q_i}{r} \right), \tag{3.3}$$

where ε_{ijk} is an anti-symmetric tensor such that $\varepsilon_{123} = 1$.

These first integrals satisfy the following commutation relations:

$$\begin{aligned} \{H, L_i\} &= 0, \\ \{H, R_i\} &= 0, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \{L_i, L_j\} &= \textstyle\sum_{k=1,2,3} \varepsilon_{ijk} L_k, \\ \{R_i, R_j\} &= \textstyle\sum_{k=1,2,3} \varepsilon_{ijk} L_k, \\ \{L_i, R_j\} &= \textstyle\sum_{k=1,2,3} \varepsilon_{ijk} R_k. \end{aligned} \tag{3.5}$$

Due to relations (3.4) and by definition of the domain U, the later is invariant under the action of Hamiltonian flows generated by the first integrals L_i and R_i.

The relations (3.5) show that these first integrals generate the Lie algebra $\mathfrak{o}(4)$.

Since $\mathfrak{o}(4) \simeq \mathfrak{o}(3) \oplus \mathfrak{o}(3)$, we can introduce two commuting triples of first integrals

$$\begin{aligned} G_i \quad &:= \tfrac{1}{2}(L_i + R_i), \text{ where } i = 1,2,3, \\ G_{3+i} &:= \tfrac{1}{2}(L_i - R_i), \text{ where } i = 1,2,3, \end{aligned} \tag{3.6}$$

satisfying the commutation relations

$$\begin{aligned} \{G_i, G_j\} \quad &= \textstyle\sum_{k=1,2,3} \varepsilon_{ijk} G_k, \text{ where } i,j = 1,2,3, \\ \{G_{3+i}, G_{3+j}\} &= \textstyle\sum_{k=1,2,3} \varepsilon_{ijk} G_{3+k}, \text{ where } i,j = 1,2,3, \\ \{G_i, G_{3+j}\} \quad &= 0, \text{ where } i,j = 1,2,3. \end{aligned} \tag{3.7}$$

4. Restrictions on the rank

Let some $3d$ mechanical system have the Hamiltonian H and 6 first integrals G_α satisfying the commutation relations Eq. (3.7).

Consider two 6×6 matrices: the Jacobi matrix J with elements

$$J_\alpha^\beta := \frac{\partial G_\alpha}{\partial z_\beta}, \text{ where } \alpha, \beta = 1, \dots, 6, \tag{4.1}$$

and the matrix P with elements

$$P_{\alpha\beta} := \{G_\alpha, G_\beta\}, \text{ where } \alpha, \beta = 1, \dots, 6. \tag{4.2}$$

Then definition (4.1) of Jacobi matrix and (2.4) of brackets imply that

$$P_{\alpha\beta} = \sum_{\gamma,\delta=1,\dots,6} J_\alpha^\gamma \omega_{\gamma\delta} J_\beta^\delta . \tag{4.3}$$

Suppose that $G_1^2 + G_2^2 + G_3^2 \neq 0$ and $G_4^2 + G_5^2 + G_6^2 \neq 0$. Then the matrix P has two independent vectors in its kernel

$$(G_1, G_2, G_3, 0, 0, 0) \text{ and } (0, 0, 0, G_4, G_5, G_6) \tag{4.4}$$

due to relations (3.7), and so $\mathrm{rank}(P) = 4$.

Since the symplectic form ω is non-degenerate, the relation Eq. (4.3) and degeneracy of the matrix P imply that

$$\mathrm{rank}(P) \leq \mathrm{rank}(J) < 6. \tag{4.5}$$

So either $\mathrm{rank}(J) = 4$ or $\mathrm{rank}(J) = 5$. Both these cases can be realized: $\mathrm{rank}(J) = 5$ for the Coulomb system while $\mathrm{rank}(J) = 4$ for some of the systems described in Section 6.

5. Not all $3d$ mechanical systems have $\mathfrak{o}(4)$ symmetry

To give an example of a $3d$ mechanical system without $\mathfrak{o}(4)$ symmetry, consider the Hamiltonian

$$H = H_1 + H_2 + H_3, \text{ where } H_i = \frac{1}{2} p_i^2 + \frac{\omega_i^2}{2} q_i^2 \tag{5.1}$$

and where the ω_i for $i = 1,2,3$ are incommensurable.

Evidently, each of the functions H_i is a first integral.

Let us show that each first integral of this system is a function of the H_i, where $i = 1,2,3$. Indeed, let F be a first integral. So, F is constant on every trajectory defined for the system under consideration by relations

$$q_i = \frac{\sqrt{2}}{\omega_i} r_i \sin(\omega_i t + \varphi_i), \quad p_i = \sqrt{2} r_i \cos(\omega_i t + \varphi_i) \quad \text{for } i = 1,2,3, \tag{5.2}$$

where the r_i and φ_i are constants specifying the trajectory. Since every trajectory given by Eq. (5.2) is everywhere dense on the torus

$$T(r_1, r_2, r_3) := \left\{ z \in \mathbb{R}^6 \mid \frac{1}{2} p_i^2 + \frac{\omega_i^2}{2} q_i^2 = r_i^2 \quad \text{for } i = 1,2,3 \right\}, \tag{5.3}$$

it follows that F is constant on every torus $T(r_1, r_2, r_3)$, and hence F is a function of the r_i. This implies $F = F(H_1, H_2, H_3)$.

Now suppose that the system has 6 first integrals G_α satisfying commutation relations (3.7) of the Lie algebra $\mathfrak{o}(4)$. Then, since $G_\alpha = G_\alpha(H_1, H_2, H_3)$, it follows that the Jacobi matrix J in Eq. (4.1) is of rank ≤ 3, and so due to Eq. (4.3) the matrix P, see Eq. (4.2), is of rank ≤ 3. But this fact contradicts the easy to verify fact that if $G_1^2 + G_2^2 + G_3^2 \neq 0$ and $G_4^2 + G_5^2 + G_6^2 \neq 0$, then $\operatorname{rank}(P) = 4$.

So, no sextuple of the first integrals of the system under consideration generates $\mathfrak{o}(4)$.

6. An example of non-Coulomb $3d$ mechanical system with Lie algebra $\mathfrak{o}(4)$ of the first integrals

Consider a particle in an homogeneous field with potential $-q_3$. This is a system with 3 degrees of freedom with Hamiltonian

$$H = \frac{\mathbf{p}^2}{2} - q_3. \tag{6.1}$$

Let

$$U := \{ z \in \mathbb{R}^6 \mid p_1^2 < a_1^2,\ p_2^2 < a_2^2 \}, \tag{6.2}$$

where each a_s is any smooth function of Hamiltonian H. We denote the boundary of U by ∂U and the closure of U by $\bar{U}$.

Then the real functions

$$\begin{aligned}
G_1 &= p_1, \\
G_2 &= \sqrt{a_1^2 - p_1^2} \cos(q_1 - p_1 p_3), \\
G_3 &= \sqrt{a_1^2 - p_1^2} \sin(q_1 - p_1 p_3), \\
G_4 &= p_2, \\
G_5 &= \sqrt{a_2^2 - p_2^2} \cos(q_2 - p_2 p_3), \\
G_6 &= \sqrt{a_2^2 - p_2^2} \sin(q_2 - p_2 p_3),
\end{aligned} \tag{6.3}$$

are the first integrals defined in $\bar{U}$ and smooth in U. Let $\mathscr{A}$ be the space generated by G_α. The space $\mathscr{A}$, with Poisson brackets as an operation, is the Lie algebra isomorphic to $\mathfrak{o}(4)$. It is subject to a direct verification that the integrals (6.3) indeed satisfy the relations (3.7) for generators of $\mathfrak{o}(4)$.

The Casimirs, defined by the formulas

$$K_1 := \sum_{i=1,2,3} G_i^2, \qquad K_2 := \sum_{i=1,2,3} G_{3+i}^2$$

are equal to

$$K_1 = a_1^2, \qquad K_2 = a_2^2$$

and do not define the Hamiltonian only if the a_s are constant. In the case where the a_s are constant, the Jacobi matrix for the functions (6.3) has rank 4 at the generic point. Otherwise, $\mathrm{rank}(J) = 5$ at the generic point.

6.1. *Non-invariance of the domain U under the flows* $\mathscr{L}_G$

For λ_2 and λ_3 real, such that $\sqrt{\lambda_2^2+\lambda_3^2} = 1$, $\lambda_2 = \cos\varphi$, and $\lambda_3 = -\sin\varphi$, we see that $G := \lambda G_1 + \lambda_2 G_2 + \lambda_3 G_3$ is of the shape

$$G = \lambda p_1 + Q\cos(q_1 - p_1 p_3 + \varphi), \text{ where } Q := \sqrt{a_1^2 - p_1^2}. \tag{6.4}$$

Set

$$Q_H := \frac{\partial Q}{\partial H} = \frac{a_1}{Q}\frac{da_1}{dH}$$

so that

$$\begin{aligned} \{z_\alpha, Q\} &= Q_H\{z_\alpha, H\} - \frac{p_1}{Q}\{z_\alpha, p_1\}, \\ \{z_\alpha, H\} &= \sum_i \{z_\alpha, p_i\}p_i - \{z_\alpha, q_3\}. \end{aligned}$$

Introduce a new variable u instead of q_1:

$$u := q_1 - p_1 p_3 + \varphi. \tag{6.5}$$

Let $z(\tau_0) \in U$. The equations of the Hamiltonian flow $\mathscr{L}_G$ are then of the form

$$\begin{aligned} &\frac{d}{d\tau}z_\alpha = \{z_\alpha, G\}, \text{ i.e.,} \\ &\frac{d}{d\tau}p_3 = Q_H\cos(u) \\ &\frac{d}{d\tau}q_3 = Qp_1\sin(u) + Q_H p_3\cos(u) \\ &\frac{d}{d\tau}p_2 = 0, \qquad \frac{d}{d\tau}q_2 = Q_H p_2\cos(u) \\ &\frac{d}{d\tau}p_1 = Q\sin(u), \\ &\frac{d}{d\tau}q_1 = \lambda + Qp_3\sin(u) - \frac{p_1}{Q}\cos(u) + Q_H p_1\cos(u). \end{aligned} \tag{6.6}$$

Since $\{G,H\} = 0$, it is clear that $\frac{dH}{d\tau} = 0$ and $\frac{da_s}{d\tau} = 0$ along the trajectories $z(\tau)$ defined by Eqs. (6.6).

Proposition 6.1. *For any $z(\tau_0) \in U$, there exists a first integral $G_z \in \mathscr{A}$ such that the Hamiltonian flow $\mathscr{L}_{G_z}$ leads the point $z(\tau_0)$ to the boundary of U for a finite time.*

Proof. We have

$$\frac{d}{d\tau} u = \lambda - \frac{p_1}{Q} \cos(u), \tag{6.7}$$

$$\frac{d}{d\tau} p_1 = Q \sin(u), \tag{6.8}$$

$$\frac{d}{d\tau} Q = -p_1 \sin(u), \tag{6.9}$$

and hence

$$\begin{aligned} \frac{d^2}{d\tau^2} p_1 &= -p_1 \sin^2(u) + Q \cos(u)(\lambda - \frac{p_1}{Q} \cos(u)) \\ &= -p_1 + \lambda Q \cos(u) \end{aligned}$$

The system (6.7)–(6.9) can be solved explicitly for any λ, but further on we consider only the case $\lambda = 0$. In this case

$$\frac{d^2}{d\tau^2} p_1 = -p_1. \tag{6.10}$$

and

$$p_1 = p_1^{\max} \sin(\tau + \psi), \tag{6.11}$$

where $p_1^{\max} \geq 0$ and ψ are constant on the trajectories.

We have

$$\begin{aligned} (p_1^{\max})^2 &= p_1^2 + \left(\frac{d}{d\tau} p_1\right)^2 = p_1^2 + (a_1^2 - p_1^2) \sin^2(u) \\ &= a_1^2 \sin^2(u) + p_1^2 \cos^2(u) = a_1^2 - (a_1^2 - p_1^2) \cos^2(u) \end{aligned}$$

and

$$(p_1^{\max})^2 = a_1^2 - (a_1^2 - p_1^2(\tau)) \cos^2(u(\tau)) \tag{6.12}$$

for any τ since $p_1^{\max}$ is constant on every trajectory.

If $\cos(u(\tau_0)) \neq 0$ and $p_1^2(\tau_0) < a_1^2(\tau_0)$, then

$$(p_1^{\max})^2 = a_1^2 - (a_1^2 - p_1^2(\tau_0)) \cos^2(u(\tau_0)) < a_1^2. \tag{6.13}$$

Eqs. (6.13) and equality (6.11) imply that

$$p_1^2(\tau) < a_1^2(\tau) \quad \text{for any } \tau \tag{6.14}$$

i.e., $z(\tau) \in U$ for any $\tau \in \mathbb{R}$. Besides, conditions (6.12) and (6.13) imply that

$$\cos(u(\tau)) \neq 0 \quad \text{for any } \tau.$$

Now, observe that for every $z(\tau_0)$ it is possible to choose λ_2 and λ_3 (i.e., φ) so that $\cos(u(\tau_0)) = 0$. Then, for this φ, we have $(p_1^{\max})^2 = a_1^2$ and $Q(\pi/2 - \psi) = 0$, i.e., $z(\pi/2 - \psi) \in \partial U$. □

Remark 6.1. The proof of Proposition 6.1 shows also that for each fixed φ, the domain

$$U_\varphi := \{z \in U \mid \cos(q_1 - p_1 p_3 + \varphi) \neq 0\} \tag{6.15}$$

is invariant under the action of Hamiltonian flow $\mathscr{L}_{Q\cos(q_1-p_1p_3+\varphi)}$ acting on U_φ as 1-dimensional Lie group.

Remark 6.2. There is no domain $U_{\text{common}} \subset U$ invariant under Hamiltonian flows $\mathscr{L}_{Q\cos(q_1-p_1p_3+\varphi)}$ for all $\varphi \in [0, 2\pi)$. Indeed, $U_{\text{common}} \subset \bigcap_\varphi U_\varphi$, and $\bigcap_\varphi U_\varphi = \varnothing$ since for any $z \in U$ there exists $\varphi \in [0, 2\pi)$ such that $\cos(q_1 - p_1 p_3 + \varphi) = 0$.

Acknowledgements

Authors are grateful to I.V. Tyutin and A.E. Shabad for useful discussions. S.K. is grateful to Russian Fund for Basic Research (grant No. 17-02-00317) for partial support of this work. S.B. was supported in part by the grant AD 065 NYUAD.

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