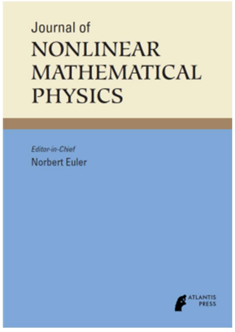

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Fredholm Property of Operators from 2D String Field Theory

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In a recent study of Landau-Ginzburg model of string field theory by Gaiotto, Moore and Witten, there appears a type of perturbed Cauchy-Riemann equation, *i.e.* the ζ-instanton equation. Solutions of ζ-instanton equation have degenerate asymptotics. This degeneracy is a severe restriction for obtaining the Fredholm property and constructing relevant homology theory. In this article, we study the Fredholm property of a sort of differential operators with degenerate asymptotics. As an application, we verify certain Fredholm property of the linearized operator of ζ-instanton equations.

Keywords: Fredholm property; Floer homology; transversal non-degeneracy.

2000 Mathematics Subject Classification: 58J37; 53D40; 47A53

1. Introduction

It is well-known that critical points of a generic smooth function $f : M \to \mathbb{R}$ on a Riemannian manifold (M,g) determine the topology of this manifold. This is the subject of Morse theory ([14]). By studying the gradient flow lines between critical points of Morse function f, one can define the so-called Morse-Smale-Witten complex, its homology is called Morse homology ([20]). Symplectic Floer homology ([5, 6, 8, 11, 13, 17]), which was originally invented as a tool of proving the Arnold conjecture ([1]) on lower bound of number of non-degenerate Hamiltonian time 1-periodic orbits on a compact symplectic manifold (X,ω), can be considered as a generalization to infinite dimension of the Morse homology. The analogous role of Morse function on a Riemannian manifold is played by the symplectic action functional defined on loop space $\mathscr{L}X$, critical points of action functional are 1-periodic solutions of Hamiltonian equation, gradient flow lines between critical points are played by Floer's connecting trajectories which are solutions of some significant elliptic partial differential equations. As a result, for some classes of symplectic manifolds, Floer constructed such infinite dimensional version of MSW complex and verified the lower bound estimation by Arnold conjecture via its homology.

The PDE studied by Floer is the perturbed Cauchy-Riemann equation

$$\bar{\partial}_{J,H} u = \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \nabla H = 0, \tag{1.1}$$

where J is an ω-compatible almost complex structure on X, $H = H_t(\cdot)$ is a time 1-periodic Hamiltonian function on X, the gradient ∇H is determined by the associated metric $\omega(\cdot, J\cdot)$, and the map $u : \mathbb{R} \times S^1 \to X$ satisfying (1.1) is called a perturbed J-holomorphic cylinder. One key point in the construction of Floer homology is to verify that, at every solution u of equation (1.1), asymptotic to

1-periodic Hamiltonian orbits as $s \to \pm\infty$, the differential of $\bar{\partial}_{J,H}$, denoted by (see (2.5))

$$D_u : W^{1,p}(\mathbb{R} \times S^1, u^*TX) \to L^p(\mathbb{R} \times S^1, u^*TX),$$

is a Fredholm operator. Recall that a bounded linear operator between Banach spaces is called a Fredholm operator if both its kernel and cokernel are of finite dimension. Floer showed that if solutions u satisfy some **non-degenerate** asymptotic conditions, then the Fredholm property holds. Based on this property, roughly, the index theorem can apply to calculate the dimension of moduli space of solutions of equation (1.1) (asymptotic to 1-periodic Hamiltonian orbits), then the differential of Floer complex can be defined.

In the recent study of Landau-Ginzburg(LG)-model for 2d string field theory([9]), there also appears perturbed Cauchy-Riemann equation, *i.e.* the physicists' so-called ζ-instanton equation $\bar{\partial}_{J,H_{ij}} u = 0$ with $u : \mathbb{R} \times \mathbb{R} \to X$ (see next section for more details). Due to the time independence of Hamiltonian H_{ij}, which is derived from the superpotential that is a J-holomorphic (time-independent) Morse function

$$W : (X,J) \to \mathbb{C},$$

however, solutions u of ζ-instanton equation only have degenerate asymptotics, which are called ζ-solitons (satisfying (2.8) and (2.9) below). Here degeneracy of ζ-solitons means that for each ζ-soliton there always exists an $\mathbb{R}$-family of solutions satisfying the ζ-soliton equation. Inspired by [9], some Floer-type homology theory is supposed to be constructed from moduli space of ζ-instantons. While Floer's argument for non-degenerate 1-periodic solutions can not apply directly due to the degeneracy of asymptotics. In this article, we study operators related to those derived from ζ-instanton equation defined on some weighted Sobolev spaces. Under some weaker non-degeneracy conditions, we obtain the following main result

Theorem 1.1. *Assume that solutions $\gamma^{\pm}(t)$ of ζ-soliton equations* (2.8) *and* (2.9) *are transversally non-degenerate. For each solution u of ζ-instanton equation satisfying some asymptotic conditions* (i.e. (2.10)-(2.12))*, the map*

$$D_u : \mathscr{V}_{+,-} \oplus W^{k,p,A}(\mathbb{R}^2; u^*TX) \to W^{k-1,p,A}(\mathbb{R}^2; u^*TX)$$

defined by (3.1) *is Fredholm for any $k \in \mathbb{R}$, $p > 1$, and its index depends neither on k nor on p, which can be given by*

$$\text{Ind}\, D_u = \mu(\gamma^+, \gamma^-, u) + 2,$$

where $W^{k,p,A}(\mathbb{R}^2; u^*TX)$ are A-weighted Sobolev spaces (*see subsection* 3.1) *and μ is the relative Maslov index.*

This result will be proved in section 4 (Corollary 4.1). This is a first step of an ongoing proposal of constructing homology theory for LG-model. Roughly, based on such Fredholn property, we will study the Morse-Bott type moduli space of ζ-instantons, define Maslov-type index for ζ-instanton and calculate the dimension of moduli space, then the Floer complex can be constructed via moduli space of certain Floer-type trajectories connecting ζ-solitons. The compactness and transversality of moduli space of Floer trajectories are crucial properties to be verified such that all constructions go through. The relevant works will appear in another paper.

It turns out that one of essential steps of overcoming the (partial) degeneracy is to study certain analytic problem presented in the following sections. In the next section, to set the problem, we introduce more notations and give more detailed description of the motivation of this work.

We remark that after submitting this work, we were informed that Jiang [12] also independently studied some analytic problems related to LG-model with different setting and method.

2. Differential operators

2.1. *Differential operators in Floer theory*

Given a compact symplectic manifold (X,ω) and a time 1-periodic function $H_t(\cdot) = H_{t+1}(\cdot)$, the Hamiltonian equation is

$$\frac{d}{dt}x(t) = X_{H_t}, \tag{2.1}$$

where X_{H_t} is the Hamiltonian vector field associated to H_t, which can be defined by $\omega(\cdot, X_{H_t}) = dH_t(\cdot)$. The solutions of (2.1) generate a family of symplectomorphisms $\phi_t : X \to X$ such that

$$\frac{d}{dt}\phi_t = X_{H_t} \circ \phi_t, \quad \phi_0 = Id.$$

We say an almost complex structure J on TX is compatible with the symplectic form ω if $\langle \cdot,\cdot \rangle = \omega(\cdot, J\cdot)$ define a Riemannian metric on X. Denote by $\mathscr{J} = \mathscr{J}(X,\omega)$ the space of ω-compatible almost complex structures. For any $J \in \mathscr{J}$ and smooth maps $u : \mathbb{R}\times S^1 \to X$, the elliptic PDE Floer studied is the perturbed Cauchy-Riemann equation

$$\frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - X_{H_t}\right) = 0 \tag{2.2}$$

satisfying boundary conditions

$$\lim_{s\to-\infty} u(s,t) = x^+(t), \quad \lim_{s\to+\infty} u(s,t) = x^-(t), \tag{2.3}$$

where $x^{\pm}(t)$ are two non-degenerate 1-periodic solutions of (2.1). A 1-periodic solution $x(t)$ of (2.1) is non-degenerate if the following condition holds

$$\det(d\phi_1(x(0)) - Id) \neq 0. \tag{2.4}$$

Denote by $\mathscr{M}(x^+,x^-,H,J)$ the (unparameterized) moduli space of solutions of (2.2) and (2.3). For generic J and H_t, these spaces are smooth finite dimensional manifolds ([7, 18]). This is called transversality property which is very important in the construction of Floer homology and can be regarded as an analog to the Morse-Smale-Thom transversality in Morse theory.

Denote the left hand side of (2.2) by

$$\bar{\partial}_{J,H}\, u = \frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - X_{H_t}\right).$$

Then $\bar{\partial}_{J,H}$ can be thought of as a section of a Banach vector bundle $\mathscr{E} \to \mathscr{B}$, where $\mathscr{B}$ is a Banach manifold which is some $W^{1,p}$-completion of the space of smooth maps $C^\infty(\mathbb{R}\times S^1, X)$. The fiber over $u \in \mathscr{B}$ is $\mathscr{E}_u = L^p(\mathbb{R}\times S^1, u^*TX)$. Then the moduli space $\mathscr{M}(x^+,x^-,H,J)$ is just the zero set of the section $\bar{\partial}_{J,H} : \mathscr{B} \to \mathscr{E}$. A key step to prove that for generic J and H_t the moduli space

$\mathscr{M}(x^+,x^-,H,J)$ is a smooth manifold is to show that the section $\bar{\partial}_{J,H}$ is a Fredholm section. That is to say, at each $u \in \mathscr{M}(x^+,x^-,H,J)$, the vertical differential $D_u = D_u\bar{\partial}_{J,H}$ of $\bar{\partial}_{J,H}$ is a Fredholm operator. Explicitly,

$$D_u : W^{1,p}(\mathbb{R}\times S^1, u^*TX) \to L^p(\mathbb{R}\times S^1, u^*TX)$$

$$D_u\xi = \nabla_s\xi + J(u)(\nabla_t\xi - \nabla_\xi X_{H_t}) + \nabla_\xi J(u)(\partial_t u - X_{H_t}(u)), \tag{2.5}$$

where ∇ denotes a Levi-Civita connection with respect to the metric $\langle\cdot,\cdot\rangle$. Floer [6] proved that if $x^{\pm}(t)$ are non-degenerate, then D_u is Fredholm operator. Moreover, the Fredholm index of D_u is the difference of Conley-Zehnder indices associated with Hamiltonian 1-periodic solutions $x^{\pm}$ (see [4, 17]).

To simplify the formula (2.5) one can choose a unitary trivialization of the vector bundle $u^*TX \to \mathbb{R}\times S^1$. This is a smooth family of vector space isomorphisms

$$\Psi(s,t) : (\mathbb{R}^{2n}, \omega_0, J_0) \to (T_{u(s,t)}X, \omega, J),$$

where ω_0 and J_0 are standard symplectic and complex structures on $\mathbb{R}^{2n}$. Under this trivialization, it is equivalent to study an operator acting on vector-valued functions

$$L : W^{1,p}(\mathbb{R}\times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R}\times S^1, \mathbb{R}^{2n})$$

$$L\,\xi = \partial_s\xi + J_0\partial_t\xi + S\xi,$$

where $2n\times 2n$ matrices $S(s,t)$ are defined by

$$S = \Psi^{-1}[\nabla_s\Psi + J(\nabla_t\Psi - \nabla_\Psi X_{H_t}) + \nabla_\Psi J(u)(\partial_t u - X_{H_t}(u))].$$

We can define $\psi_{\pm}(t) = \lim_{s\to\mp\infty}\Psi(s,t)$ and

$$S_{\pm}(t) = \lim_{s\to\mp\infty} S(s,t) = \psi_{\pm}^{-1}J(\nabla_t\psi_{\pm} - \nabla_{\psi_{\pm}}X_{H_t}).$$

By [18] we know that both $S_{\pm}$ and $J_0S_{\pm}$ are symmetric, and up to a compact perturbation one can assume S is symmetric for all s and t. Then one can associate to the symmetric matrix-valued function a symplectic matrix-valued function $P : \mathbb{R}\times\mathbb{R} \to Sp(2n)$, where $Sp(2n) := \{\Phi \in M_{2n\times 2n} \mid \Phi^T J_0 \Phi = J_0\}$ is the group of symplectic matrices, defined by

$$\partial_t P(s,t) - J_0 S(s,t) P(s,t) = 0, \quad P(s,0) = Id. \tag{2.6}$$

Thus we have symplectic paths $P_{\pm}(t) = \lim_{s\to\mp\infty} P(s,t)$. Then to show that operator D_u is Fredholm is equivalent to verifying that under the non-degenerate conditions

$$\det(P_{\pm}(1) - Id) \neq 0, \tag{2.7}$$

the operator $L = \partial_s + J_0\partial_t + S$ is Fredholm.

Definition 2.1. Symplectic paths $P_{\pm}(t)$ are called asymptotics of the operator L. We say the operator L has non-degenerate asymptotics if (2.7) hold, which is equivalent to that 1-periodic solutions $x^{\pm}$ are non-degenerate.

Denote by $l_\pm = \partial_t - J_0 S_\pm(t)$. $l_\pm$ can be regarded as an expression under the trivialization of the linearized operator of Hamiltonian equation (2.1) at solutions $x^\pm$. Then the non-degenerate condition (2.7) is equivalent to that operators $l_\pm$ are invertible. Also we note that another condition which is equivalent to (2.7) is that matrices $S_\pm$ have no vanishing eigenvalues.

2.2. *Differential operators in LG-model*

Let $(X, \omega = d\lambda)$ be an exact symplectic manifold with boundary and with a compatible almost complex structure J, and $W : X \to \mathbb{C}$ be a Lefschetz fibration ([19]) which is a J-holomorphic Morse function. To this data physicists associate a Landau-Ginzburg (LG) model.

For any pair of distinct critical points x_i, x_j of W, denote the space of smooth paths connecting x_i and x_j by

$$\mathscr{P}M_{i,j} := \left\{ \gamma \in C^\infty(\mathbb{R}, X) \;\middle|\; \lim_{t\to-\infty} \gamma(t) = x_i,\ \lim_{t\to+\infty} \gamma(t) = x_j \right\}$$

The action functional

$$\mathscr{A}_W : \mathscr{P}M_{i,j} \to \mathbb{R}$$

can be defined as

$$\mathscr{A}_W(\gamma) = \int_{-\infty}^{+\infty} \gamma^* \lambda + \int_{-\infty}^{+\infty} \mathrm{Re}(\zeta_{ij}^{-1} W \circ \gamma(t)) dt,$$

where ζ_{ij} is the phase such that $\mathrm{i}\zeta_{ij} = \frac{W(x_j) - W(x_i)}{|W(x_j) - W(x_i)|}$ is the unit in $\mathbb{C}$.

The critical curves of the action functional $\mathscr{A}_W$ are solutions of physicists' ζ-soliton equation

$$\frac{d}{dt}\gamma(t) = J\nabla[\mathrm{Re}(\zeta_{ij}^{-1} W(\gamma(t)))], \tag{2.8}$$

satisfying boundary conditions

$$\lim_{t\to-\infty} \gamma(t) = x_i, \qquad \lim_{t\to+\infty} \gamma(t) = x_j. \tag{2.9}$$

which can be considered as an equation for phase flow of autonomous Hamiltonian $H_{ij} := \mathrm{Re}(\zeta_{ij}^{-1} W)$[a]. The gradient is defined with respect to the associated Riemannian metric $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$.

We generically assume that, to critical points x_i and x_j, the associated Lefschetz thimbles L_i and L_j intersect transversally in the fiber $W^{-1}(z_0)$ over a regular value z_0 of W on the line segment in $\mathbb{C}$ between $W(x_i)$ and $W(x_j)$. In this case, there will be a finite number of geometrically different critical curves of $\mathscr{A}_W$, one for each intersection point $p \in L_i \cap L_j$. Denote such a unparameterized curve by Γ_p which actually represents a $\mathbb{R}$-family of solutions of (2.8) and (2.9) because (2.8) is an autonomous Hamiltonian equation. Denote by $\mathscr{P}_{ij} = \bigcup_{p\in L_i\cap L_j} \Gamma_p$.

Definition 2.2. We say a solution $\gamma(t)$ of (2.8) and (2.9) are transversally non-degenerate if for each intersection point $p \in L_i \cap L_j$ the unparameterized curve Γ_p are isolated. This is equivalent

[a]Generally, the functional $\mathscr{A}_W$ and equation (2.8) can be defined for any phase ζ, while there might be no solution for general ζ since solutions to (2.8) and (2.9) project to the unique straight line of slope $\mathrm{i}\zeta_{ij}$ in the complex W-plane.

to the assumption that the associated Lefschetz thimbles L_i and L_j intersect transversally in a fiber $W^{-1}(z_0)$ over a regular value of W.

For two distinct $\Gamma_+, \Gamma_- \in \mathscr{P}_{ij}$, the ζ-instanton equation is the perturbed Cauchy-Riemann equation

$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \nabla H_{ij} = 0, \tag{2.10}$$

satisfying

$$\lim_{t\to -\infty} u(s,t) = x_i, \quad \lim_{t\to +\infty} u(s,t) = x_j, \tag{2.11}$$

$$\lim_{s\to +\infty} u(s,t) = \gamma^-(t) \in \Gamma_-, \quad \lim_{s\to -\infty} u(s,t) = \gamma^+(t) \in \Gamma_+. \tag{2.12}$$

In [9], Physicists formulate an outline of construction of invariants via solutions of ζ-instanton equation (2.10) satisfying various boundary conditions. In particular, when solutions u of (2.10) satisfy boundary conditions (2.11) and (2.12), some sort of Morse-Floer type homology might be defined. It is a challenge for mathematicians to realize these constructions. One may expect that, by using Floer's arguments on the study of the moduli space of solutions of (2.10), (2.11) and (2.12), similar construction will go through provided that some non-degenerate conditions hold. However, it is clear that H_{ij} is time independent and (2.8) is an autonomous Hamiltonian equation which has no non-degenerate solutions. Therefore, the argument of Floer can not apply directly. Explicitly, in current LG-model setting, the differential D_u of the equation (2.10) is

$$D_u : W^{1,p}(\mathbb{R}\times\mathbb{R}, u^*TX) \to L^p(\mathbb{R}\times\mathbb{R}, u^*TX)$$

$$D_u\xi = \nabla_s\xi + J(u)(\nabla_t\xi - \nabla_\xi X_{H_{ij}}) + \nabla_\xi J(u)(\partial_t u - X_{H_{ij}}(u)), \tag{2.13}$$

and, under a unitary trivialization Ψ of the vector bundle $u^*TX \to \mathbb{R}^2$, the equivalently expressed operator L is of similar form

$$L : W^{1,p}(\mathbb{R}^2, \mathbb{R}^{2n}) \to L^p(\mathbb{R}^2, \mathbb{R}^{2n})$$

$$L\,\xi = \partial_s\xi + J_0\partial_t\xi + S\xi, \tag{2.14}$$

where $S(s,t)$ are defined by

$$S = \Psi^{-1}[\nabla_s\Psi + J(\nabla_t\Psi - \nabla_\Psi X_{H_{ij}}) + \nabla_\Psi J(\partial_t u - X_{H_{ij}})]. \tag{2.15}$$

Similarly, we can define $\psi_\pm(t) = \lim_{s\to\mp\infty} \Psi(s,t)$ and

$$S_\pm(t) = \lim_{s\to\mp\infty} S(s,t) = \psi_\pm^{-1} J(\nabla_t\psi_\pm - \nabla_{\psi_\pm} X_{H_{ij}}). \tag{2.16}$$

Since solutions of autonomous Hamiltonian equation (2.8) and (2.9) always occur in an $\mathbb{R}$-family, the asymptotics of L are not non-degenerate any more. Fortunately, if solutions $\gamma(t)$ of equations (2.8) and (2.9) are still of some non-degeneracy in the direction normal to the subspace $T\gamma \oplus JT\gamma$, which we call transversal non-degeneracy, then some Fredhom property may also be obtained. In this paper, we will show that, working with some weighted Sobolev norm, we can do some perturbation to get non-degenerate asymptotics and prove the following result

Theorem 2.1 (Theorem 4.1). *Given a strandard symplectic space with the standard complex structure $(\mathbb{R}^{2n},\omega_0,J_0)$. Let $S:\mathbb{R}^2\to M_{2n\times 2n}(\mathbb{R})$ be a smooth bounded matrix-valued function such that $S(s,t)$ are symmetric for all $(s,t)\in\mathbb{R}^2$. Assume that the following conditions hold:*

(i) *As $s\to\mp\infty$, $S(s,t)$ converges to $S_\pm(t)$ in the C^0-topology such that for each $t\in\mathbb{R}$, the matrix $J_0S_\pm(t)$ is symmetric;*
(ii) *As $t\to\mp\infty$, $S(s,t)$ converges to constant symmetric matrices in C^0-topology.*
(iii) *There are splittings for those two paths of matrices as*

$$S_\pm(t):=\begin{pmatrix}0 & 0\\ 0 & \widetilde{S}_\pm(t)\end{pmatrix},$$

where $\widetilde{S}_\pm(t):=-J_0\dot{P}_\pm(t)P_\pm(t)^{-1}$ and $P_\pm$ are two non-degenerate smooth paths in symplectic group $Sp(2n-2)$ (see Definition 3.2).

Then the operator $L=\partial_s+J_0\partial_t+S$ between some weighted Sobolev spaces (see Definitions 3.1, 3.3 and (3.2)) is Fredholm.

As a result, Theorem 1.1 holds. That is, under the assumption that solutions of (2.8) and (2.9) are transversally non-degenerate and in the sense of weighted norm, the operator D_u is Fredholm. This is the first step for succeeding works on constructing Morse-Floer type homology and even A_∞-category in LG-model.

We remark that in [10] Haydys studied the moduli space of solutions of a modified version of ζ-instanton equation. After suitable modification of the ζ-soliton equation the solutions of the new equation turn to be non-degenerate, while the physical meaning of the new equation seems not so obvious. However, some new methods are introduced in [10] and the results are inspiring for our current research.

3. Operators on weighted spaces

3.1. *Weighted spaces*

In the setting of LG-model, the domain of pseudo-holomorphic maps u is $\mathbb{R}^2$ with coordinates s and t. We can obtain a compactification $\overline{\mathbb{R}^2}$ of the $\mathbb{R}^2$ by extending the coordinate s and t to $\pm\infty$ such that $(s,\pm\infty)=(s',\pm\infty)$ for $\forall\ s,s'\in[-\infty,+\infty]$. Topologically, $\overline{\mathbb{R}^2}$ is identified with a digon. Denote by $\ell_\pm$ the 1-dimensional boundary component $\{(\mp\infty,t),\ t\in[-\infty,+\infty]\}$. Let E be a symplectic vector bundle of rank $2n$ over $\mathbb{R}^2$ together with two families of symplectic vector spaces $E_t^\pm$ which are symplectic-linearly identified with a standard symplectic space with symplectic splitting $(\mathbb{R}^2,\omega_0)\oplus(\mathbb{R}^{2n-2},\omega_0)$ for all $t\in\mathbb{R}$. These give rise to a symplectic vector bundle $\overline{E}$ over the compactification $\overline{\mathbb{R}^2}$ such that fibres over the compactification boundary have fixed symplectic splittings and trivialization, that is we have symplectic linear identifications

$$\overline{E}_{(+\infty,t)}\cong\overline{E}_{(-\infty,t)}\cong(\mathbb{R}^2,\omega_0)\oplus(\mathbb{R}^{2n-2},\omega_0)$$

for each $t\in[-\infty,+\infty]$.

We choose a particular smooth function $F(t) > 0$ such that $\dot{F}(t) = \dfrac{dF}{dt} = e^{-t^2}$ and $F(0) = 1$. Recall that for a matrix B, the exponential of B is $e^B = Id + B + \dfrac{1}{2!}B^2 + \cdots + \dfrac{1}{n!}B^n + \cdots$.

Definition 3.1. Given a symplectic vector bundle E over $\mathbb{R}^2$ and symplectic vector spaces $E_{(\pm\infty,t)}$ as above. Let J be a compatible complex structure on E. For $A > 0$, an A-weighted (or (A,J)-twisted) Sobolev space of sections ξ of E, denoted by $W^{k,p,A}(E,J)$ or simply $W^{k,p,A}(E)$, satisfies the following:

(1) ξ are locally in $W^{k,p}(E)$;

(2) as $s \to \pm\infty$, $e^{\frac{A}{p}F(t)J}\xi(s,t) \in W^{k,p}(E)$;

(3) the Banach norm on $W^{k,p,A}(E)$ is defined as $\|\xi\|_{W^{k,p,A}} = \|e^{\frac{A}{p}F(t)J}\xi\|_{W^{k,p}}$. For instance, when $k = 1$,

$$\|\xi\|^p_{W^{1,p,A}} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |e^{\frac{A}{p}F(t)J}\xi|^p + |\nabla_s(e^{\frac{A}{p}F(t)J}\xi)|^p + |\nabla_t(e^{\frac{A}{p}F(t)J}\xi)|^p ds dt.$$

(4) $\xi(s,\pm\infty)$ are independent of s.

We remark that the method of studying operators in weighted Sobolev spaces is partially motivated by the work of [3], in which the problem of degenerate asymptotics also appears (for example, near the cylindrical ends asymptotic to the Reeb orbits) in the study of analytic foundation of symplectic field theory. In section 2 of [3], since the usual Fredholm theory cannot apply directly to the case of differential operators with degenerate asymptotics, a kind of d-weighted Sobolev space $L^{p,d}_k(E)$ of sections of the bundle E over closed surface is constructed. While in the case of LG-model, since we consider Riemann surface with boundary, the appropriate weight function $AF(t)$ is chosen to overcome degeneracy. The calculation in the next section shows that the Fredholm property can be also obtained in the sense of such weighted spaces.

3.2. *Symplectic paths*

Let $P : (-\infty,+\infty) \to Sp(2n)$ be a smooth path in the symplectic group, J_0 be the standard complex structure on $\mathbb{R}^{2n}$. Then $S(t) := -J_0\dot{P}(t)P(t)^{-1}$ is a path of matrices which are symmetric. Consider an operator

$$l : W^{k,p}(\mathbb{R},\mathbb{R}^{2n}) \to W^{k-1,p}(\mathbb{R},\mathbb{R}^{2n})$$

defined by

$$l := \frac{d}{dt} - J_0 S(t).$$

Definition 3.2. We say a $2n$-order smooth path in the symplectic group

$$P : (-\infty,+\infty) \to Sp(2n), \quad P(0) = Id$$

is (k,p)-**non-degenerate** or simply non-degenerate if the operator l is invertible. Denote by $\mathcal{P}_{k,p}(2n)$ or simply $\mathcal{P}(2n)$ the set of above defined non-degenerate $2n$-order symplectic paths.

Let $\theta^\pm$ be two sections of the restricted bundle $\overline{E}|_{\ell_\pm}$ such that at each point $(\pm\infty,t)$ they are in the first summand of $\overline{E}_{(\pm\infty,t)} \cong (\mathbb{R}^2,\omega_0) \oplus (\mathbb{R}^{2n-2},\omega_0)$, respectively, and $\theta^-(\cdot,\pm\infty) = \theta^+(\cdot,\pm\infty)$.

Choose a smooth cut-off function $\rho : \mathbb{R}^2 \to [0,1]$ such that $\rho(s,t) = 0$ near $s = +\infty$, $\rho(s) = 1$ near $s = -\infty$.

Definition 3.3. Denote by $\mathscr{V}_{+,-}$ the vector space generated by the sections

$$\sigma^+(s,t) = \rho(s,t)\theta^+ \quad \text{and} \quad \sigma^-(s,t) = (1-\rho(s,t))\theta^-.$$

3.3. *Linear operators*

Let E be the symplectic vector bundle of rank $2n$ defined above. For two non-degenerate $(2n-2)$-order symplectic paths $P_\pm \in \mathcal{P}(2n-2)$, denote by $\mathcal{O}(\mathbb{R}^2, E, P_\pm)$ the the set of bounded linear operators

$$D : \mathscr{V}_{+,-} \oplus W^{k,p,A}(E) \to W^{k-1,p,A}(\Lambda^{0,1}(E)), \tag{3.1}$$

with $k \geq 1$, $p > 2$, $A > 0$, such that under a suitable unitary trivialization $\Psi(s,t) : (\mathbb{R}^{2n}, J_0) \to (E, J)$ the following are satisfied:

(1) the operator D is equivalent to a matrix-valued function

$$L : \mathscr{V}_{+,-} \oplus W^{k,p,A}(\mathbb{R}^2, \mathbb{R}^{2n}, J_0) \to W^{k-1,p,A}(\mathbb{R}^2, \mathbb{R}^{2n}, J_0) \tag{3.2}$$

$$L = \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S(s,t), \tag{3.3}$$

where $S(s,t)$ extend continuously to the compactification $\overline{\mathbb{R}^2}$ such that, in the interior of $\mathbb{R}^2$, they are locally in $W^{k,p}$, and near a boundary $S(s,t) - S(\pm\infty, t)$ and $S(s,t) - S(s, \pm\infty)$ are in $W^{k-1,p,A}(\mathbb{R}^2, \mathbb{R}^{2n} \times \mathbb{R}^{2n})$;

(2) those two paths of matrices $S(\pm\infty, t)$ split respectively in each $E_{(\pm\infty,t)} = \mathbb{R}^2 \oplus \mathbb{R}^{2n-2}$ as

$$S_\pm(t) := S(\pm\infty, t) = 0 \oplus \widetilde{S}_\pm(t) = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{S}_\pm(t) \end{pmatrix}, \tag{3.4}$$

where $\widetilde{S}_\pm(t) := -J_0 \dot{P}_\pm(t) P_\pm(t)^{-1}$ is a path of $(2n-2) \times (2n-2)$-matrices which are symmetric with respect to the Euclidean structure determined by the symplectic and the complex structure.

Recall

$$l_\pm : W^{k,p}(\mathbb{R}, \mathbb{R}^{2n}) \to W^{k-1,p}(\mathbb{R}, \mathbb{R}^{2n}) \tag{3.5}$$

$$l_\pm = \frac{d}{dt} - J_0 S_\pm(t).$$

Denote by

$$\tilde{l}_\pm := \frac{d}{dt} - J_0 \widetilde{S}_\pm(t).$$

Since $P_\pm$ are non-degenerate $(2n-2)$-order symplectic paths, by the Definition 3.2, the operator $\tilde{l}_\pm$ are invertible. Note that the differential operators we consider have degenerate asymptotics, since the matrices S vanish on the first summand of each $E_{(\pm\infty,t)} = \mathbb{R}^2 \oplus \mathbb{R}^{2n-2}$, so the operator $l_\pm$ are not invertible. Hence, we can not directly apply to them the argument of Fredholm theory as in [10].

We remark that when we consider our actual example of ζ-instanton equation (2.10), let $E = u^*TX$, then in some sense of weighted norm, its linearized operator D_u is just as the one in (3.1) which after trivialization also looks like L in (3.3). It is clear that the Definition 2.2 of transversal non-degeneracy is equivalent to the following

Definition 3.4. We say a solution $\gamma(t)$ of (2.8) and (2.9) are transversally non-degenerate if, under a suitable trivialization, for the operator L in (3.3) the following hold

(i) $S(s,t)$ is independent of s and of the form $S(t) = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{S}(t) \end{pmatrix}$,

(ii) the operator $\tilde{l} = \dfrac{d}{dt} - J_0\widetilde{S}(t)$ is invertible.

4. Fredholm property

Assumption 1. We assume that the following conditions hold:

(1) $S : \mathbb{R}^2 \to M_{2n\times 2n}(\mathbb{R})$ is C^∞-bounded;
(2) $S(s,t)$ converges to $S_\pm(t)$ in the $C^0(\mathbb{R})$-topology as $s \to \mp\infty$;
(3) The operators $\dfrac{d}{dt} - J_0 S_\pm(t)$ are invertible;

4.1. *Works by Haydys*

Here we recall some conclusions drawn by Haydys in [10] to get Fredholm property for certain operators with non-degenerate asymptotics. Such operators appears when one considers the modified version of ζ-instanton equations. The following lemma is just the Lemma 2.28 in [10].

Lemma 4.1. *Let*

$$A = \sum_{|\alpha|\le\mu} f_\alpha(x)\frac{\partial}{\partial x^\alpha},\ \ x \in \mathbb{R}^n$$

be a uniformly elliptic C^∞ bounded differential operator of order μ, where f_α takes values in the space of $l\times l$-matrices. If $A : W^{\mu,2}(\mathbb{R}^n;\mathbb{R}^l) \to L^2(\mathbb{R}^n;\mathbb{R}^l)$ is Fredholm, then $A : W^{k+\mu,p}(\mathbb{R}^n;\mathbb{R}^l) \to W^{k,p}(\mathbb{R}^n;\mathbb{R}^l)$ is Fredholm for all $k \in \mathbb{R}$, $p > 1$ and its index depends neither on k nor on p.

The following proposition is just the Lemma 2.29 in [10].

Proposition 4.1. *Assume that in addition to conditions* (1)-(3) *of Assumption* 1 *the following holds:*

(i) *For each $t \in \mathbb{R}$ the matrix $J_0 S_\pm(t)$ is symmetric;*
(ii) *$S(s,t)$ converges to constant matrices $\Sigma_\pm$ in the C^0-topology as $t \to \mp\infty$. Moreover,*

$$\Sigma_+ = \lim_{t\to-\infty} S_+(t) = \lim_{t\to-\infty} S_-(t) \quad \textit{and} \quad \Sigma_- = \lim_{t\to+\infty} S_+(t) = \lim_{t\to+\infty} S_-(t)$$

are symmetric matrices.

Then $L: W^{k,p}(\mathbb{R}^2;\mathbb{R}^{2n}) \to W^{k-1,p}(\mathbb{R}^2;\mathbb{R}^{2n})$ *is Fredholm for any* $k \in \mathbb{R}$*,* $p > 1$ *and its index depends neither on* k *nor on* p*.*

Proof. We outline the key points of proof. First consider special case when $k = 1$, $p = 2$.
1. Since

$$l_{\pm} : W^{1,2}(\mathbb{R},\mathbb{R}^{2n}) \to L^2(\mathbb{R},\mathbb{R}^{2n})$$

are isomorphisms, one can show that the s-independent operators

$$L_{\pm} : W^{1,2}(\mathbb{R}^2,\mathbb{R}^{2n}) \to L^2(\mathbb{R}^2,\mathbb{R}^{2n}),$$
$$L_{\pm} = \partial_s + J_0\partial_t + S_{\pm}(t)$$

are invertible.
2. Verify that the operators

$$k_{\pm} = \frac{d}{ds} + \Sigma_{\pm} : W^{1,2}(\mathbb{R},\mathbb{R}^{2n}) \to L^2(\mathbb{R},\mathbb{R}^{2n})$$

are isomorphisms, which implies the operators with constant coefficients

$$J_0K_{\pm} : W^{1,2}(\mathbb{R}^2,\mathbb{R}^{2n}) \to L^2(\mathbb{R}^2,\mathbb{R}^{2n}),$$
$$J_0K_{\pm} = -\partial_t + J_0(\partial_s + \Sigma_{\pm})$$

and hence $K_{\pm}$ are invertible.
3. Verify that any limit operator $L_* = \partial_s + J_0\partial_t + S_*(s,t)$ of L must be $K_{\pm}$ or

$$L^{\tau}_{\pm} := \partial_s + J_0\partial_t + S_{\pm}(t+\tau),$$

each of which is also an isomorphism.

Then the proposition follows from a result in the theory of limit operators (Theorem 5.6 of [15]) and the Lemma 4.1. □

4.2. *Fredholm property*

Note in our case, however, the condition (3) of Assumption 1 does not hold since our ζ-solitons are degenerate. So we can not apply the Proposition 4.1 directly.

Theorem 4.1. *Assume that in addition to conditions* (1)-(2) *of Assumption* 1 *the following conditions hold:*

(a) *For each* $t \in \mathbb{R}$*, the matrix* $J_0S_{\pm}(t)$ *is symmetric;*
(b) $S(s,t)$ *converges to constant matrices* $\Sigma_{\pm}$ *in the* C^0*-topology as* $t \to \mp\infty$*. Moreover,*

$$\Sigma_+ = \lim_{t\to-\infty} S_+(t) = \lim_{t\to-\infty} S_-(t), \quad \Sigma_- = \lim_{t\to+\infty} S_+(t) = \lim_{t\to+\infty} S_-(t)$$

are symmetric matrices;
(c) $P_{\pm} \in \mathcal{P}(2n-2)$*,* $D \in \mathcal{O}(\mathbb{R}^2, E, P_{\pm})$*.*

Then the operator L in (3.3) (*and hence D*) *is Fredholm for any* $k \in \mathbb{R}$, $p > 1$ *and its index* $\operatorname{Ind} L$ *depends neither on k nor on p.*

Proof. First we fix $k = 1$, $p = 2$.

Consider the restriction of L to $W^{1,2,A}(\mathbb{R}^2, \mathbb{R}^{2n})$. Note that the space $W^{1,2,A}(\mathbb{R}^2, \mathbb{R}^{2n})$ is isomorphic to the space $W^{1,2}(\mathbb{R}^2, \mathbb{R}^{2n})$, via a transformation that is identity away from the boundaries and given by $e^{\frac{A}{2}F(t)J_0}$ near the boundaries $\ell_\pm$. Note that the matrix $e^{\frac{A}{2}F(t)J_0}$ commutes with J_0 and $\dot{F}(t) = e^{-t^2}$, then

$$\begin{aligned}
L(e^{\frac{A}{2}F(t)J_0}\xi) &= e^{\frac{A}{2}F(t)J_0}\partial_s\xi + J_0[e^{\frac{A}{2}F(t)J_0}J_0(\tfrac{A}{2}\dot{F}(t))\xi + e^{\frac{A}{2}F(t)J_0}\partial_t\xi] + Se^{\frac{A}{2}F(t)J_0}\xi \\
&= e^{\frac{A}{2}F(t)J_0}[\partial_s + J_0\partial_t + (e^{\frac{A}{2}F(t)J_0})^{-1}Se^{\frac{A}{2}F(t)J_0} - (\tfrac{A}{2}e^{-t^2})Id]\ \xi \\
&= e^{\frac{A}{2}F(t)J_0}[\partial_s + J_0\partial_t + (e^{\frac{A}{2}F(t)J_0})^{-1}(S - (\tfrac{A}{2}e^{-t^2})Id)e^{\frac{A}{2}F(t)J_0}]\ \xi \\
&= e^{\frac{A}{2}F(t)J_0}[\partial_s + J_0\partial_t + S_A]\ \xi,
\end{aligned}$$

where denote by $S_A = (e^{\frac{A}{2}F(t)J_0})^{-1}(S - (\frac{A}{2}e^{-t^2})Id)e^{\frac{A}{2}F(t)J_0}$. Since the matrix $J_0S_\pm(t)$ is symmetric, we can as well assume that J_0S is symmetric for all s and t. Hence it is clear that S_A is symmetric and S_A goes to S as $A \to 0$. Using the isomorphism one can conjugate our restricted operator to an operator

$$L' : W^{1,2}(\mathbb{R}^2, \mathbb{R}^{2n}) \to L^2(\mathbb{R}^2, \mathbb{R}^{2n}),$$

which looks like

$$L' := \partial_s + J_0\partial_t + S_A.$$

Let $S'(s,t) = S_(s,t) - (\dfrac{A}{2}e^{-t^2})Id$. Then

$$S'_\pm(t) := S'(\pm\infty, t) = S_\pm(t) - (\frac{A}{2}e^{-t^2})Id = -J_0\dot{P}'_\pm(t)P'_\pm(t)^{-1}, \tag{4.1}$$

where $S_\pm(t) = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{S}_\pm(t) \end{pmatrix}$, $P'_\pm$ are $2n$-order symplectic paths. Since from (3.4) $\widetilde{S}_\pm(t) := -J_0\dot{P}_\pm(t)P_\pm(t)^{-1}$, the operator L has asymptotics $\begin{pmatrix} 0 & 0 \\ 0 & P_\pm(t) \end{pmatrix}$, then a calculation shows that, up to a similarity transformation, the new operator L' has asymptotics

$$P'_\pm(t) = e^{-\frac{A}{2}F(t)} \begin{pmatrix} Id & 0 \\ 0 & P_\pm(t) \end{pmatrix},$$

which are not degenerate anymore.

Let $S_{A\pm}(t) = \lim\limits_{s \to \pm\infty} S_A(s,t)$. We consider the operators

$$\begin{gathered}
l_{A\pm} : W^{1,2,A}(\mathbb{R}, \mathbb{R}^{2n}) \to L^{2,A}(\mathbb{R}, \mathbb{R}^{2n}), \\
l_{A\pm} := \frac{d}{dt} - J_0S_{A\pm}(t).
\end{gathered}$$

Note that $l_{A\pm}$ are isomorphisms, and for each $t \in \mathbb{R}$, the matrix $J_0S_{A\pm}(t)$ are symmetric. Since $S(s,t)$ converges to $\Sigma_\pm$, $S_A(s,t)$ also converges to constant symmetric matrices, denoted by $\Sigma_{A\pm}$ in the C^0-topology as $t \to \mp\infty$. Thus, we can apply the Proposition 4.1 to L'. As a consequence, we obtain a Fredholm operator L'.

We remark that in our actual example, we consider bounded linear operators

$$D_u : \mathscr{V}_{+,-} \oplus W^{k,p,A}(u^*TX) \to W^{k-1,p,A}(\Lambda^{0,1}(u^*TX)) \tag{4.2}$$

for a solution u of the ζ-instanton equation asymptotic to a pair of ζ-solitons $\gamma^\pm$. The restriction of D_u to $W^{k,p,A}(\mathbb{R}^2, u^*TX)$ is conjugated to a Cauchy-Riemann operator

$$D'_u : W^{k,p}(u^*TX) \to W^{k-1,p}(\Lambda^{0,1}(u^*TX)), \tag{4.3}$$

which under trivialization is equivalently expressed by L'. The index of D'_u or L' can be given by relative Maslov index ([16])

$$\mu(\gamma^+, \gamma^-; u) = \mu(\Lambda_0^+, \Lambda_0^-)$$

which is the Robin-Salamon index for a pair of paths of Lagrangian subspaces $\Lambda_0^\pm : \mathbb{R} \to \mathscr{L}ag(\mathbb{R}^{2n})$. This will be explained in the next subsection.

Let us now consider the summand $\mathscr{V}_{+,-}$. The image of these sections have their support near a boundary and decay exponentially, hence L maps the new elements into $L^{2,A}(E)$. The space $\mathscr{V}_{+,-}$ is of dimension 2. Thus the operator L is Fredholm. Its Fredholm index is

$$\text{Ind}\, L = \text{Ind}(L|_{W^{1,p,A}(E)}) + 2 = \text{Ind}\, L' + 2$$

the sum of the index of the operator restricted to $W^{1,p,A}(E)$ and the dimension of $\mathscr{V}_{+,-}$.

When we consider arbitrary k and p, the last statement of the Theorem follows from Lemma 4.1. □

As an application, we consider the symplectic manifold (X, ω, J) with superpotential W and solutions of ζ-instanton equation. The following corollary is just our main result theorem 1.1

Corollary 4.1. *Assume that solutions $\gamma^\pm(t)$ of* (2.8) *and* (2.9) *are transversally non-degenerate. For each solution u of* (2.10)–(2.12) *the map*

$$D_u : \mathscr{V}_{+,-} \oplus W^{k,p,A}(\mathbb{R}^2; u^*TX) \to W^{k-1,p,A}(\mathbb{R}^2; u^*TX)$$

defined by (2.13) *is Fredholm for any $k \in \mathbb{R}$, $p > 1$, and its index depends neither on k nor on p, which can be given by*

$$\text{Ind} D_u = \mu(\gamma^+, \gamma^-, u) + 2.$$

Proof. The conclusions hold by applying the Theorem 4.1 to the case that $E = u^*TX$ and the matrix-valued function $S(s,t)$ defined by (2.15). The space $\mathscr{V}_{+,-}$ now is generated by Hamiltonian vector field along the solutions $\gamma^\pm(t)$ of (2.8) and (2.9). We can verify that $J_0 S_\pm(t)$ is symmetric by computation via (2.16). Note that matrices $\Sigma_\mp$ are the Hessians of the function $\text{Re}(\xi_{ij}^{-1} W)$ at x_i and x_j, so they are symmetric. □

4.3. *Relative Maslov index*

Now for our concrete example of symplectic manifold (X, ω, J) with a superpotential W, for a pair of solutions $\gamma^\pm$ of (2.8) and (2.9) and a solution u of (2.10)–(2.12), we study the computation of the index of D'_u in (4.3) based on the method in [10]. Since the index depends neither on k nor on p, we can set $k = 1$, $p = 2$. With an arbitrary smooth curve $\gamma(t)$ in X satisfying

$$\lim_{t\to-\infty}\gamma(t) = x_i, \quad \lim_{t\to+\infty}\gamma(t) = x_j, \quad \text{and} \quad \lim_{t\to\pm\infty}\dot{\gamma}(t) = 0, \tag{4.4}$$

we can associate a pair of Lagrangian subspaces $\Lambda^\pm$ in $T_{\gamma(0)}X$. Moreover, the kernel of the operator

$$\mathscr{A} : W^{1,2}(\gamma^*TX) \to L^2(\gamma^*TX)$$
$$\mathscr{A}\,\xi = J_0\partial_t\xi + S\xi$$

can be identified with $\Lambda^+ \cap \Lambda^-$. In particular, $\ker\mathscr{A}$ is nontrivial if and only if $\Lambda^+ \cap \Lambda^- \neq \{0\}$. In precise, the Lagrangian subspaces are defined as

$$\Lambda^\pm = \left\{ v \in T_{\gamma(0)}X \mid \lim_{t\to\mp\infty}\xi_v(t) = 0 \right\},$$

where ξ_v is the solution of Cauchy problem $\mathscr{A}\,\xi_v = 0$, $\xi_v(0) = v$.

For a pair of curves $\gamma^\pm$ satisfying (4.4), let $u : \mathbb{R}^2 \to X$ be any C^1-map such that each curve $\gamma_s(t) := u(s,t)$ also satisfies (4.4) and $\gamma_s \to \gamma^\pm$ as $s \to \mp\infty$ in the C^1-topology. Then as s goes from $-\infty$ to $+\infty$, we obtain two paths of Lagrangian subspaces $\Lambda^\pm(s)$. If for the two curves γ^+ and γ^-, the associated pair of Lagrangian subspaces $\Lambda^+(\pm\infty)$ and $\Lambda^-(\pm\infty)$ intersect transversally, respectively, then applying the argument of Robbin-Salamon [16] to the pair of paths of Lagrangian subspaces $(\Lambda^+(s), \Lambda^-(s))$ of $T_{\gamma_s(0)}X$, one can associate with the triple (γ^+, γ^-, u) an integer $\mu(\gamma^+, \gamma^-, u) = \mu(\Lambda^+, \Lambda^-)$, called relative Maslov index (see section 2.8 of [10]). Moreover, in this non-degenerate case, the number $\mu(\Lambda^+, \Lambda^-)$ is a homotopy invariant (see Corollary 3.3 of [16]).

Although solutions of ζ-soliton equation (2.8) are degenerate, which implies the associated two pairs of Lagrangian subspaces $(\Lambda^+(+\infty), \Lambda^-(+\infty))$ and $(\Lambda^+(-\infty), \Lambda^-(-\infty))$ might not intersect transversally, the relative Maslov index $\mu(\Lambda^+, \Lambda^-)$ is still well-defined. Then the actual Fredholm index of D'_u would be given by $\mu(\gamma^+, \gamma^-, u)$.

Proposition 4.2. *Assume that solutions $\gamma^\pm(t)$ of* (2.8) *and* (2.9) *are transversally non-degenerate. For each solution u of* (2.10)–(2.12)*, the restriction of D_u, defined by* (4.2) *and* (2.13)*, to $W^{k,p,A}(\mathbb{R}^2; u^*TX)$ is conjugate to a Fredholm operator D'_u whose index can be given by the relative maslov index*

$$\mathrm{Ind}D'_u = \mu(\gamma^+, \gamma^-, u).$$

Proof. The argument is almost the same as the one in Proposition 2.32 in [10]. The point is by [2] the index of D'_u (*i.e.* under a unitary trivialization the index of $L' = \partial_s + \mathscr{A}(s)$) can be computed with the help of the spectral flow of $\mathscr{A}(s)$ and, under the identification $\Lambda^+(s_0) \cap \Lambda^-(s_0) = \ker\mathscr{A}(s_0)$, the associated crossing forms $\Gamma(\Lambda^+, \Lambda^-, s_0)$ and $\Gamma(\mathscr{A}, s_0)$ coincide at each regular crossing s_0. The difference of our case from the one in [10] is that the subspaces $\Lambda^+(s)$ and $\Lambda^-(s)$ are not transverse for $s = \pm\infty$, while the definition of relative Maslov index still does work for the case that $\Lambda^+(\pm\infty) \cap \Lambda^-(\pm\infty) \neq 0$ but with only regular crossings. The contributions at $\pm\infty$ to the index are $\frac{1}{2}\mathrm{sign}\,\Gamma(\Lambda^+, \Lambda^-, +\infty) + \frac{1}{2}\mathrm{sign}\,\Gamma(\Lambda^+, \Lambda^-, -\infty)$. So the index $\mu(\gamma^+, \gamma_-, u)$ generally might be a half integer. This is induced by the one-dimensional degeneracy of γ^+ and γ^-. We remark that, in actual computation, one would perturb slightly the ζ-solitons to get non-degenerate curves $\widetilde{\gamma}^\pm$ satisfying

(4.4), and calculate the index Ind$D_{\tilde{u}}$ at the perturbed map $\tilde{u}$ which is close to ζ-instanton u. Then one can verify that Ind$D_{\tilde{u}} = \mu(\widetilde{\gamma}^{+}, \widetilde{\gamma}^{-}, \tilde{u}) - 1$. □

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