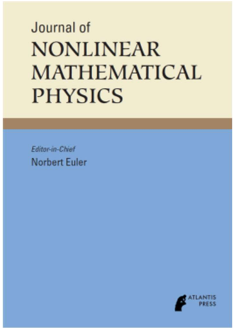

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

Solution formulas for the two-dimensional Toda lattice and particle-like solutions with unexpected asymptotic behaviour

Tomas Nilson, Cornelia Schiebold

To cite this article: Tomas Nilson, Cornelia Schiebold (2020) Solution formulas for the two-dimensional Toda lattice and particle-like solutions with unexpected asymptotic behaviour, Journal of Nonlinear Mathematical Physics 27:1, 57–94, DOI: https://doi.org/10.1080/14029251.2020.1683978

To link to this article: https://doi.org/10.1080/14029251.2020.1683978

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 27, No. 1 (2020) 57–94

Solution formulas for the two-dimensional Toda lattice and particle-like solutions with unexpected asymptotic behaviour

Tomas Nilson [a] and Cornelia Schiebold [a,b]

[a] *Department of Mathematics and Science Education, Mid Sweden University, S-851 70 Sundsvall, Sweden*

[b] *Instytut Matematyki, Uniwersytet Jana Kochanowskiego w Kielcach, Poland*

Received 19 December 2018

Accepted 7 July 2019

The first main aim of this article is to derive an explicit solution formula for the scalar two-dimensional Toda lattice depending on *three* independent operator parameters, ameliorating work in [31]. This is achieved by studying a noncommutative version of the 2d-Toda lattice, generalizing its soliton solution to the noncommutative setting.

The purpose of the applications part is to show that the family of solutions obtained from matrix data exhibits a rich variety of asymptotic behaviour. The first indicator is that web structures, studied extensively in the literature, see [4] and references therein, are a subfamily. Then three further classes of solutions (with increasingly unusual behaviour) are constructed, and their asymptotics are derived.

1. Introduction

In a similar way as the Toda lattice is related to the KdV equation, the two-dimensional Toda (2d-Toda) lattice

$$\frac{\partial^2}{\partial x \partial y} \log(1+w_n) = w_{n+1} - 2w_n + w_{n-1} \tag{1.1}$$

is the discretization of the Kadomtsev-Petviashvili equation

$$\big(-4u_t + 6uu_x + u_{xxx}\big)_x + 3\sigma^2 u_{yy} = 0, \tag{1.2}$$

more precisely the KP-II where $\sigma = 1$ (in contrast to the KP-I where $\sigma = \mathrm{i}$) [4]. Note that the unknown function $w_n = w_n(x,y)$ in (1.1) depends on two continuous variables x, $y \in \mathbb{R}$ and a discrete variable $n \in \mathbb{Z}$. Although (1.1) was already considered by Darboux [10] in 1915, its integrability was only established by Mikhailov [20] in 1979. For a review of recent research and extensive references, we refer to [37].

In the present article we develop an operator theoretic approach to the solution theory of the 2d-Toda lattice. Our model is the treatment of the KP equation given in [8,32]. To be more precise, [8] establishes a solution formula for (1.2) depending on two parameters A,B which are bounded linear operators on some Banach space, satisfying the strong assumption $[A,B] = 0$. However it is shown in [32] how this assumption can be dropped, allowing for more flexibility in applications. For the 2d-Toda lattice (1.1), a solution formula with commuting parameters is obtained in [31], but the question whether the extra condition is necessary has remained open so far. Our main result in Theorem 3.6 fills in this gap (and introduces and additional operator parameter D which will become significant in the applications).

Co-published by Atlantis Press and Taylor & Francis

Inserting the dependent variable transformation $w_n = (1+v_n)/(1+v_{n-1}) - 1$ in (1.1) yields

$$\frac{\partial^2}{\partial x \partial y} \log(1+v_n) = \frac{1+v_{n+1}}{1+v_n} - \frac{1+v_n}{1+v_{n-1}}, \tag{1.3}$$

which admits the noncommutative interpretation

$$\frac{\partial}{\partial y}\Big((I+V_n)^{-1}\frac{\partial}{\partial x}V_n\Big) = (I+V_n)^{-1}(I+V_{n+1}) - (I+V_{n-1})^{-1}(I+V_n). \tag{1.4}$$

Here we can view $V_n = V_n(x,y)$ as an unknown function taking its values in $\mathscr{L}(F)$, the space of bounded linear operators on a Banach space F, or more generally in a noncommutative Banach algebra. In Theorem 2.1 we generalize the soliton of (1.3) to the operator level. The proof of this (which makes extensive use of the tool box of operator identities provided in Appendix D) is the most involved step towards achieving our final goal, a solution formula for the scalar equation depending on operator parameters. Here it should be observed that discrete integrable systems have the tendency to require considerably more intricate computations than their continuous counterparts, as will also become apparent in our case. Once the operator soliton is known, we can follow more familiar roads to extract the desired solution formula for the scalar equation. This is elaborated in Section 3, building on related work in [2,27,31]. In Appendix C we extend our results to Hirota's bilinear form of (1.1). We would like to mention a quite parallel recent approach, called Cauchy matrix approach in [22], see also [39,40], where this approach is extended to a general method. We refer in particular to [39, Section 5.2] for explanations on the connection between the Cauchy matrix approach and the operator method employed in the present paper.

The solution formula we use in our applications, see Proposition 4.1, depends on four linear mappings A, B, C and D. In the finite-dimensional case all of these are induced by matrices (denoted by the same symbols), A and B are quadratic of size $M \times M$ and $N \times N$, C is of size $M \times N$ and satisfies a 1-dimensionality condition depending on A and B, and D is of size $N \times M$ but otherwise arbitrary. Simplifying slightly for the sake of exposition, we may think of A, B and D as free parameters and of C as essentially determined[a] by A and B. The motivation of our applications part stems from the broader problem to classify the family $\mathscr{F}$ of solutions obtained by all possible choices of these matrices in terms of their asymptotic behaviour.

The second author has treated the analogous problem for solution formulas for soliton equations in one space dimension. For the KdV, the classification is complete [9] and yields multiple pole solutions besides the familiar N-solitons. For the sine-Gordon (sG), the modified KdV (mKdV), and the Nonlinear Schrödinger (NLS) equations, the picture is reasonably complete: For the sG and mKdV besides solitons one obtains breathers (or pulsating solitons, constituting a bound state between a soliton and an antisoliton [23]) as solutions with particle character, to which the concept of multiple pole solutions extends naturally [28,36]; for the NLS one has solitons being complex in nature [35]. Note that whereas in the KdV case different solitons necessarily have different velocities, this is no longer the case for the NLS. General results on such degeneracies are obtained in [18,35].

Most relevant for the present article are the results on the 1d-Toda lattice [29], which are roughly analogous to the KdV case. It should be stressed that in the latter cases wave packets of multiple pole solutions always consist of both *regular* solitons and *singular* antisolitons.

[a]The 1-dimensionality condition involves two further parameters, which are less significant for the dynamic behaviour and can be neglected in this discussion.

In the applications, we will provide evidence that the phenomena observable within the family of solutions coming from matrices are structurally much richer than what we have mentioned in one space dimension. We start from web structures (roughly speaking these are solutions built from line solitons with X- and Y-crossings), which exist for KP-II and the 2d-Toda lattice. Their asymptotic classification is a major contribution to the topic, see the articles [3, 4, 19] and the recent monograph [17]. We will show that the solutions studied in [4, 19] (including generalized cases with singularities, whose asymptotics are not studied yet) are included in $\mathscr{F}$. For the reader's convenience, we summarise some background from [4] in Appendix B. The link between the Wronskian constructions used originally to obtain web structures and our formulas is of independent interest, see Appendix A.

In the remaining applications we study three solution classes in $\mathscr{F}$ which differ from web-structures essentially. They are obtained by choosing A, B and D as 2×2-matrices with appropriate structure. To get a guideline to what may be interesting, we observe that one can assume A and B in Jordan normal form (Lemma 4.8).

Taking A, B as Jordan blocks (hence commuting) and $D = I_2$, we obtain solutions whose intersection with a y-slice $\{y = c\}$ looks similar to the familiar 2-pole solution of the KdV: A weakly bound wave packet moving with constant velocity as $n \to \pm\infty$, which consists of a regular soliton and a singular antisoliton deviating logarithmically from the common center, switching sides under collision, see Figure 6.

For A, B as before, we continue with the case that D is arbitrary. Asymptotic analysis shows that the entry in the left lower corner of D plays a privileged role, leading to solutions which are essentially different from the case $D = I_2$. In a y-slice, such a solution is a 2-pole, but moving on steeper logarithmic curves, and not changing sides any longer. Moreover, it can be arranged that the solution is asymptotically regular. It should be mentioned that singularities occur at the place where the solitons collide, see Figure 7.

Finally, taking A as Jordan block but B as diagonal matrix with two different entries leads to even more intriguing solutions: Here a y-slice shows two solitons moving on logarithmic paths, but these do *not* deviate from a *common* center, but each from a center of its own. Hence the solution looks as if it consists of the superposition of two 2-poles, where each of the 2-poles has lost one of its partners, see Figure 8. Splitting the asymptotic solitons into two pairs, regularity can be arranged for each of the pairs independently.

For the above mentioned three solution classes, the asymptotic analysis is given.

Similar phenomena are to be expected for the KP-II where comparable solution formulas in [32] can be exploited.

2. Solving the noncommutative two-dimensional Toda lattice

The aim of the first section is to find a general solution formula for its noncommutative (nc) version (1.4). Our result on the general operator level is the following.

Theorem 2.1. *Let E, F be Banach spaces, and $A \in \mathscr{L}(E)$, $B \in \mathscr{L}(F)$ invertible. Assume that the families of operators $L_n = L_n(x,y) \in \mathscr{L}(F,E)$, $M_n = M_n(x,y) \in \mathscr{L}(E,F)$ satisfy the following set* *of base equations*

$$
\begin{aligned}
L_{n+1} &= AL_n, & \frac{\partial}{\partial x} L_n &= AL_n, & \frac{\partial}{\partial y} L_n &= -A^{-1} L_n, \\
M_{n+1} &= BM_n, & \frac{\partial}{\partial x} M_n &= -B^{-1} M_n, & \frac{\partial}{\partial y} M_n &= BM_n,
\end{aligned} \tag{2.1}
$$

and that $(I_F + M_n L_n)$ is invertible for all $n \in \mathbb{Z}$ and all $(x,y) \in \Omega$, where Ω is an open subset of $\mathbb{R}^2$. Then the $\mathscr{L}(F)$-valued function

$$
V_n = B(I_F + M_n L_n)^{-1} M_n (AL_n - L_n B^{-1}). \tag{2.2}
$$

is a solution of the nc 2d-Toda lattice (1.4) *on $\mathbb{Z} \times \Omega$.*

Theorem 2.1 generalizes [31, Theorem 2.1], where the result was obtained under the additional assumptions $[A,B] = 0$, which is very restrictive in applications[b]. One of our motivations was a similar generalization obtained for the noncommutative Kadomtsev-Petviashvili equations in [32].

The proof of Theorem 2.1 relies on a rather general tool box of operator identities which is presented in Appendix D.

Proof. Let us start with some preparations. To utilize the appendix, we introduce the following operator-functions

$$
\begin{aligned}
S_1^{\pm}(n) &= (I_E + L_n M_n)^{-1} (A^{\pm 1} L_n - L_n B^{\mp 1}), \\
S_2^{\pm}(n) &= (I_F + M_n L_n)^{-1} (B^{\mp 1} M_n - M_n A^{\pm 1}), \\
T_1^{\pm}(n) &= (I_E + L_n M_n)^{-1} (A^{\pm 1} + L_n B^{\mp 1} M_n), \\
T_2^{\pm}(n) &= (I_F + M_n L_n)^{-1} (B^{\mp 1} + M_n A^{\pm 1} L_n),
\end{aligned}
$$

and collect the identities needed for the proof.

The first three identities, which follow on application[c] of Lemma D.5, are the derivation rules

$$
\frac{\partial}{\partial x} T_2^{+}(n) = -S_2^{+}(n) S_1^{+}(n), \tag{2.3a}
$$

$$
\frac{\partial}{\partial y} S_1^{+}(n) = -T_1^{-}(n) S_1^{+}(n), \tag{2.3b}
$$

---

[b] The lattice (1.4) differs from the one studied in [31] only by the coordinate change $y \mapsto -y$.

[c] For both (2.3a) and (2.3c), use $L_1(n,x,y) = M_n(x,y)$, $L_2(n,x,y) = L_n(x,y)$ and $A_1 = B$, $A_2 = A$. Then, for (2.3a) one sets

$$
f_1^{[a]}(z) = f_1^{[b]}(z) = -z^{-1}, \; f_2^{[a]}(z) = f_2^{[b]}(z) = z,
$$

with which one has $T_2^{+}(n) = -T_1^{[b]}$. By Lemma D.5 b), we get $\frac{\partial}{\partial x} T_1^{[b]} = -S_1^{[a]} S_2^{[b]}$ and one identifies $S_1^{[a]} = -S_2^{+}(n)$, $S_2^{[b]} = S_1^{+}(n)$.

For (2.3c) on the other hand, with

$$
f_1^{[a]}(z) = f_2^{[b]}(z) = z, \; f_1^{[b]}(z) = f_2^{[a]}(z) = -z^{-1}
$$

we find $S_2^{+}(n) = -S_1^{[b]}$. Hence Lemma D.5 a) gives $\frac{\partial}{\partial y} S_1^{[b]} = T_1^{[a]} S_1^{[b]}$, where $T_1^{[a]} = T_2^{-}(n)$.

Turning to (2.3b), we now use $L_1(n,x,y) = L_n(x,y)$, $L_2(n,x,y) = M_n(x,y)$, $A_1 = A$, $A_2 = B$, and

$$
f_1^{[a]}(z) = f_2^{[b]}(z) = -z^{-1}, \; f_1^{[b]}(z) = f_2^{[a]}(z) = z.
$$

Then $S_1^{+}(n) = S_1^{[b]}$, and again $\frac{\partial}{\partial y} S_1^{[b]} = T_1^{[a]} S_1^{[b]}$ by Lemma D.5 a), where now $T_1^{[a]} = -T_1^{-}(n)$.

$$\frac{\partial}{\partial y} S_2^+(n) = \quad T_2^-(n) S_2^+(n). \tag{2.3c}$$

The fourth identity, which follows[d] from Lemma D.2 a) and Corollary D.3, allows us to handle a certain product of the operator-functions,

$$S_2^+(n) T_1^-(n) = -T_2^+(n) S_2^-(n). \tag{2.3d}$$

We are now in the position to prove Theorem 2.1. Let us begin with the left-hand side of the nc 2d-Toda lattice (1.4). Observe first that

$$\begin{aligned} I_F + V_n &= I_F + B(I_F + M_n L_n)^{-1} M_n (A L_n - L_n B^{-1}) \\ &= B(I_F + M_n L_n)^{-1} \left((I_F + M_n L_n) B^{-1} + M_n (A L_n - L_n B^{-1}) \right) \\ &= B(I_F + M_n L_n)^{-1} \left(B^{-1} + M_n A L_n \right) && (2.4a) \\ &= B(I_F + M_n L_n)^{-1} B^{-1} (I_F + M_{n+1} L_{n+1}). && (2.4b) \end{aligned}$$

Note that (2.4a) implies that

$$I_F + V_n = B T_2^+(n). \tag{2.4c}$$

Hence, we find

$$\frac{\partial}{\partial x} V_n = \frac{\partial}{\partial x}(I_F + V_n) \overset{(2.4c)}{=} B \frac{\partial}{\partial x} T_2^+(n) \overset{(2.3a)}{=} -B S_2^+(n) S_1^+(n).$$

Moreover,

$$\begin{aligned} (I_F + V_n)^{-1} B S_2^{\pm}(n) &\overset{(2.4b)}{=} (I_F + M_{n+1} L_{n+1})^{-1} B (B^{\mp 1} M_n - M_n A^{\pm 1}) \\ &= \quad S_2^{\pm}(n+1) && (2.5) \end{aligned}$$

so that

$$(I_F + V_n)^{-1} \frac{\partial}{\partial x} V_n = -S_2^+(n+1) S_1^+(n).$$

Starting from this identity, and using (2.3b), (2.3c), we get for the left-hand side of the nc 2d-Toda lattice (1.4)

$$\begin{aligned} \frac{\partial}{\partial y}\Big((I_F + V_n)^{-1} \frac{\partial}{\partial x} V_n \Big) &= -\Big(\frac{\partial}{\partial y} S_2^+(n+1) \Big) S_1^+(n) + S_2^+(n+1) \Big(\frac{\partial}{\partial y} S_1^+(n) \Big) \\ &= -\Big(T_2^-(n+1) S_2^+(n+1) - S_2^+(n+1) T_1^-(n) \Big) S_1^+(n). && (2.6) \end{aligned}$$

We next turn to the right-hand side of the nc 2d-Toda lattice (1.4). To compute the second term, we use

$$V_n - V_{n-1} = B\Big((I_F + M_n L_n)^{-1} M_n - (I_F + M_{n-1} L_{n-1})^{-1} M_{n-1} A^{-1} \Big)$$

---

[d]Use again $L_1(n,x,y) = M_n(x,y)$, $L_2(n,x,y) = L_n(x,y)$ and $A_1 = B$, $A_2 = A$. Furthermore one sets

$$f_1^{[a]}(z) = f_2^{[b]}(z) = z^{-1}, \; f_1^{[b]}(z) = f_2^{[a]}(z) = -z,$$

such that $S_1^{[a]} = S_2^+(n)$, $T_2^{[b]} = T_1^-(n)$. Note that $f_1^{[a]} f_1^{[b]} = f_2^{[a]} f_2^{[b]} = -1$, hence Corollary D.3 applies, yielding $S_1^{[a]} T_2^{[b]} = T_1^{[a]} S_1^{[b]}$. Finally, we identify $T_1^{[a]} = T_2^+(n)$, $S_1^{[b]} = -S_2^-(n)$.

$$
\begin{aligned}
&(AL_n - L_nB^{-1})\\
&= B\Big(M_n(I_E+L_nM_n)^{-1} - (I_F+M_{n-1}L_{n-1})^{-1}M_{n-1}A^{-1}\Big)\\
&\quad (AL_n - L_nB^{-1}) \qquad\qquad \text{(see Lemma D.1 a))}\\
&= B(I_F+M_{n-1}L_{n-1})^{-1}\\
&\quad \Big((I_F+M_{n-1}L_{n-1})M_n - M_{n-1}A^{-1}(I_E+L_nM_n)\Big)\\
&\quad (I_E+L_nM_n)^{-1}(AL_n - L_nB^{-1})\\
&= B(I_F+M_{n-1}L_{n-1})^{-1}\big(M_n - M_{n-1}A^{-1}\big)S_1^+(n)\\
&= B(I_F+M_{n-1}L_{n-1})^{-1}\big(BM_{n-1} - M_{n-1}A^{-1}\big)S_1^+(n)\\
&= B\,S_2^-(n-1)S_1^+(n),
\end{aligned}
$$

yielding

$$
\begin{aligned}
(I_F+V_{n-1})^{-1}(V_n - V_{n-1}) &= (I_F+V_{n-1})^{-1}BS_2^-(n-1)S_1^+(n)\\
&\overset{(2.5)}{=} S_2^-(n)S_1^+(n).
\end{aligned}
\tag{2.7}
$$

Analogously,

$$
\begin{aligned}
V_{n+1} - V_n &= B\Big((I_F+M_{n+1}L_{n+1})^{-1}M_{n+1}A - (I_F+M_nL_n)^{-1}M_n\Big)\\
&\quad (AL_n - L_nB^{-1})\\
&= B\Big((I_F+M_{n+1}L_{n+1})^{-1}M_{n+1}A - M_n(I_E+L_nM_n)^{-1}\Big)\\
&\quad (AL_n - L_nB^{-1})\\
&= B(I_F+M_{n+1}L_{n+1})^{-1}\\
&\quad \Big(M_{n+1}A(I_E+L_nM_n) - (I_F+M_{n+1}L_{n+1})M_n\Big)\\
&\quad (I_E+L_nM_n)^{-1}(AL_n - L_nB^{-1})\\
&= B(I_F+M_{n+1}L_{n+1})^{-1}\big(M_{n+1}A - M_n\big)S_1^+(n)\\
&= B(I_F+M_{n+1}L_{n+1})^{-1}\big(M_{n+1}A - B^{-1}M_{n+1}\big)S_1^+(n)\\
&= -B\,S_2^+(n+1)S_1^+(n).
\end{aligned}
$$

Note that (2.4b) implies that $(I_F+V_n)^{-1} = (I_F+M_{n+1}L_{n+1})^{-1}B(I_F+M_nL_n)\ B^{-1} = (I_F + M_{n+1}L_{n+1})^{-1}(B+M_{n+1}A^{-1}L_{n+1})\ B^{-1} = T_2^-(n+1)B^{-1}$, showing

$$
(I_F+V_n)^{-1}(V_{n+1}-V_n) = -T_2^-(n+1)S_2^+(n+1)S_1^+(n).
\tag{2.8}
$$

Thus, for the right-hand side of the nc 2d-Toda lattice (1.4) we obtain

$$
\begin{aligned}
&(I_F+V_n)^{-1}(I_F+V_{n+1}) - (I_F+V_{n-1})^{-1}(I_F+V_n)\\
&= \Big((I_F+V_n)^{-1}(I_F+V_{n+1}) + I_F\Big) - \Big((I_F+V_{n-1})^{-1}(I_F+V_n) + I_F\Big)\\
&= (I_F+V_n)^{-1}\Big((I_F+V_{n+1}) - (I_F+V_n)\Big)\\
&\qquad -(I_F+V_{n-1})^{-1}\Big((I_F+V_n) - (I_F+V_{n-1})\Big)
\end{aligned}
\tag{2.9}
$$

$$\begin{aligned} &= (I_F + V_n)^{-1}(V_{n+1} - V_n) - (I_F + V_{n-1})^{-1}(V_n - V_{n-1}) \\ &\overset{(2.7),(2.8)}{=} -\Big(T_2^-(n+1)S_2^+(n+1) + S_2^-(n)\Big)S_1^+(n). \end{aligned} \tag{2.10}$$

Comparing (2.6) with (2.10) it remains to show $S_2^+(n+1)T_1^-(n) = -S_2^-(n)$ to complete the proof. The obstacle here is that properties as in Lemma D.2 can only be applied if the operator-functions have the same arguments. Hence we need a little detour.

$$\begin{aligned} S_2^+(n+1)T_1^-(n) &\overset{(2.5)}{=} (I_F + V_n)^{-1} B S_2^+(n) T_1^-(n) \\ &\overset{(2.3d)}{=} -(I_F + V_n)^{-1} B T_2^+(n) S_2^-(n) \\ &\overset{(2.4c)}{=} -S_2^-(n), \end{aligned}$$

which completes the proof. □

3. A solution formula for the scalar 2d-Toda lattice depending on two independent operator parameters

In this section a scalarization process is introduced with which, starting from a solution of the non-commutative 2d-Toda lattice (1.4), a solution formula for the scalar lattice (1.3) can be constructed.

After introducing the necessary background from functional analysis in Subsection 3.1, we explain the main idea behind scalarization in Subsection 3.2. In Subsection 3.3, this strategy is carried out explicitely for the 2d-Toda lattice. Finally, in Subsection 3.4 we explain how the theory of elementary operators can be exploited to meet the requirements of scalarization and to simplify the solution formulas considerably.

3.1. *Terminology*

Before explaining our general strategy, we need some terminology. Let E, F be Banach spaces. A one-dimensional operator $T \in \mathscr{L}(E,F)$ is an operator whose range is contained in a one-dimensional subspace of F. Every such operator T can be written as $b \otimes c$ for a vector $c \in F$ and a functional $b \in E'$, where

$$b \otimes c\,(x) := \langle x, b\rangle c \quad \forall x \in E$$

($\langle\,,\rangle$ is the usual dual pairing: for the functional $b \in E'$, $\langle x, b\rangle$ is the evaluation of b at the vector $x \in E$).

A finite-rank operator is an operator $T \in \mathscr{L}(E,F)$ with finite-dimensional range, and the space of all finite-rank operators from E into F is denoted by $\mathscr{F}(E,F)$. We set $\operatorname{rank}(T) = \dim(\operatorname{ran}(T))$. Note that

$$\mathscr{F}(E,F) = \Big\{ \sum_{j=1}^N b_j \otimes c_j \;\Big|\; b_j \in E', c_j \in F \Big\},$$

The class $\mathscr{F} = \cup_{E,F}\mathscr{F}(E,F)$ of finite-rank operators forms an operator ideal. Moreover, it is well-known that there is a unique trace $\operatorname{tr}_{\mathscr{F}} : \cup_F \mathscr{F}(F) \to \mathbb{C}$, which is given by

$$\operatorname{tr}_{\mathscr{F}}(T) = \sum_{j=1}^N \langle c_j, b_j\rangle,$$

where $T = \sum_{j=1}^{N} b_j \otimes c_j$, $b_j \in F'$, $c_j \in F$, is an arbitrary (finite) representation of $T \in \mathscr{F}(F)$ (see [24] for more detailed information on the concept of traces and determinants on quasi-Banach operator ideals).

3.2. *Strategy*

Let us briefly explain the idea of the scalarization process. Let us assume that $V_n = V_n(x,y) \in \mathscr{L}(F)$ is an operator-valued solution of the nc 2d-Toda lattice (1.4). A natural ansatz to derive a solution v_n for the scalar lattice (1.3) is to apply a continuous linear functional τ to V_n, i.e. to try $v_n = \tau(V_n)$. Of course the functional τ has to be chosen in a way that the solution property is maintained under its application. Since the 2d-Toda lattice is nonlinear, τ needs to be multiplicative at least in a certain sense.

To meet this requirement we introduce, for a fixed functional $b \in F'$, the subalgebra

$$\mathscr{S}_b(F) = \{ b \otimes c \,|\, c \in F \}$$

of $\mathscr{L}(F)$ consisting of one-dimensional operators whose behavior is governed by the functional b. Observe that the restriction of the trace $\mathrm{tr}_{\mathscr{F}}$ on the finite-rank operators $\mathscr{F}(F)$ to $\mathscr{S}_b(F)$ coincides with the evaluation of the functional b. Now the crucial observation is that $\mathrm{tr}_{\mathscr{F}}$ is multiplicative on $\mathscr{S}_b(F)$:

Lemma 3.1. *For $T_1, T_2 \in \mathscr{S}_b(F)$, it holds* $\mathrm{tr}_{\mathscr{F}}\big(T_1T_2\big) = \mathrm{tr}_{\mathscr{F}}(T_1)\mathrm{tr}_{\mathscr{F}}(T_2)$.

Proof. We verify

$$T_1T_2 = \big(\mathrm{tr}_{\mathscr{F}}(T_2)\big)T_1,$$

then the assertion follows from the linearity of the trace. Indeed, for $T_1 = b \otimes c_1$, $T_2 = b \otimes c_2$ with $c_1, c_2 \in F$, we have

$$\begin{aligned}(T_1T_2)(x) = T_1(T_2(x)) \;&=\; T_1\big((b\otimes c_2)x\big) \;=\; T_1\big(\langle x,b\rangle c_2\big)\\ &= \langle x,b\rangle T_1(c_2) \;=\; \langle x,b\rangle (b\otimes c_1)c_2\\ &= \langle x,b\rangle\,\langle c_2,b\rangle c_1 \;=\; \langle c_2,b\rangle T_1(x)\end{aligned}$$

for $x \in F$. □

This motivates the following choices:

(1) We assume that the operator solution V_n belongs to $\mathscr{S}_b(F)$ for a constant, fixed $b \in F'$. In other words, V_n is one-dimensional with fixed kernel.
(2) For scalarization, we use the functional $\mathrm{tr}_{\mathscr{F}}$.

Then application of $\mathrm{tr}_{\mathscr{F}}$ to V_n maintains the solution property.

Proposition 3.2. *Let $V_n = V_n(x,y) \in \mathscr{S}_b(F)$ be a solution of* (1.4). *Then $v_n = \mathrm{tr}_{\mathscr{F}}(V_n)$ solves* (1.3).

Remark 3.3. For a more systematic explanation of the choices (1), (2) above, we refer to [2].

For the proof of Proposion 3.2, we need two more properties for operators in $\mathscr{S}_b$.

Lemma 3.4. *Let T be an operator-valued function depending $\mathscr{C}^1$-smoothly on some variable ξ such that $T(\xi) \in \mathscr{S}_b(F)$. Then also $\operatorname{tr}_{\mathscr{F}}(T)$ depends $\mathscr{C}^1$-smoothly on ξ with*

$$\frac{d}{d\xi}\big(\operatorname{tr}_{\mathscr{F}}(T)\big) = \operatorname{tr}_{\mathscr{F}}\Big(\frac{d}{d\xi}T\Big).$$

Proof. By assumption, we can write $T = b \otimes c$ with a $\mathscr{C}^1$-smooth vector-function $c : \xi \to c(\xi) \in F$. Hence $\frac{d}{d\xi}T = b \otimes \big(\frac{d}{d\xi}c\big)$, in particular also $\frac{d}{d\xi}T$ takes its values in $\mathscr{S}_b(F)$. Since evaluation of the functional b is continuous, we get

$$\begin{aligned}
\frac{d}{d\xi}(\langle c(\xi), b\rangle) &= \lim_{h\to 0} \frac{\langle c(\xi+h), b\rangle - \langle c(\xi), b\rangle}{h} \\
&= \lim_{h\to 0} \Big\langle \frac{c(\xi+h)-c(\xi)}{h}, b\Big\rangle \ = \ \Big\langle \lim_{h\to 0} \frac{c(\xi+h)-c(\xi)}{h}, b\Big\rangle \\
&= \Big\langle \frac{d}{d\xi}c(\xi), b\Big\rangle,
\end{aligned}$$

showing

$$\begin{aligned}
\frac{d}{d\xi}\big(\operatorname{tr}_{\mathscr{F}}(T)\big) &= \frac{d}{d\xi}\big(\operatorname{tr}_{\mathscr{F}}(b\otimes c)\big) \ = \ \frac{d}{d\xi}\big(\langle c, b\rangle\big) = \Big\langle \frac{d}{d\xi}c, b\Big\rangle \\
&= \operatorname{tr}_{\mathscr{F}}\big(b \otimes \big(\frac{d}{d\xi}c\big)\big) \ = \ \operatorname{tr}_{\mathscr{F}}\big(\frac{d}{d\xi}T\big).
\end{aligned}$$

□

Lemma 3.5. *For $T_1, T_2 \in \mathscr{S}_b(F)$ with $I_F + T_1$ invertible, it holds*

$$\operatorname{tr}_{\mathscr{F}}\big((I_F+T_1)^{-1}T_2\big) = \frac{\operatorname{tr}_{\mathscr{F}}(T_2)}{1+\operatorname{tr}_{\mathscr{F}}(T_1)}.$$

Proof. The assumption follows from

$$\begin{aligned}
&\big(1+\operatorname{tr}_{\mathscr{F}}(T_1)\big)\operatorname{tr}_{\mathscr{F}}\big((I_F+T_1)^{-1}T_2\big) \ = \\
&= \operatorname{tr}_{\mathscr{F}}\big((I_F+T_1)^{-1}T_2\big) + \boxed{\operatorname{tr}_{\mathscr{F}}(T_1)\operatorname{tr}_{\mathscr{F}}\big((I_F+T_1)^{-1}T_2\big)} \\
&= \operatorname{tr}_{\mathscr{F}}\big((I_F+T_1)^{-1}T_2\big) + \boxed{\operatorname{tr}_{\mathscr{F}}\big(T_1(I_F+T_1)^{-1}T_2\big)} \\
&= \operatorname{tr}_{\mathscr{F}}\big((I_F+T_1)(I_F+T_1)^{-1}T_2\big) \\
&= \operatorname{tr}_{\mathscr{F}}(T_2)
\end{aligned}$$

and the fact that $1+\operatorname{tr}_{\mathscr{F}}(T_1) = \det_{\mathscr{F}}(I_F+T_1) \neq 0$ since I_F+T_1 is invertible. Note that we have used the multiplicity of $\operatorname{tr}_{\mathscr{F}}$ in the boxed reformulation. □

Proof of Propostion 3.2. Using the tools collected above, we can directly verify that $v_n = \operatorname{tr}_{\mathscr{F}}(V_n)$ satisfies (1.3).

$$\frac{\partial^2}{\partial x \partial y}\log(1+v_n) \ = \ \frac{\partial}{\partial y}\,\frac{\frac{\partial}{\partial x}v_n}{1+v_n}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial y}\,\frac{\frac{\partial}{\partial x}\mathrm{tr}_{\mathscr{F}}(V_n)}{1+\mathrm{tr}_{\mathscr{F}}(V_n)} \overset{L3.4}{=} \frac{\partial}{\partial y}\,\frac{\mathrm{tr}_{\mathscr{F}}(\frac{\partial}{\partial x}V_n)}{1+\mathrm{tr}_{\mathscr{F}}(V_n)} \\
&\overset{L3.5}{=} \frac{\partial}{\partial y}\,\mathrm{tr}_{\mathscr{F}}\Big((I_F+V_n)^{-1}\frac{\partial}{\partial x}V_n\Big) \overset{L3.4}{=} \mathrm{tr}_{\mathscr{F}}\Big(\frac{\partial}{\partial y}\big((I_F+V_n)^{-1}\frac{\partial}{\partial x}V_n\big)\Big) \\
&\overset{(1.4)}{=} \mathrm{tr}_{\mathscr{F}}\Big((I_F+V_n)^{-1}(I_F+V_{n+1})-(I_F+V_{n-1})^{-1}(I_F+V_n)\Big) \\
&= \mathrm{tr}_{\mathscr{F}}\Big((I_F+V_n)^{-1}(V_{n+1}-V_n)-(I_F+V_{n-1})^{-1}(V_n-V_{n-1})\Big) \\
&\overset{(\star)}{=} \mathrm{tr}_{\mathscr{F}}\Big((I_F+V_n)^{-1}(V_{n+1}-V_n)\Big)-\mathrm{tr}_{\mathscr{F}}\Big((I_F+V_{n-1})^{-1}(V_n-V_{n-1})\Big) \\
&\overset{L3.5}{=} \frac{\mathrm{tr}_{\mathscr{F}}(V_{n+1})-\mathrm{tr}_{\mathscr{F}}(V_n)}{1+\mathrm{tr}_{\mathscr{F}}(V_n)} - \frac{\mathrm{tr}_{\mathscr{F}}(V_n)-\mathrm{tr}_{\mathscr{F}}(V_{n-1})}{1+\mathrm{tr}_{\mathscr{F}}(V_{n-1})} \\
&= \frac{v_{n+1}-v_n}{1+v_n}-\frac{v_n-v_{n-1}}{1+v_{n-1}} \\
&= \frac{1+v_{n+1}}{1+v_n}-\frac{1+v_n}{1+v_{n-1}}.
\end{aligned}
$$

Note that for Step $(\star)$ above, linearity of $\mathrm{tr}_{\mathscr{F}}$, more precisely $\mathrm{tr}_{\mathscr{F}}(T_1+T_2)=\mathrm{tr}_{\mathscr{F}}(T_1)+\mathrm{tr}_{\mathscr{F}}(T_2)$ for $T_1,T_2\in\mathscr{S}_b(F)$, is used, to which end we observe that $(I_F+V_n)^{-1}(V_{n+1}-V_n)\in\mathscr{S}_b(F)$ for all n. □

3.3. *Solution formulas*

Application of the scalarization technique from last subsection to the operator solution in Theorem 2.1 provides us with a first solution formula for (1.3).

Theorem 3.6. *Let E, F be Banach spaces, and $A\in\mathscr{L}(E)$, $B\in\mathscr{L}(F)$ invertible. Let $L_n=L_n(x,y)\in\mathscr{L}(F,E)$, $M_n=M_n(x,y)\in\mathscr{L}(E,F)$ be operator functions which satisfy the base equations* (2.1) *and the one-dimensionality condition*

$$M_n(AL_n-L_nB^{-1})\in\mathscr{S}_b(F) \tag{3.1}$$

for some $b\in F'$.

If $(I_F+M_nL_n)$ is invertible for all $n\in\mathbb{Z}$ and all $(x,y)\in\Omega$, where Ω is an open subset of $\mathbb{R}^2$, then

$$v_n=\mathrm{tr}_{\mathscr{F}}\Big(B(I_F+M_nL_n)^{-1}M_n(AL_n-L_nB^{-1})\Big) \tag{3.2}$$

is a solution of the scalar 2d-Toda lattice (1.3) *on $\mathbb{Z}\times\Omega$.*

Proof. Since $L_n(x,y)$, $M_n(x,y)$ satisfy the base equations (2.1), Theorem 2.1 implies that $V_n=B(I_F+M_nL_n)^{-1}M_n(AL_n-L_nB^{-1})$ solves of the nc 2d-Toda lattice (1.4) on $\mathbb{Z}\times\Omega$. Now the one-dimensionality condition (3.1) tells $M_n(AL_n-L_nB^{-1})=b\otimes c_n$ with some vector function $c_n=c_n(x,y)\in F$, and hence

$$
\begin{aligned}
V_n &= B(I_F+M_nL_n)^{-1}M_n(AL_n-L_nB^{-1}) \\
&= B(I_F+M_nL_n)^{-1}(b\otimes c_n) \\
&= b\otimes\big((I_F+M_nL_n)^{-1}c_n\big).
\end{aligned}
$$

This shows $V_n\in\mathscr{S}_b(F)$, and the assertion follows from Proposition 3.2. □

If the operator function $L_n = L_n(x,y)$ even takes its values in a quasi-Banach operator ideal $\mathscr{A}$ which is equipped with a nice, generalized determinant, the solution formula in Theorem 3.6 can be improved considerably.

Theorem 3.7. *Let the assumptions of Theorem 3.6 be satisfied. Let, in addition, $L_n = L_n(x,y) \in \mathscr{A}(F,E)$, where $\mathscr{A}$ is a quasi-Banach operator ideal admitting a continuous determinant δ. Then the solution* (3.2) *can be expressed as*

$$v_n = \frac{\delta(I_F + M_{n+1}L_{n+1})}{\delta(I_F + M_nL_n)} - 1.$$

Proof. Let τ be the trace associated to the determinant δ by the trace-determinant theorem [24], and note this trace coincides with $\mathrm{tr}_{\mathscr{F}}$ on $\mathscr{F}(F)$, by uniqueness of $\mathrm{tr}_{\mathscr{F}}$. Hence we get

$$\begin{aligned} v_n &= \mathrm{tr}_{\mathscr{F}} \left(B(I_F + M_nL_n)^{-1} M_n(AL_n - L_nB^{-1}) \right) \\ &= \tau \left(B(I_F + M_nL_n)^{-1} M_n(AL_n - L_nB^{-1}) \right). \end{aligned}$$

Next, using the fact that $1 + \tau(T) = \delta(I + T)$ holds for one-dimensional operators T, we obtain

$$\begin{aligned} v_n &= \delta \left(I_F + B(I_F + M_nL_n)^{-1} M_n(AL_n - L_nB^{-1}) \right) - 1 \\ &= \delta\Big(B(I_F + M_nL_n)^{-1}(B^{-1} + M_nAL_n)\Big) - 1 \\ &= \delta\Big((I_F + M_nL_n)^{-1}(B^{-1} + M_nAL_n)B\Big) - 1 \\ &= \delta\Big((I_F + M_nL_n)^{-1}(I_F + M_nAL_nB)\Big) - 1 \\ &= \frac{\delta(I_F + M_nAL_nB)}{\delta(I_F + M_nL_n)} - 1 \\ &= \frac{\delta(I_F + BM_nAL_n)}{\delta(I_F + M_nL_n)} - 1 \\ &= \frac{\delta(I_F + M_{n+1}L_{n+1})}{\delta(I_F + M_nL_n)} - 1, \end{aligned}$$

which was to be shown. □

3.4. *Elementary operators*

A natural way to satisfy the base equations (2.1) is by choosing

$$\begin{aligned} L_n(x,y) &= A^n \exp\big(Ax - A^{-1}y\big)C, \\ M_n(x,y) &= B^n \exp\big(-B^{-1}x + By\big)D \end{aligned}$$

with arbitrary constant operators $C \in \mathscr{L}(F,E)$, $D \in \mathscr{L}(E,F)$. Then the one-dimensionality condition (3.1) is met provided that $AC - CB^{-1}$ is one-dimensional or, equivalently, provided that C is a solution of the Sylvester equation

$$AXB - X = b \otimes c \tag{3.3}$$

for some $b \in F'$ and $c \in E$. In the present subsection we review some relevant results on the Sylvester equation and their impact on our solution formulas.

To this end, let $\mathscr{A}$ be a quasi-Banach operator ideal, and $\Phi_{A,B} : \mathscr{A}(F,E) \longrightarrow \mathscr{A}(F,E)$ the operator defined by

$$\Phi_{A,B}(X) = AXB - X. \tag{3.4}$$

The operator $\Phi_{A,B}$ belongs to the larger class of so-called elementary operators the structural properties of which have been extensively studied in the literature (see the survey [26] and references therein).

For the spectrum of $\Phi_{A,B}$ the following striking formula

$$\begin{aligned} \operatorname{spec}(\Phi_{A,B}) &= \operatorname{spec}(A)\cdot\operatorname{spec}(B) - 1 \\ &:= \{\alpha\beta - 1 \mid \alpha \in \operatorname{spec}(A),\ \beta \in \operatorname{spec}(B)\} \end{aligned} \tag{3.5}$$

holds (see [11] for fundamental work on the spectra of elementary operators on Banach operator ideals, and [1] for the extension to quasi-Banach operator ideals). It shows that invertibility of $\Phi_{A,B}$ can be read off from the spectra of A and B, independently of the underlying quasi-Banach operator ideal $\mathscr{A}$.

In particular, since we are interested in solving the Sylvester equation (3.3) with right-hand side $b \otimes c \in \mathscr{F}(F,E)$, and the finite-dimensional operators $\mathscr{F}$ are contained in any quasi-Banach operator ideal, we are then free to choose $\mathscr{A}$ arbitrarily.

> *Provided that* $1 \notin \operatorname{spec}(A)\cdot\operatorname{spec}(B)$*, the Sylvester equation* (3.3) *has the (unique) solution* $\Phi_{A,B}^{-1}(b\otimes c)$*. Moreover,* $\Phi_{A,B}^{-1}(b\otimes c) \in \mathscr{A}(F,E)$ *for any quasi-Banach operator ideal* $\mathscr{A}$*.*

A particularly convenient choice for $\mathscr{A}$ is the quasi-Banach operator ideal $\mathscr{N}_{2/3}$ of $2/3$-nuclear operators in the sense of Grothendieck. Recall that a bounded operator $T : F \to E$ belongs to $\mathscr{N}_{2/3}(F,E)$ iff there is a representation

$$T = \sum_{j=1}^{\infty} b_j \otimes c_j \text{ with } \sum_{j=1}^{\infty} (\|b_j\|\,\|c_j\|)^{2/3} < \infty,$$

where $b_j \in F'$, $c_j \in E$. Note that $\mathscr{N}_{2/3}$ becomes a quasi-Banach operator ideal with respect to the quasi-norm

$$\| T \mid \mathscr{N}_{2/3} \| = \inf\left\{ \left(\sum_{j=1}^{\infty} (\|b_j\|\,\|c_j\|)^{2/3} \right)^{3/2} \right\},$$

where the infimum is taken over all possible representations.

It was already shown by Grothendieck [12] that $\mathscr{N}_{2/3}$ is of eigenvalue-type ℓ_1, i.e. operators in $\mathscr{N}_{2/3}$ possess absolutely summing eigenvalues. For quasi-Banach operator ideals of eigenvalue-type ℓ_1, a deep result of White [38] states that the spectral sum $\operatorname{tr}_\lambda$ (i.e. the sum of the eigenvalues) is a continuous trace. By the trace extension theorem [24], this trace is unique. Using the relationship between traces and determinants on quasi-Banach operator ideals [24, Chapter 4.6], there also is a unique continuous determinant $\det_\lambda$ on $\mathscr{N}_{2/3}$, which is spectral.

We sum up:

Proposition 3.8. *Let E, F be Banach spaces and $A \in \mathscr{L}(E)$, $B \in \mathscr{L}(F)$ invertible with $1 \notin \operatorname{spec}(A) \cdot \operatorname{spec}(B)$. Let $b \in F'$, $c \in E$, and $D \in \mathscr{L}(E,F)$, and set*

$$\begin{aligned} L_n(x,y) &= A^n \exp\left(Ax - A^{-1}y \right) \Phi_{A,B}^{-1}(b \otimes c), \\ M_n(x,y) &= B^n \exp\left(-B^{-1}x + By \right) D, \end{aligned}$$

such that $M_n L_n \in \mathscr{N}_{2/3}(F)$.

If $\det_\lambda(I_F + M_n L_n) \neq 0$ for all $n \in \mathbb{Z}$ and all $(x,y) \in \Omega$, where Ω is an open subset of $\mathbb{R}^2$, then

$$v_n = \frac{\det_\lambda(I_F + M_{n+1}L_{n+1})}{\det_\lambda(I_F + M_n L_n)} - 1$$

is a solution of the scalar 2d-Toda lattice (1.3) *on $\mathbb{Z} \times \Omega$ (and $\det_\lambda$ denotes the unique continuous and spectral determinant on $\mathscr{N}_{2/3}$).*

4. Applications

In the concluding section we discuss first applications of the solution formula in Proposition 3.8. First we re-derive line solitons [14, 16] in Subsection 4.1, then we study how resonance and web structure phenomena [4, 19] fit into the picture in Subsection 4.2. Finally, Subsection 4.3 contains some examples beyond the soliton solutions.

Usually one visualizes solutions for (1.1). Recall from the introduction that the connection between (1.3) and (1.1) is given by the dependent variable transformation $w_n = (1+v_n)/(1+v_{n-1}) - 1$. One then plots the solutions for fixed values of y, the time variable, as functions of (x,n). To facilitate comparison with the literature, we start with summing up the contents of Propositions 3.8 and C.1 for (1.1).

Proposition 4.1. *Let A be an $M \times M$-matrix, B an $N \times N$-matrix, both invertible with $1 \notin \operatorname{spec}(A) \cdot \operatorname{spec}(B)$, and D an $N \times M$-matrix. Let $b \in \mathbb{C}^N$, $c \in \mathbb{C}^M$, and let the $M \times N$-matrix C satisfy*

$$ACB - C = c\,b^{\mathrm{t}}$$

(where b^{t} denotes the transposed of b)[e]*. Set*

$$\begin{aligned} L_n(x,y) &= A^n \exp\left(Ax - A^{-1}y \right) C, \\ M_n(x,y) &= B^n \exp\left(-B^{-1}x + By \right) D, \end{aligned}$$

and assume that $\det(I_N + M_n L_n) \neq 0$ for all $n \in \mathbb{Z}$ and all $(x,y) \in \Omega$, where Ω is an open subset of $\mathbb{R}^2$. Then

$$w_n(x,y) = \frac{\partial^2}{\partial x \partial y} \log p(n,x,y)$$

with

$$p(n,x,y) = \det\Big(I_N + M_n(x,y) L_n(x,y) \Big), \tag{4.1}$$

is a solution of (1.1) *on $\mathbb{Z} \times \Omega$.*

[e]Recall that all functionals β on $\mathbb{C}^N$ are of the form $\beta : x \mapsto b^{\mathrm{t}}x$ for some $b \in \mathbb{C}^N$, and $\beta \otimes c(x) = \langle x, \beta \rangle c = (b^{\mathrm{t}}x)c = (c\,b^{\mathrm{t}})x$.

Note that we formulated Proposition 4.1 in the finite-dimensional setting, which is sufficient for the applications we have in mind, but it should be mentioned that there are numerous applications relying on analysis in infinite-dimensional spaces [5, 7, 8, 33].

4.1. *Line solitons*

Let us start with explicitly evaluating the solution formula in Proposition 4.1 for the following choices: Let $M = N$ and let A, B are diagonal matrices, say

$$A = \begin{pmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_N \end{pmatrix}, \quad B = \begin{pmatrix} 1/q_1 & & 0 \\ & \ddots & \\ 0 & & 1/q_N \end{pmatrix}$$

with $p_j, q_j \neq 0$ and $p_i/q_j \neq 1$ for all i, j. Note that this implies that A, B are invertible, and the spectral condition $1 \notin \operatorname{spec}(A) \cdot \operatorname{spec}(B)$ is satisfied.

Furthermore let $D = I_N$.

Given $b, c \in \mathbb{C}^N$, one verifies directly that the unique solution of the one-dimensionality condition $ACB - C = c\,b^{\mathrm{t}}$ is given by

$$C = \Big(\frac{b_j c_i}{p_i/q_j - 1} \Big)_{i,j=1}^{N}.$$

Finally, $A^n \exp(Ax - A^{-1}y) = \operatorname{diag}\{\ell[p_j] \,|\, j\}$, $B^n \exp(-B^{-1}x + By) = \operatorname{diag}\{1/\ell[q_j] \,|\, j\}$, where we have set

$$\ell[p] = \ell[p](n,x,y) = p^n \exp\big(px - y/p\big). \tag{4.2}$$

Then we get for the determinant (4.1) in Proposition 4.1 that

$$\begin{aligned} p &= \det\left(\delta_{ij} + \frac{b_j c_i}{p_i/q_j - 1} \ell[p_i]/\ell[q_i] \right)_{i,j=1}^{N} \\ &= 1 + \sum_{n=1}^{N} \sum_{i_1 < \cdots < i_n} \det\left(\frac{b_{i_{j'}} c_{i_j}}{p_{i_j}/q_{i_{j'}} - 1} \ell[p_{i_j}]/\ell[q_{i_j}] \right)_{j,j'=1}^{n} \\ &= 1 + \sum_{n=1}^{N} \sum_{i_1 < \cdots < i_n} \det\left(\frac{1}{p_{i_j}/q_{i_{j'}} - 1} \right)_{j,j'=1}^{n} \prod_{j=1}^{n} b_{i_j} c_{i_j} \frac{\ell[p_{i_j}]}{\ell[q_{i_j}]} \\ &= 1 + \sum_{n=1}^{N} \sum_{i_1 < \cdots < i_n} \prod_{j=1}^{n} \frac{b_{i_j} c_{i_j}}{p_{i_j}/q_{i_j} - 1} \frac{\ell[p_{i_j}]}{\ell[q_{i_j}]} \prod_{\substack{j,j'=1 \\ j<j'}}^{n} \frac{(p_{i_j} - p_{i_{j'}})(1/q_{i_j} - 1/q_{i_{j'}})}{(p_{i_j}/q_{i_{j'}} - 1)(p_{i_{j'}}/q_{i_j} - 1)}. \end{aligned} \tag{4.3}$$

The calculation of $\det\Big(\frac{1}{p_i/q_j - 1}\Big)_{i,j=1}^{n}$ in the last step is a slight modification of the calculation of $\det\Big(\frac{1}{p_i + q_j}\Big)_{i,j=1}^{n}$, which can be traced back to a classical remark of Cauchy [6], see also [25]. For details and generalizations with impact on integrable systems see [34].

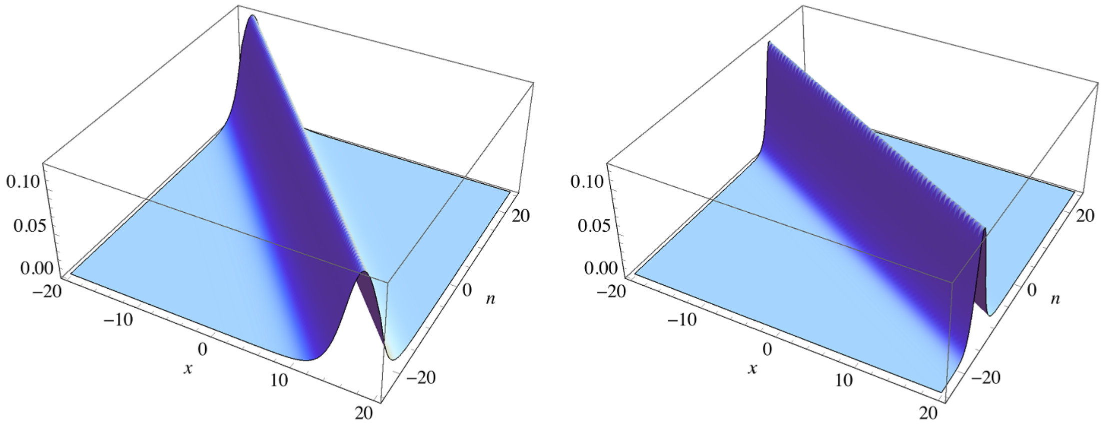

Fig. 1. Line-solitons plotted for $y = 0$. The parameters in the plot to the left are $(p_1, q_1) = (1, 0.5)$, in the plot to the right, $(p_2, q_2) = (2, 1)$, and $\varphi_1 = \varphi_2 = 0$. Note that the solitons have the same height.

In particular, for $N = 1$ we get

$$p = 1 + \frac{b_1 c_1}{p_1/q_1 - 1} \frac{\ell[p_1]}{\ell[q_1]}.$$

Taking $p_1 > q_1 > 0$ and $b_1 c_1 > 0$, to avoid singularities, we recover the 1-soliton solution (line soliton)

$$w = -\frac{1}{4}(p_1 - q_1)\Big(\frac{1}{p_1} - \frac{1}{q_1}\Big)\operatorname{sech}^2\Big(\frac{1}{2}(\theta[p_1] - \theta[q_1] + \varphi_1)\Big),$$

where we have set

$$\theta[p] = \log(p)n + px - y/p \quad \text{and} \quad \varphi_1 = \log \frac{b_1 c_1}{p_1/q_1 - 1}.$$

For fixed y, the line soliton is situated on a line in xn-space with slope d. (For visualization it is convenient to regard n as continuous variable). It is characterized by the latter and its amplitude a, where we have

$$a = \frac{(p_1 - q_1)^2}{4 p_1 q_1},$$
$$d = -\frac{p_1 - q_1}{\log(p_1) - \log(q_1)} < 0,$$

see Fig. 1 for illustration.

For $N = 2$, inspection of (4.3) shows that the solution is regular if $p_j > q_j > 0$, $b_j c_j > 0$ for $j = 1, 2$, and

$$A_{12} = \frac{(p_1 - p_2)(1/q_1 - 1/q_2)}{(p_1/q_2 - 1)(p_2/q_1 - 1)} > 0. \tag{4.4}$$

Note that A_{12} encodes the position shift of the asymptotic paths of the line-solitons, which is caused by their collision.

We may assume $p_1 > p_2$ without loss of generality. Then (4.4) can be satisfied by either one of the following conditions

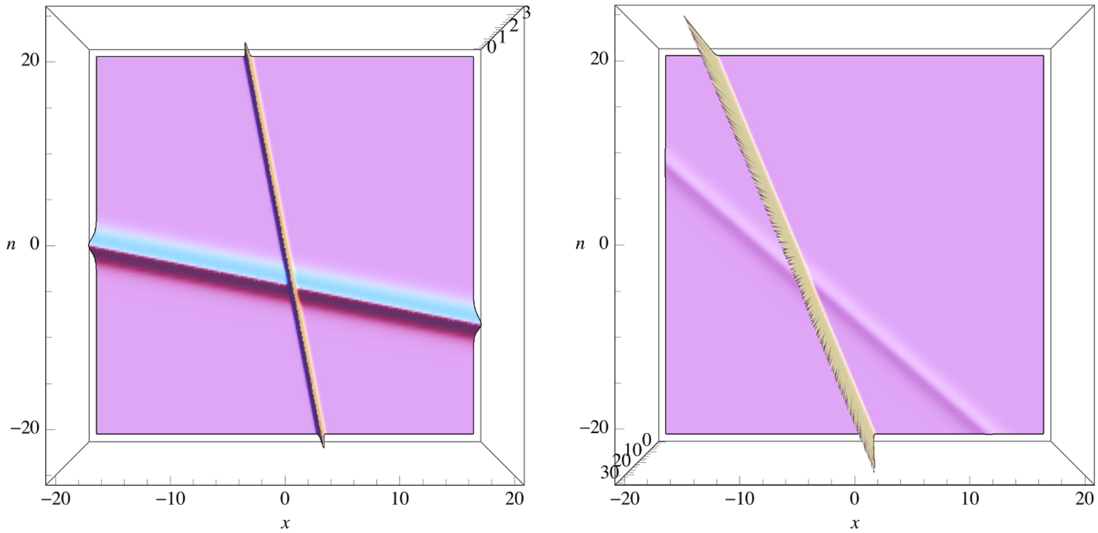

Fig. 2. Elastic collisions of two line-solitons: ordinary type (on the left) and asymmetric type (on the right), see Example 4.2 for details.

(O) $\quad 0 < q_2 < p_2 < q_1 < p_1,$
(A) $\quad 0 < q_1 < q_2 < p_2 < p_1.$

The resulting solution describes the elastic collision of two line-solitons. For condition (O), the involved solitons belong to the parameter constellations (p_1, q_1) and (p_2, q_2), constituting a collision of so-called *ordinary type*. For condition (A), they belong to (p_1, q_2) and (p_2, q_1), a collision of so-called *asymmetric type*, see [4] for details.

Example 4.2. In Fig. 2 we provide an illustration for

$$\{p_1,\ p_2,\ q_1,\ q_2\} = \{15,\ 2,\ 0.5,\ 0.1\}, \tag{4.5}$$

compare [4, Figure 3], and $\varphi_1 = \varphi_2 = 0$.

The plot on the right corresponds to condition (O), which implies that the parameter constellation is $q_2 = 0.1$, $p_2 = 0.5$, $q_1 = 2$, $p_1 = 15$. The resulting solution shows the collision of two solitons with parameters $(15, 2)$, $(0.5, 0.1)$. It is plotted for $y = 1$.

In contrast, the plot on the left corresponds to condition (A) with the parameter constellation $q_1 = 0.1$, $q_2 = 0.5$, $p_2 = 2$, $p_1 = 15$. In this case the solution shows the collision of two solitons with parameters $(15, 0.1)$, $(2, 0.5)$. It is plotted for $y = 10$.

In [31] it is shown that the line solitons are comprised in the solution class discussed above by establishing the link to their representation in terms of Casorati determinants [14, 16].

4.2. *Resonance and web structures*

In [4] a solution class was studied, which not only includes elastic collision of line-solitons [15], but also soliton resonances and web structures [19]. In this subsection we first briefly recall the construction in [4]. Then we show how this class can be realized in terms of Proposition 4.1. Finally we give some examples.

The following solution class for the 2d Toda lattice (1.1) was studied in [4]:

$$w_n = \frac{\partial^2}{\partial x \partial y} \log\det\left(Ce^{\Theta}K\right), \tag{4.6}$$

where C is a (real) $N\times M$-matrix with $N \leq M$, and K the $M\times N$-matrix

$$K = \left(\, p_j^{i-1} \,\right)_{\substack{j=1\dots,M\\ i=1,\dots,N}} = \begin{pmatrix} 1 & p_1 & \dots & p_1^{N-1} \\ \vdots & \vdots & & \vdots \\ 1 & p_M & \dots & p_M^{N-1} \end{pmatrix}$$

with $0 < p_1 < \cdots < p_M$. Finally, Θ is the $M\times M$-diagonal matrix with $\theta[p_j]$ on the diagonal, where

$$\theta[p_j](n,x,t) = \log(p_j)n + p_j x - y/p_j + \theta_j.$$

In fact, this class contains line solitons [15] for $C = C_1$, and fully resonant solutions [19] for $C = C_2$, where

$$C_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & \\ & \vdots & & & \ddots \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 1 & \dots & 1 \\ p_1 & \dots & p_M \\ \vdots & & \vdots \\ p_1^{N-1} & \dots & p_M^{N-1} \end{pmatrix}.$$

For this class, the following assumptions can be made:

(1) $N < M$.
(2) C is in reduced row echelon form (RREF) with $\mathrm{rank}(C) = N$.
(3) The matrix C in addition satisfies that (i) each column of C contains at least one nonzero element, and (ii) each row of C contains at least one nonzero element in addition to the pivot.

Assumption (1) is no restriction since for $N = M$ one obtains the trivial solution. Assumption (2) can be made without loss of generality. The reason is that the transformation $C \to C' = GC$ with $G \in \mathrm{GL}_N(\mathbb{R})$ leaves the solution invariant. Note also that unless C has full rank, the determinant in (4.6) vanishes identically, and the solution is undefined. Assumption (3) serves to avoid redundancies.

Remark 4.3. The additional assumption that

(4) all $N\times N$-minors of C are non-negative

implies regularity of the solutions (which can be easily seen by expanding the determinant).The corresponding solution class is characterised asymptotically in [4]. For the reader's convenience, we describe in Appendix B how the asymptotics of the solution depends on the RREF of C. □

The main aim of the subsection is to explain how this solution class can be realized in terms of Proposition 4.1. Note that we will not need Assumption (4). We start with the following simplification of the coefficient matrix C.

Lemma 4.4. *Without loss of generality, $C = \big(I_N\ D\big)$ with some $N \times (M-N)$-matrix D. Here I_N denotes the $N \times N$-unit matrix.*

Proof. Since C is in RREF with $\operatorname{rank}(C) = N$, there is a matrix Π, of size $M \times M$, such that $C\Pi = (I_N\ D)$ with some $N \times (M-N)$-matrix D. Now

$$\det\big(Ce^{\Theta}K\big) = \det\Big((C\Pi)\big(\Pi^{-1}e^{\Theta}\Pi\big)(\Pi^{-1}K)\Big) = \det\Big(\big(I_N\ D\big)e^{\Pi^{-1}\Theta\Pi}(\Pi^{-1}K)\Big).$$

Inspection shows that $\Pi^{-1}\Theta\Pi$ corresponds to a permutation of $p_1, \dots, p_M$, and that $\Pi^{-1}K$ changes the Vandermonde matrix K accordingly. $\square$

Similar to the decomposition of the coefficient matrix C in an $N \times N$-matrix and an $N \times (M-N)$-matrix in the above lemma, we write

$$K = \begin{pmatrix} K_N \\ K_{M-N} \end{pmatrix} \text{ and } \Theta = \operatorname{diag}\{\Theta_N, \Theta_{M-N}\}$$

with $N \times N$-matrices K_N, Θ_N, an $(M-N) \times N$-matrix K_{M-N}, and an $(M-N) \times (M-N)$-matrix Θ_{M-N}. Then

$$\begin{aligned}
\det\big(Ce^{\Theta}K\big) &= \det\left(\big(I_N\ D\big)\begin{pmatrix} e^{\Theta_N} & 0 \\ 0 & e^{\Theta_{M-N}} \end{pmatrix}\begin{pmatrix} K_N \\ K_{M-N} \end{pmatrix}\right) \\
&= \det\big(e^{\Theta_N}K_N + De^{\Theta_{M-N}}K_{M-N}\big) \\
&= \det\Big(e^{\Theta_N}\Big(I_N + e^{-\Theta_N}De^{\Theta_{M-N}}K_{M-N}K_N^{-1}\Big)K_N\Big),
\end{aligned}$$

which generates same solution w_n as

$$\det\big(I_N + e^{-\Theta_N}De^{\Theta_{M-N}}K_{M-N}K_N^{-1}\big).$$

To see how the corresponding solution is realized using Proposition 4.1, we set $E = \mathbb{R}^{M-N}$, $F = \mathbb{R}^N$, and define

$$A = \operatorname{diag}\big\{p_{N+1}, \dots, p_M\big\}, \qquad B = \operatorname{diag}\Big\{\frac{1}{p_1}, \dots, \frac{1}{p_N}\Big\}.$$

By Proposition A.4, see Appendix A, the $(M-N) \times N$-matrix $K_{M-N}K_N^{-1}$ is a solution of the Sylvester equation $AXB - X = cb^{\mathrm{t}}$ for appropriately chosen $b \in \mathbb{R}^N, c \in \mathbb{R}^{M-N}$. Hence,

$$\begin{aligned}
e^{\Theta_{M-N}}K_{M-N}K_N^{-1} &= A^n \exp\big(Ax - A^{-1}y\big)\,C_0, \\
e^{-\Theta_N}D &= B^n \exp\big(-B^{-1}x + By\big)\,D_0,
\end{aligned}$$

where

$$C_0 = \operatorname{diag}\{e^{\theta_{N+1}}, \dots e^{\theta_M}\}K_{M-N}K_N^{-1} \quad \text{and} \quad D_0 = \operatorname{diag}\{e^{-\theta_1}, \dots e^{-\theta_N}\}D.$$

As a consequence, the solution class (4.6) is included in Propsition 4.1 with the two generating matrices A, B being diagonal, but not necessarily of the same size.

Let us illustrate the above by having a closer look at collisions of two line-solitons. Recall $0 < p_1 < p_2 < p_3 < p_4$, and let $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$. Elastic collision of two line-solitons correponds to the following choices for C [4, Lemma 4.1]:

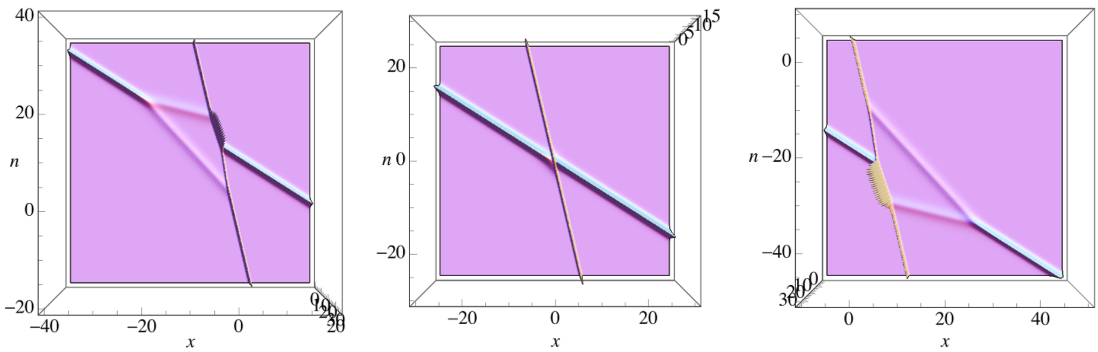

Fig. 3. Elastic collision of two line-solitons: resonant type, plotted for $y = -4$, $y = 0$, and $y = 6$ (from left to right). See Example 4.5 for details.

(O) *ordinary type*: $C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$,

(A) *asymmetric type*: $C = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$,

(R) *resonant type*: $C = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & c & d \end{pmatrix}$ with $c > d > 0$.

Note that in the case (O), the second and third column of C have to be interchanged in order to bring C into the form of Lemma 4.4. Hence the following settings are needed in our formalism:

(O) $A = \begin{pmatrix} p_2 & 0 \\ 0 & p_4 \end{pmatrix}$, $B = \begin{pmatrix} 1/p_1 & 0 \\ 0 & 1/p_3 \end{pmatrix}$, and $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

(A) $A = \begin{pmatrix} p_3 & 0 \\ 0 & p_4 \end{pmatrix}$, $B = \begin{pmatrix} 1/p_1 & 0 \\ 0 & 1/p_2 \end{pmatrix}$, and $D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

(R) $A = \begin{pmatrix} p_3 & 0 \\ 0 & p_4 \end{pmatrix}$, $B = \begin{pmatrix} 1/p_1 & 0 \\ 0 & 1/p_2 \end{pmatrix}$, and $D = \begin{pmatrix} -1 & -1 \\ c & d \end{pmatrix}$ with $c > d > 0$.

Note however, that the constants θ_j and the φ_j from our formalism not necessarily coincide.

Example 4.5. For $p_4 = 15$, $p_3 = 2$, $p_2 = 0.5$, $p_1 = 0.1$, the elastic collision of two line-solitons of resonant type is visualized in Fig. 3 with $c = 2$, $d = 1$, compare [4, Figure 4].

A similar transition can be made for the inelastic collision of two line-solitons. The conditions given in [4, Lemma 4.1] are:

type I: $C = \begin{pmatrix} 1 & 1 & 0 & -r \\ 0 & 0 & 1 & 1 \end{pmatrix}$, *type II*: $C = \begin{pmatrix} 1 & 0 & -r & -r \\ 0 & 1 & 1 & 1 \end{pmatrix}$,

type III: $C = \begin{pmatrix} 1 & 0 & 0 & -r \\ 0 & 1 & 1 & 1 \end{pmatrix}$, *type IV*: $C = \begin{pmatrix} 1 & 0 & -r & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

with $r > 0$. The corresponding settings for the realization in our formalism are the following:

type I: $D = \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix}$, *type II*: $D = \begin{pmatrix} -r & -r \\ 1 & 1 \end{pmatrix}$,

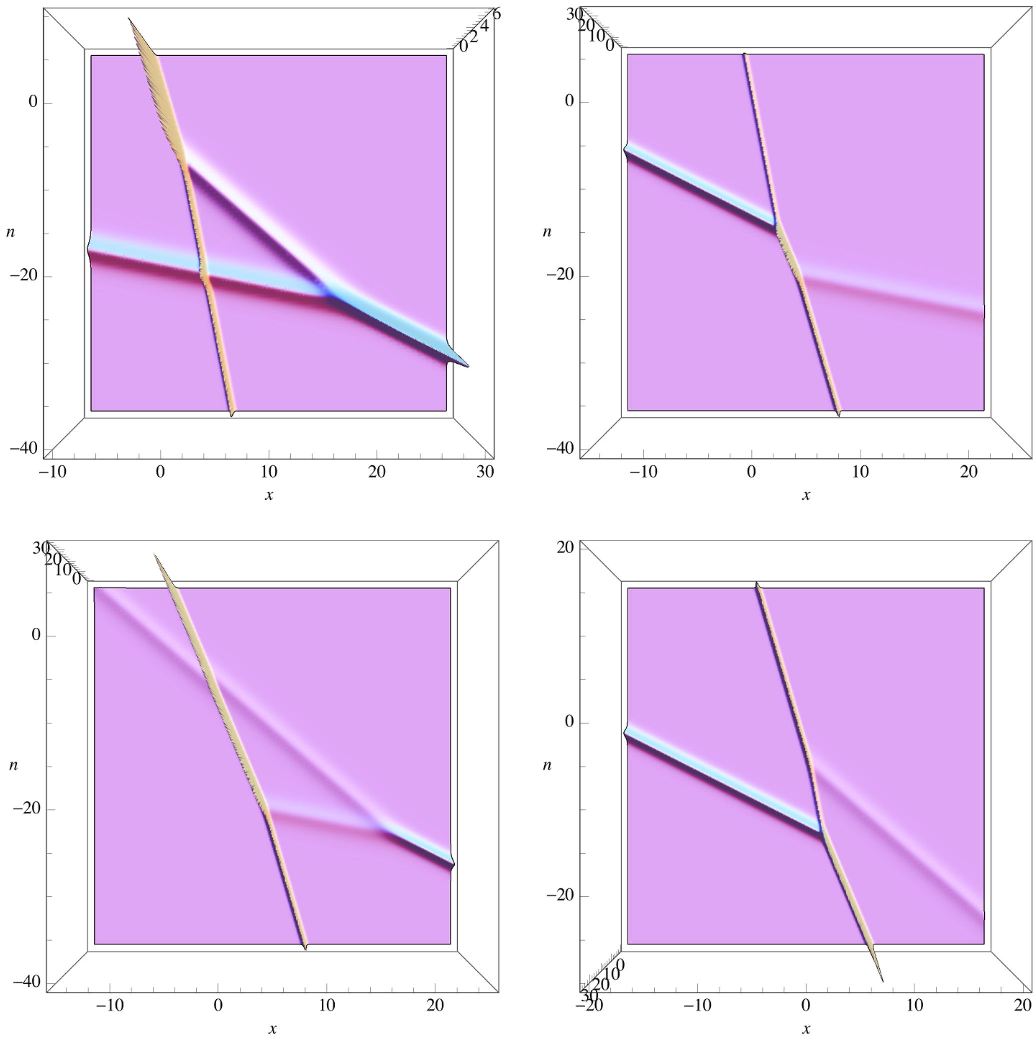

Fig. 4. Inelastic collision of two line-solitons, plotted for $y = 4$. Top left: type I, top right: type II, bottom left: type III, bottom right: type IV. See Example 4.6 for details.

$$\textit{type III}: \quad D = \begin{pmatrix} 0 & -r \\ 1 & 1 \end{pmatrix}, \qquad \textit{type IV}: \quad D = \begin{pmatrix} -r & -1 \\ 1 & 0 \end{pmatrix},$$

with

$$A = \begin{pmatrix} p_2 & 0 \\ 0 & p_4 \end{pmatrix}, B = \begin{pmatrix} 1/p_1 & 0 \\ 0 & 1/p_3 \end{pmatrix} \text{ for type I,}$$
$$A = \begin{pmatrix} p_3 & 0 \\ 0 & p_4 \end{pmatrix}, B = \begin{pmatrix} 1/p_1 & 0 \\ 0 & 1/p_2 \end{pmatrix} \text{ for types II – IV.}$$

Example 4.6. For $p_4 = 15,\ p_3 = 2,\ p_2 = 0.5,\ p_1 = 0.1$, inelastic collision of two line-solitons is visualized in Fig. 4 with $r = 1$, compare [4, Figure 5].

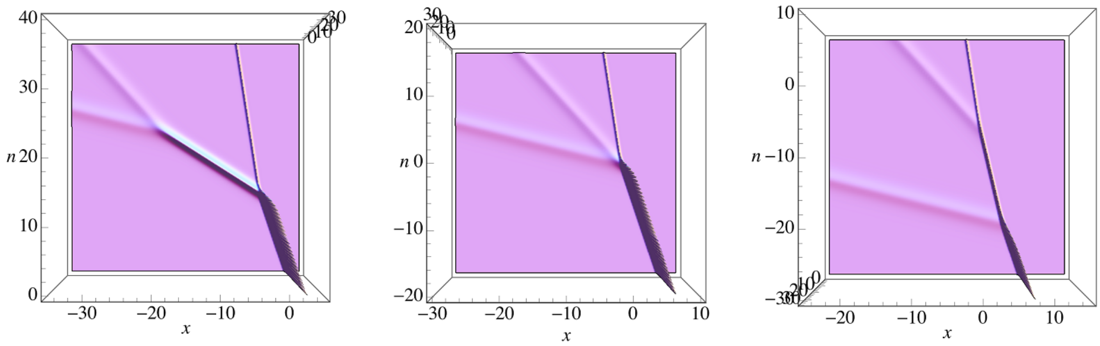

Fig. 5. A Miles structure, plotted for $y = -4$, $y = 0$, and $y = 4$ (from left to right). See Example 4.7 for details.

We conclude the present subsection with the realization of Miles structures [21]. Choosing

$$A = \mathrm{diag}\{p_1, \ldots, p_M\}, \qquad B = (1/q),$$

and the $1 \times M$-matrix $D = \big(1, 1, \ldots, 1\big)$ in Proposition 4.1, we get for the solution of the one-dimensionality condition $ACB - C = c\, b^{\mathrm{t}}$ with $b \in \mathbb{R}$, $c \in \mathbb{R}^M$ the $M \times 1$-matrix

$$\left(\frac{bc_j}{p_j/q - 1} \right)_{j=1}^{M},$$

leading to

$$p = 1 + \sum_{j=1}^{N} \frac{bc_j}{p_j/q - 1} \ell[p_j]/\ell[q], \tag{4.7}$$

where $\ell[p]$ is defined as in (4.2). Regularity of the corresponding solution can for example be guaranteed by $p_j > q > 0$ and $bc_j > 0$ for all j.

Note that (4.7) can be considered as the degenerated case $q_1 = \cdots = q_M = q$ of (4.3). In fact, starting from $A = \mathrm{diag}\{p_1, \ldots, p_M\}$, $B = \mathrm{diag}\{1/q_1, \ldots, 1/q_M\}$, one arrives at the same solution class. This class has also been discussed in [31]. For fixed values of y the solutions represent tree-like structures with M line solitons for $n \to \infty$ and 1 line soliton for $n \to -\infty$.

A similar discussion can be made for solutions generated from

$$A = (p), \qquad B = \mathrm{diag}\{1/q_1, \ldots, 1/q_N\}.$$

Example 4.7. The solutions in Fig. 5 correspond to the case $M = 3$ and the parameter constellation $p_1 = 0.5$, $p_2 = 2$, $p_3 = 15$ and $q = 0.1$. The initial phase shifts are determined by $(bc_j)/(p_j/q - 1) = 1$.

4.3. *Further examples*

In this subsection we give some selected applications of Proposition 4.1 beyond the case of diagonal matrices A, B.

First we will discuss the case that A, B are 2×2 Jordan blocks and $D = I_2$. Note that the corresponding solutions can be easily realized in the framework of [31], where a solution formula was established for commuting A and B, without the additional matrix parameter D. In the next case we drop the assumption $D = I_2$. Finally we consider the case that A is a 2×2 Jordan block and B a 2×2-diagonal matrix with two different eigenvalues.

We start with the observation that the assumption that A, B are in Jordan form can be made without loss of generality.

Lemma 4.8. *Let U, V be the transformation matrices of the $M\times M$-matrix A, the $N\times N$-matrix B into Jordan canonical form J_A, J_B, respectively (i.e., we have $A = U^{-1}J_AU$ and $B = V^{-1}J_BV$).*

Then the solution in Proposition 4.1 is not altered if we replace simultaneously A, B by J_A, J_B, the matrix D by VDU^{-1} and the vectors b, c by $(V^{-1})^{\mathrm{t}}b$, Uc.

Proof. Let $\tilde{D} = VDU^{-1}$, $\tilde{b} = (V^{-1})^{\mathrm{t}}b$, $\tilde{c} = Uc$, and $\tilde{C} = UCV^{-1}$.

First we show that replacing A, B by J_A, J_B and b, c by $\tilde{b}$, $\tilde{c}$ amounts to replacing C by $\tilde{C}$. Since C solves the Sylvester equation $AXB - X = cb^{\mathrm{t}}$,

$$\begin{aligned}\tilde{c}\tilde{b}^{\mathrm{t}} &= (Uc)\big((V^{-1})^{\mathrm{t}}b\big)^{\mathrm{t}} = U(cb^{\mathrm{t}})V^{-1} = U(ACB-C)V^{-1}\\ &= (UAU^{-1})(UCV^{-1})(VBV^{-1}) - UCV^{-1} = J_A\tilde{C}J_B - \tilde{C},\end{aligned}$$

implying that $\tilde{C}$ is the (unique) solution of $J_AXJ_B - X = \tilde{c}\tilde{b}^{\mathrm{t}}$.

Furthermore, by definition of the exponential function as power series, we have $A^n\exp\big(Ax - A^{-1}y\big) = U^{-1}J_A^n\exp\big(J_Ax - J_A^{-1}y\big)U$, and $B^n\exp\big(-B^{-1}x + By\big) = V^{-1}J_B^n\exp\big(-J_B^{-1}x + J_By\big)V$. This yields

$$\begin{aligned}UL_n(x,y)V^{-1} &= J_A^n\exp\big(J_Ax - J_A^{-1}y\big)\tilde{C},\\ VM_n(x,y)U^{-1} &= J_B^n\exp\big(-J_B^{-1}x + J_By\big)\tilde{D}.\end{aligned}$$

By multiplicity of the determinant, $p(n,x,y) = \det(I_N + M_n(x,y)L_n(x,y))$ hence coincides with the determinant of

$$I_N + J_B^n\exp\big(-J_B^{-1}x + J_By\big)\tilde{D}J_A^n\exp\big(J_Ax - J_A^{-1}y\big)\tilde{C},$$

generated using the data J_A, J_B, $\tilde{D}$, and $\tilde{b}$, $\tilde{c}$ in Proposition 4.1. □

4.3.1. *A, B (commuting)* 2×2*-Jordan blocks and* $D = I_2$

Let us now first consider the case that

$$A = \begin{pmatrix} p & 1\\ 0 & p\end{pmatrix},\quad B = \begin{pmatrix} 1/q & 1\\ 0 & 1/q\end{pmatrix},$$

with $p,q > 0$. It is not too hard to compute the determinant in Proposition 4.1 explicitly. The result is

$$p(n,x,y) = 1 + f(n,x,y)\ell(n,x,y) - \ell(n,x,y)^2,$$

where

$$\ell(n,x,y) = \varepsilon\exp\big(\theta[p] - \theta[q] + \varphi\big),$$

$$f(n,x,y) = (p-q)\big(P+Q+\gamma\big)/q.$$

Here we have introduced the ploynomials

$$P = P(n,x,y) = \big(\chi[p] - p/(p-q)\big)/p, \qquad Q = Q(n,x,y) = q\big(\chi[q] - p/(p-q)\big),$$

with

$$\theta[p] = \log(p)n + px - y/p, \qquad \chi[p] = n + px + y/p,$$

and[f] the parameters φ, γ and the sign ε are given by

$$\varepsilon \exp(\varphi) = \frac{b_1c_2}{(p/q-1)^2}, \quad \gamma = \frac{b_1c_1+b_2c_2}{b_1c_2} - \frac{1+pq}{p-q}.$$

In order to do an asymptotic investigation, we fix y and look at the solution for $n \to \pm\infty$. Following the kind of arguments laid down in [29] for the one-dimensional Toda lattice, we arrive at[g]

$$w_n(x,y) \approx w_1^{\pm}(n,x,y) + w_2^{\pm}(n,x,y) \text{ for } n \approx \pm\infty,$$

with solitons

$$w_j^{\pm}(n,x,y) = \frac{(p-q)^2}{pq} \frac{\ell_j^{\pm}(n,x,y)}{\big(1+\ell_j^{\pm}(n,x,y)\big)^2},$$

where

$$\begin{aligned} \ell_1^{\pm}(n,x,y) &= \ \ \varepsilon C^{\pm 1} \exp\big(\theta[p] - \theta[q] \pm \log|n| + \varphi\big), \\ \ell_2^{\pm}(n,x,y) &= -\varepsilon C^{\mp 1} \exp\big(\theta[p] - \theta[q] \mp \log|n| + \varphi\big), \end{aligned}$$

and

$$C = \frac{p-q}{q}\big((1+p/d)/p + q(1+q/d)\big), \qquad d = -\frac{p-q}{\log p - \log q}.$$

This shows that, for fixed y, the solution constitutes a wave packet consisting of two partners. As a whole, the wave packet has slope d in xn-space, with the partners deviating logarithmically as $n \to \pm\infty$.

Note that the asymptotics also shows that one of the partners is a (regular) soliton and the other a (singular) antisoliton. Identifying them by their regularity, one also observes that, from $n \to -\infty$ to $n \to \infty$, the solitons cross sides. In fact, for the one-dimensional Toda lattice, which can be obtained from (1.1) by reduction [4], this kind of appearance of negaton solutions is well reflected [29].

Example 4.9. For $p = 1$, $q = 2$, the solution discussed above is visualized in Figure 6, with $b_1 = 1/10$, $b_2 = -1$, $c_1 = 2$, $c_2 = 1$. Note that the solution is plotted in the frame $(n, x - \log(2)n, 0)$, corresponding to the fact that a line-soliton with parameters $p = 1$, $q = 2$ has slope $-1/\log(2)$ in xn-space.

[f] assuming $b_1c_2 \neq 0$

[g] Here we use the following notion of convergence, see [29]: For fixed x, $f_x(n) := f(n,x)$ is viewed as a mapping to the Riemann number sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Equipping $\hat{\mathbb{C}}$ with the metric $d^\infty(w,z) = |\pi^{-1}(w) - \pi^{-1}(z)|$, where π denotes the stereographic projection $\pi : S^2 \to \hat{\mathbb{C}}$, we say that two functions $f(n,x)$ and $g(n,x)$ have the same asymptotic behaviour for $n \to \pm\infty$, briefly $f(n,x) \approx g(n,x)$ for $n \approx \pm\infty$, if the family $(f-g)_n : \mathbb{R} \to \hat{\mathbb{C}}$ converges to zero as $n \to \pm\infty$ uniformly with respect to the metric d^∞.

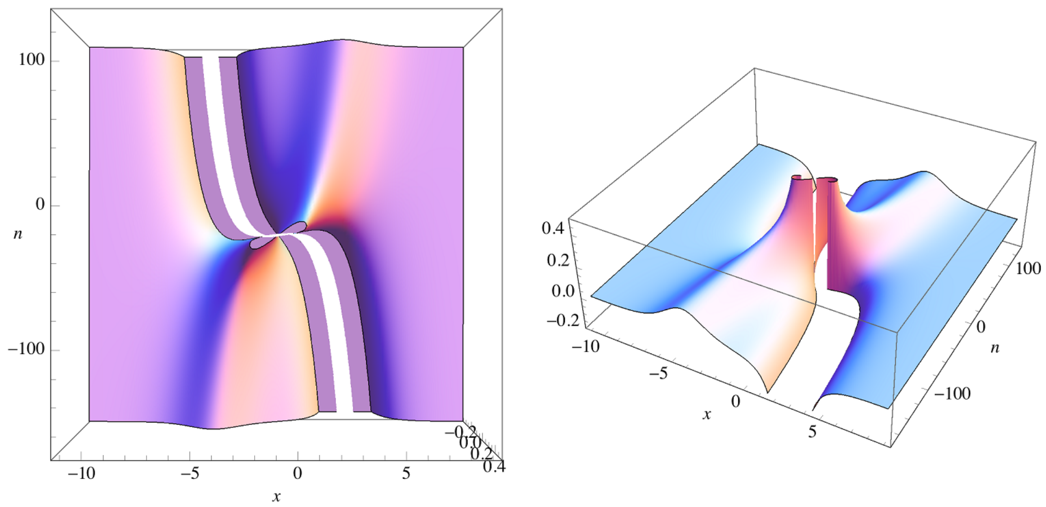

Fig. 6. The plots depict the solution from Example 4.9 for $y = 0$, differing only in the viewpoint. Note that the white regions indicate singularities.

4.3.2. *The general case of (commuting)* 2×2*-Jordan blocks A, B*

In the next case we once more consider

$$A = \begin{pmatrix} p & 1 \\ 0 & p \end{pmatrix}, \quad B = \begin{pmatrix} 1/q & 1 \\ 0 & 1/q \end{pmatrix}$$

with p, $q > 0$, but now we take advantage of the additional matrix parameter D. To this end, let

$$D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}.$$

Inspection of the determinant in Proposition 4.1 in this case shows that in order to capture the leading terms in the asymptotic analysis it suffices to choose

$$b = (b_1, 0)^{\mathrm{t}}, \quad c = (0, c_2)^{\mathrm{t}}$$

with $b_1 c_2 \neq 0$. For the determinant we then get

$$p(n,x,y) = 1 + f(n,x,y)\ell(n,x,y) - \det(D)\,\ell(n,x,y)^2,$$

where all ingredients are defined as in the case before except

$$f(n,x,y) = (p-q)\Big(d_2 + d_1 P + d_4 Q + d_3\big(PQ + q^2/(p-q)^2\big)\Big)/q.$$

Assume $d_3 \neq 0$ and $\det(D) \neq 0$. Asymptotic analysis (for fixed y) confirms that, for $n \to \pm\infty$, the solution constitutes a wave packet consisting of two solitons $w_j^{\pm}(n,x,y)$ as before, but now build with the exponentials

$$\ell_1(n,x,y) = \quad \varepsilon C \exp\big(\theta[p] - \theta[q] + 2\log|n| + \varphi + \log\delta\big),$$

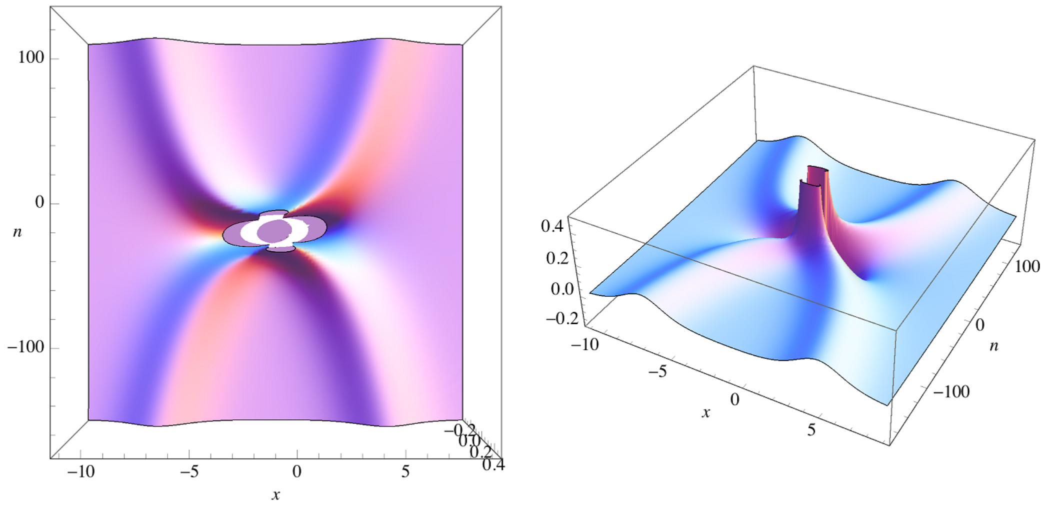

Fig. 7. The plots depict the solution from Example 4.10 for $y = 0$, differing only in the viewpoint, with white regions indicating singularities.

$$\ell_2(n,x,y) = -\sigma\varepsilon C^{-1}\exp\left(\theta[p]-\theta[q]-2\log|n|+\varphi+\log\delta\right),$$

where

$$C = \frac{d_3}{\delta}\,\frac{p-q}{p}(1+p/d)(1+q/d)$$

and we have set $\det(D) = \sigma\delta^2$ with a sign σ and $\delta > 0$.

Observe the additional factor 2 in front of the logarithm, which is due to the fact that the polynomial appearing in the determinant $p(n,x,y)$ is of degree 2. The even degree of the polynomial is also responsible for the fact that the solitons no longer cross sides. It is remarkable that choosing d_3 and $\det(D)$ appropriately, we now can arrange that *both* solitons are *regular*.

It should be stressed that the first case ($D = I_2$) actually is degeneration of the case above.

Example 4.10. For $p = 2$ and $q = 1$, the solution discussed above is visualized in Figure 7, with $b_1 = c_2 = 1$ and $d_1 = -10$, $d_2 = -2$, $d_3 = -1$, $d_4 = 1$. It is plotted in the frame $(n, x-\log(2)n, 0)$.

4.3.3. *Noncommuting A, B (of same size)*

Finally we look at the case

$$A = \begin{pmatrix} p & 1 \\ 0 & p \end{pmatrix}, \quad B = \begin{pmatrix} 1/q_1 & 0 \\ 0 & 1/q_2 \end{pmatrix},$$

(i.e. with non-commuting matrix parameters A and B) for $p, q_1, q_2 > 0$ and, without loss of generality, $q_1 > q_2$.

Computing the determinant in Proposition 4.1 for this parameter choice, one sees that to keep the leading terms it is sufficient to choose

$$b = (b_1, b_2)^{\mathrm{t}}, \quad c = (0, c_2)^{\mathrm{t}}$$

with $b_1 c_2$ and $b_2 c_2 \neq 0$. For the determinant we then get

$$\begin{aligned} p(n,x,y) = {} & 1 + d_1 P^{(1)}(n,x,y)\ell^{(1)}(n,x,y) \\ & + d_3 P^{(2)}(n,x,y)\ell^{(2)}(n,x,y) - \det(D)A_{12}\,\ell^{(1)}(n,x,y)\ell^{(2)}(n,x,y), \end{aligned}$$

where

$$\begin{aligned} \ell^{(j)}(n,x,y) &= \varepsilon_j \exp(\theta[p] - \theta[q_j] + \varphi_j), \\ P^{(j)}(n,x,y) &= (p - q_j)\Big(\chi[p] - p/(p - q_j)\Big)/p, \end{aligned}$$

with the parameters φ_j and the signs ε_j given by $\varepsilon_j \exp(\varphi_j) = b_j c_2\, q_j/(p - q_j)$, and where

$$A_{12} = \frac{q_1 - q_2}{(p - q_1)(p - q_2)}.$$

Assume $d_1, d_3 \neq 0$ and $\det(D) \neq 0$. Recall that without loss of generality $q_1 > q_2$. For the asymptotic analysis we also fix $p > q_1$ (note that $q_1 > p > q_2$ and $q_2 > p$ can be treated similarly). As a result, for $n \to \pm\infty$ the solution consists of two solitons $w^{(j),\pm}(n,x,y)$, build with

$$\begin{aligned} \ell^{(1),\pm}(n,x,y) &= -\varepsilon_1 \sigma_1^{\pm}(C^{(1)})^{\mp 1} \exp\big(\,\theta[p] - \theta[q_1] \mp \log|n| + \varphi_1 + \log\delta_1 + \varphi_1^{\pm}\,\big), \\ \ell^{(2),\pm}(n,x,y) &= \;\;\varepsilon_2 \sigma_2^{\pm}(C^{(2)})^{\pm 1} \exp\big(\,\theta[p] - \theta[q_2] \pm \log|n| + \varphi_2 + \log\delta_2 + \varphi_2^{\pm}\,\big), \end{aligned}$$

where

$$C^{(j)} = \sqrt{\left|\frac{d_1 d_3}{\det(D)}\right|}\left(\frac{1}{p} + \frac{1}{d^{(j)}}\right), \qquad d^{(j)} = -\frac{p - q_j}{\log p - \log q_j},$$

$$\begin{aligned} &\sigma_1^+ = \operatorname{sgn}(d_3 \det(D)), \quad \sigma_1^- = \operatorname{sgn}(d_1), \quad \sigma_2^+ = \operatorname{sgn}(d_3), \quad \sigma_2^- = \operatorname{sgn}(d_1 \det(D)), \\ &\qquad \delta_1 = \sqrt{|d_1 \det(D)/d_3|}, \qquad\qquad\qquad \delta_2 = \sqrt{|d_3 \det(D)/d_1|}. \end{aligned}$$

and

$$\varphi_1^+ = \varphi_2^- = \log A_{12}, \qquad \varphi_1^- = \varphi_2^+ = 0.$$

Hence the solution consists of two *different* solitons, corresponding to the parameter constellations (p, q_1) and (p, q_2), respectively, which deviate logarithmically from their respective slopes $d^{(1)}$ and $d^{(2)}$.

Note that the collision ot the two solitons causes a phase-shift determined by $A_{12} = (1/q_1 - 1/q_2)/((p/q_2 - 1)(p/q_1 - 1))$, compare with (4.4).

Observe also that regularity of the solitons can be manipulated by the signs of d_1, d_3 and $\det(D)$ in the following way: One can pair the solitons arbitrarily and assign to each pair separately whether both solitons of the pair are regular or singular.

Example 4.11. For $p = 2$, $q_1 = 1$, $q_2 = 1/2$, the corresponding solution is depicted in Figure 8 with $b_1 = c_2 = 1$ and $d_1 = 1$, $d_2 = -1$, $d_3 = -10$, $d_4 = -1$.

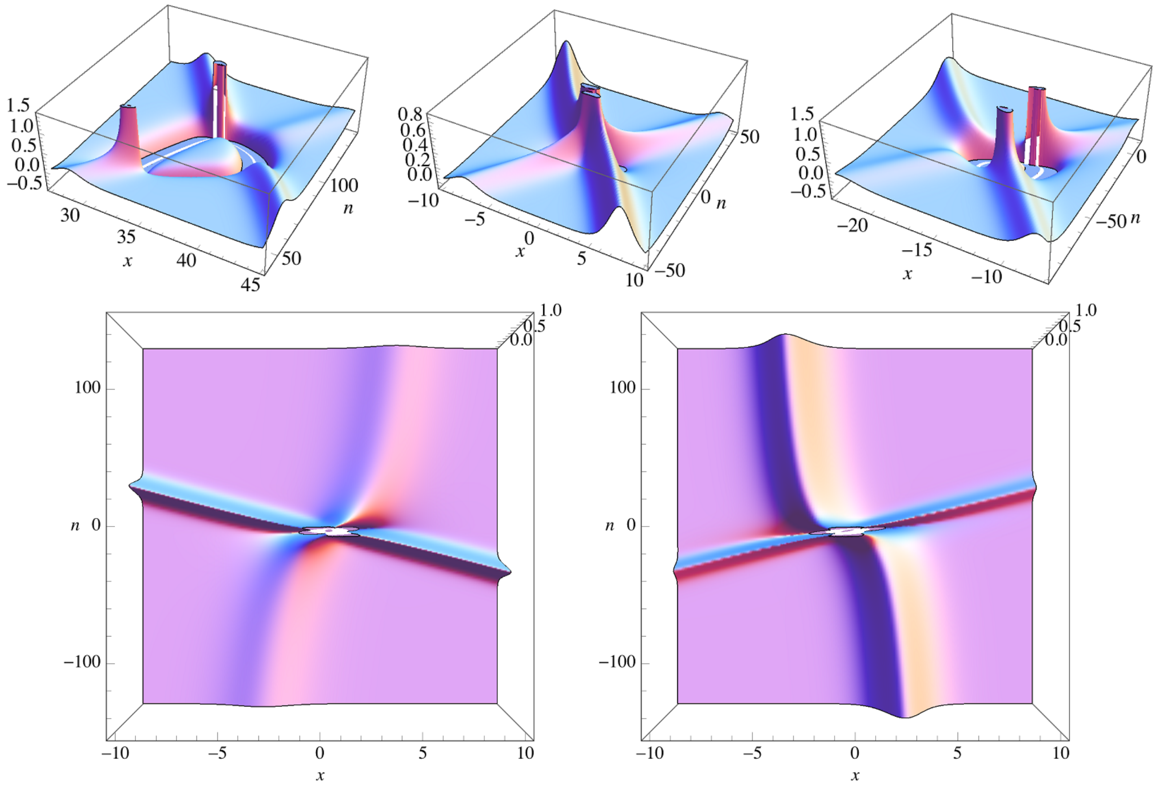

Fig. 8. In the upper row the solution from Example 4.11 is depicted for $y = -50$, $y = 0$, and $y = 20$ (from left to right). The two plots in the lower row show the solution for $y = 0$, but in different coordinate systems, see Example 4.11 for details.

The two involved solitons correspond to the parameters $(2,1)$ and $(2,1/2)$. Observe that a line-soliton with parameters $(2,1)$ has slope $-1/\log 2$ in xn-space and one with parameters $(2,1/2)$ has slope $-3/(4\log 2)$. The plots in the lower row of Figure 8 show the solution, plotted for $y = 0$, in the corresponding frames $(n, x-\log(2)n, 0)$ (left) and $(n, x-4\log(2)n/3, 0)$ (right), the logarithmic paths of the solitons being clearly visible.

The plots in the upper row are plotted in $(n, x-7\log(2)n/6, 0)$, where $-7\log(2)/6$ is the arithmetic middle of $-\log(2)$ and $-4\log(2)/3$.

A. Factorizing solutions of the Sylvester equation $AX + XB = C$ by Vandermonde matrices

Let $A = \operatorname{diag}\{\alpha_1, \ldots, \alpha_m\}$, $B = \operatorname{diag}\{\beta_1, \ldots, \beta_n\}$ be diagonal matrices, not necessarily of the same size, with $0 \notin \operatorname{spec}(A) + \operatorname{spec}(B)$. Note that this spectral condition guarantees the invertibility of the elementary operator

$$\Psi_{A,B}(X) = AX + XB \tag{A.1}$$

on the $m \times n$-matrices over $\mathbb{C}$.

In what follows we give a factorization of $\Psi_{A,B}^{-1}(C)$ for arbitrary one-dimensional matrices C in terms of Vandermonde-type matrices. The results below generalize work in [30] for A, B of the same size.

To the data given above, we assign the $m \times n$-Vandermonde matrix $V_{A,n}$ and the $n \times n$-matrix W_B as follows

$$V_{A,n} = \Big(\alpha_i^{j-1} \Big)_{\substack{i=1,\dots,m \\ j=1,\dots,n}} = \begin{pmatrix} 1 & \alpha_1 & \dots & \alpha_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \alpha_m & \dots & \alpha_m^{n-1} \end{pmatrix}$$

and

$$W_B = \Bigg(\sum_{\substack{\kappa_1 < \dots < \kappa_{i-1} \\ j \notin \{\kappa_1, \dots, \kappa_{i-1}\}}} \beta_{\kappa_1} \dots \beta_{\kappa_{i-1}} \Bigg)_{i,j=1}^{n} = \begin{pmatrix} 1 & \dots & 1 \\ \sum_{j \neq 1} \beta_j & \dots & \sum_{j \neq n} \beta_j \\ \sum_{\substack{j<j' \\ j,j' \neq 1}} \beta_j \beta_{j'} & \dots & \sum_{\substack{j<j' \\ j,j' \neq n}} \beta_j \beta_{j'} \\ \vdots & & \vdots \\ \prod_{j \neq 1} \beta_j & \dots & \prod_{j \neq n} \beta_j \end{pmatrix}.$$

Furthermore, we introduce the involution $J_n = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$ of size $n \times n$.

Lemma A.1. *The following identity holds:*

$$V_{A,n} J_n W_B = \Bigg(\prod_{\substack{\kappa=1 \\ \kappa \neq j}}^{n} (\alpha_i + \beta_\kappa) \Bigg)_{\substack{i=1,\dots,m \\ j=1,\dots,n}}.$$

Proof. By direct verification, $V_{A,n} J_n = \big(\alpha_i^{n-j} \big)_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$.

Hence, denoting the ij-th entry of a matrix T as usual by T_{ij}, matrix calculation shows

$$\begin{aligned} \big(V_{A,n} J_n W_B \big)_{ij} &= \alpha_i^{n-1} + \alpha_i^{n-2} \sum_{\kappa \neq j} \beta_\kappa + \alpha_i^{n-3} \sum_{\kappa, \kappa' \neq j} \beta_\kappa \beta_{\kappa'} + \dots + \prod_{\kappa \neq j} \beta_\kappa \\ &= \prod_{\kappa \neq j} (\alpha_i + \beta_\kappa). \end{aligned}$$

□

As a consequence, $V_{A,n} J_n W_B$ is a solution of the Sylvester equation $AX + XB = C$ for an appropriately chosen one-dimensional right-hand side C.

Corollary A.2. *It holds that $A(V_{A,n} J_n W_B) + (V_{A,n} J_n W_B) B = D\ (f_m f_n^{\mathrm{t}})$, where $f_k \in \mathbb{C}^k$ is the vector with all entries equal to* 1 *and*

$$D = \operatorname{diag} \Bigg\{ \prod_{\kappa=1}^{n} (\alpha_i + \beta_\kappa) \ \Bigg| \ i = 1, \dots, m \Bigg\}. \tag{A.2}$$

In summary, we have shown the following factorization result.

Theorem A.3. *Let* $A = \mathrm{diag}\{\alpha_1, \dots, \alpha_m\}$ *and* $B = \mathrm{diag}\{\beta_1, \dots, \beta_n\}$ *such that* $0 \notin \mathrm{spec}(A) + \mathrm{spec}(B)$. *Then, for any* $b \in \mathbb{C}^n$, $c \in \mathbb{C}^m$, *the following factorization holds:*

$$\Psi_{A,B}^{-1}(c\,b^{\mathrm{t}}) = D_c D^{-1}(V_{A,n} J_n W_B) D_b,$$

where, for $f = (f_\kappa)_{\kappa=1}^k \in \mathbb{C}^k$, D_f *denotes the* $k \times k$*-diagonal matrix with the entries* $f_1, \dots, f_k$ *on the diagonal and* D *is given in* (A.2).

Proof. From Corollary A.2 we have $V_{A,n}J_nW_B = \Psi_{A,B}^{-1}\big(D(f_m f_n^{\mathrm{t}})\big)$. Since D commutes with A, it immediately follows that $D^{-1}(V_{A,n}J_nW_B) = \Psi_{A,B}^{-1}\big(f_m f_n^{\mathrm{t}}\big)$, and, similarly,

$$D_c D^{-1}(V_{A,n}J_nW_B)D_b = \Psi_{A,B}^{-1}\,\big(\,D_c(f_m f_n^{\mathrm{t}})D_b\,\big) = \Psi_{A,B}^{-1}(c\,b^{\mathrm{t}}).$$

□

The following variant of the Theorem A.3 for the elementary operator $\Phi_{A,B}$ defined in (3.4) is used in Subsection 4.2.

Proposition A.4. *Let* $0 < p_1 < \dots < p_M$, *and let* $A = \mathrm{diag}\{p_{N+1}, \dots, p_M\}$ *and* $B = \mathrm{diag}\{1/p_1, \dots, 1/p_N\}$ *with* $N < M$. *Furthermore, set*

$$K_N = \big(\,p_i^{j-1}\,\big)_{i,j=1}^N, \qquad K_{M-N} = \big(\,p_{N+i-1}^{j-1}\,\big)_{\substack{i=1,\dots,M-N \\ j=1,\dots,N}}.$$

Then

$$K_{M-N}K_N^{-1} = \Phi_{A,B}^{-1}(C)$$

for an appropriately chosen one-dimensional matrix C.

For the proof it is useful to express the inverse of a Vandermonde matrix in the notation at hand. Note that this is a direct consequence of Lemma A.1 for $n = m$ and $B = -A$.

Corollary A.5. *Let the* α_j *be pairwise different. Then*

$$V_{A,m}^{-1} = J_m W_{-A} D_A^{-1},$$

where $D_A = \mathrm{diag}\Big\{ \prod_{\kappa \neq i} (\alpha_i - \alpha_\kappa) \,\Big|\, i = 1, \dots, m \Big\}$.

Proof of Proposition A.4. Note first that, in the notation of this section, we have $m = M - N$, $n = N$, and $K_{M-N} = V_A$ and $K_N = V_{B^{-1}}$. By Corollary A.5,

$$(V_{B^{-1}})^{-1} = J_N W_{-B^{-1}} D_1^{-1}, \quad \text{where } D_1 = \mathrm{diag}\Big\{ \prod_{\substack{\kappa=1 \\ \kappa \neq i}}^{N} (p_i - p_\kappa) \,\Big|\, i = 1, \dots, N \Big\}.$$

Hence $K_{M-N}K_N^{-1}D_1 = V_A J_N W_{-B^{-1}}$, and by Corollary A.2 the latter is a solution of the Sylvester equation

$$AX + X(-B^{-1}) = D_2(f_{M-N} f_N^{\mathrm{t}}),$$

$$\text{where } D_2 = \operatorname{diag}\left\{ \prod_{\kappa=1}^{N} (p_{N+i} + 1/p_\kappa) \;\middle|\; i = 1, \ldots, M-N \right\}.$$

Since B and D_1 commute, it follows that $K_{M-N}K_N^{-1}$ solves

$$\begin{aligned} AXB - X &= D_2(f_{M-N}f_N^{\mathrm{t}})\,(D_1B)^{-1} \\ &= \big(D_2 f_{M-N}\big)\big((D_1B)^{-1} f_N^{\mathrm{t}}\big) \end{aligned}$$

with the right-hand side being one-dimensional. □

B. Classification of line solitons

For the reader's convenience, we rehearse results from [4], where a classification of the solutions (4.6) under Assumptions (1)–(4) is given. More precisely, their asymptotic behaviour for $n \to \pm\infty$ at a fixed time y is obtained in dependence of the coefficient matrix C. Note that the asymptotics is essentially independent of the particular value of y.

Recall that a *single* line soliton is determined by *two* of the parameters $0 < p_1 < \cdots < p_M$ (up to a possible initial shift), and we use the abbreviation $[i, j]$ for the soliton determined by $p_i < p_j$. Note that the p_i are assumed to be generic in an appropriate sense, see [4]. Line solitons appearing asymptotically for $n \to -\infty$ are called *incoming*, for $n \to \infty$ *outgoing* line solitons.

1. There are precisely N incoming line solitons, which are characterised by $[i_1^-, j_1^-], \ldots, [i_N^-, j_N^-]$ (with $i_k^- < j_k^-$), where $i_1^-, \ldots, i_N^-$ are the numbers of the pivot columns of C.
2. There are precisely $M-N$ outgoing line solitons, which are characterised by $[i_1^+, j_1^+], \ldots, [i_{M-N}^+, j_{N-M}^+]$ (with $i_k^+ < j_k^+$), where $j_1^+, \ldots, j_{N-M}^+$ are the numbers of the non-pivot columns of C.

To identify the incoming and outgoing line solitons completely, we introduce the matrices $C_{ij}^{\pm}$ build from the following columns of C:

columns no. $1, \ldots, i-1$ and $j+1, \ldots, M$ for C_{ij}^-,
columns no. $i+1, \ldots, j-1$ for C_{ij}^+.

Observe that we admit empty matrices with rank 0. The following statements hold:

1. If $[i, j]$ is an incoming line soliton, then
   a) $\operatorname{rank}(C_{ij}^-) < N$,
   b) augmenting C_{ij}^- by (i) column no. i, (ii) column no. j or (iii) columns no. i and j from C, increases the rank by 1.
2. If $[i, j]$ is an outgoing line soliton, then
   a) $\operatorname{rank}(C_{ij}^+) < N$,
   b) augmenting C_{ij}^+ by (i) column no. i, (ii) column no. j or (iii) columns no. i and j from C, increases the rank by 1.

For the justification of the above statements we refer to [4]. Here we confine ourselves to mentioning that the above conditions yield the asymptotic behaviour of collisions of two solitons (i.e. solutions with 2 incoming and 2 outgoing solitons) in Table 1.

type	asymptotic line solitons	
	$n \to -\infty$	$n \to \infty$
ordinary (O)	$[1,2],[3,4]$	$[1,2],[3,4]$
asymmetric (A)	$[1,4],[2,3]$	$[2,3],[1,4]$
resonant (R)	$[1,3],[2,4]$	$[1,3],[2,4]$
I	$[1,3],[3,4]$	$[1,2],[2,4]$
II	$[1,2],[2,4]$	$[1,3],[3,4]$
III	$[1,3],[2,4]$	$[2,3],[1,4]$
IV	$[1,4],[2,3]$	$[1,3],[2,4]$

Table 1. Summary of the asymptotics of (elastic and inelastic) 2-soliton solutions.

C. The two-dimensional Toda lattice in bilinear form

The aim of this appendix is to present a solution formula for the bilinear form of the 2d-Toda lattice

$$\tau_n \frac{\partial^2 \tau_n}{\partial y \partial x} - \frac{\partial \tau_n}{\partial y}\,\frac{\partial \tau_n}{\partial x} = \tau_{n+1}\tau_{n-1} - \tau_n^2. \tag{C.1}$$

(see [13] and references therein) in terms of a generalized determinant. Note that the dependent variable transformation

$$v_n = \frac{\tau_{n+1}}{\tau_n} - 1$$

maps solutions $\tau_n(x,y)$ of (C.1) to solutions of the 2d Toda lattice (1.3).

Our main result is

Proposition C.1. *Let E, F be a Banach spaces, $A \in \mathscr{L}(E)$, $B \in \mathscr{L}(F)$ invertible, and $\mathscr{A}$ a quasi-Banach operator ideal admitting a continuous determinant δ. Let $L_n = L_n(x,y) \in \mathscr{A}(F,E)$, $M_n = M_n(x,y) \in \mathscr{A}(E,F)$ be operator-functions with values in $\mathscr{A}$, which satisfy the base equations* (2.1) *and the one-dimensionality condition* (3.1) *with some $b \in F'$. Then*

$$\tau_n = \delta(I_F + M_n L_n)$$

is a solution of (C.1) *on* $\Omega = \{(n,x,y) \in \mathbb{Z} \times \mathbb{R}^2 \mid \tau_n(x,y) \neq 0,\ \tau_{n-1}(x,y) \neq 0\}$.

Proof. Let τ be the trace associated to the determinant δ according to the trace-determinant theorem [24]. Recall that for a smooth operator-function $\xi \to T(\xi)$, with values in the endomorphisms on some Banach space, the following derivation rule holds [24]

$$\Big(\frac{d}{d\xi}\delta(I+T)\Big)\Big/\delta(I+T) = \tau\Big((I+T)^{-1}\frac{d}{d\xi}T\Big). \tag{C.2}$$

We will verify that τ_n solves

$$\frac{\partial}{\partial y}\Big(\Big(\frac{\partial \tau_n}{\partial x}\Big)\Big/\tau_n\Big) = \Big(\frac{\tau_{n+1}}{\tau_n}\Big)\Big/\Big(\frac{\tau_n}{\tau_{n-1}}\Big) - 1, \tag{C.3}$$

which implies (C.1). Let us start with the left-hand side of (C.3). Application of (C.2), the base equations (2.1), and linearity of the trace yield

$$\begin{aligned}
\Big(\frac{\partial \tau_n}{\partial x}\Big)/\tau_n &= \tau\Big((I_F+M_nL_n)^{-1}\frac{\partial}{\partial x}(M_nL_n)\Big)\\
&= \tau\Big((I_F+M_nL_n)^{-1}\big(-B^{-1}M_n+M_nA\big)L_n\Big)\\
&= \tau\Big((I_F+M_nL_n)^{-1}M_nAL_n\Big)-\tau\Big((I_F+M_nL_n)^{-1}B^{-1}M_nL_n\Big).
\end{aligned}$$

Using the trace property $\tau(ST)=\tau(TS)$ we get $\tau\big((I_F+M_nL_n)^{-1}B^{-1}M_nL_n\big)=\tau\big(M_nL_n(I_F+M_nL_n)^{-1}B^{-1}\big)=\tau\big((I_F+M_nL_n)^{-1}M_nL_nB^{-1}\big)$, which gives

$$\begin{aligned}
\Big(\frac{\partial \tau_n}{\partial x}\Big)/\tau_n &= \tau\Big((I_F+M_nL_n)^{-1}M_n(AL_n-L_nB^{-1})\Big)\\
&= \tau(B^{-1}V_n),
\end{aligned}$$

where V_n is the operator-soliton defined in (2.2). By continuity of τ, we therefore find

$$\frac{\partial}{\partial y}\bigg(\Big(\frac{\partial \tau_n}{\partial x}\Big)/\tau_n\bigg)=\tau\Big(B^{-1}\frac{\partial}{\partial y}V_n\Big), \tag{C.4}$$

Next we turn to the right-hand side of (C.3). Using again the base equations (2.1), we find

$$\begin{aligned}
\frac{\tau_{n+1}}{\tau_n} &= \frac{\delta(I_F+M_{n+1}L_{n+1})}{\delta(I_F+M_nL_n)} = \frac{\delta(I_F+BM_nAL_n)}{\delta(I_F+M_nL_n)} = \frac{\delta(I_F+M_nAL_nB)}{\delta(I_F+M_nL_n)}\\
&= \delta\Big((I_F+M_nL_n)^{-1}(I_F+M_nAL_nB)\Big)\\
&= \delta\Big((I_F+M_nL_n)^{-1}\big((I_F+M_nL_n)+(M_nAL_nB-M_nL_n)\big)\Big)\\
&= \delta\Big(I_F+(I_F+M_nL_n)^{-1}M_n(AL_nB-L_n)\Big)\\
&= \delta\Big(I_F+(I_F+M_nL_n)^{-1}M_n(AL_n-L_nB^{-1})B\Big)\\
&= \delta\Big(I_F+B(I_F+M_nL_n)^{-1}M_n(AL_n-L_nB^{-1})\Big)\\
&= \delta\big(I_F+V_n\big).
\end{aligned}$$

Hence, the right-hand side of (C.3) becomes

$$\begin{aligned}
\Big(\frac{\tau_{n+1}}{\tau_n}\Big)\Big/\Big(\frac{\tau_n}{\tau_{n-1}}\Big)-1 &= \frac{\delta(I_F+V_n)}{\delta(I_F+V_{n-1})}-1 \;=\; \delta\Big((I_F+V_{n-1})^{-1}(I_F+V_n)\Big)-1\\
&= \delta\Big((I_F+V_{n-1})^{-1}\big((I_F+V_{n-1})+(V_n-V_{n-1})\big)\Big)-1\\
&= \delta\Big(I_F+(I+V_{n-1})^{-1}(V_n-V_{n-1})\Big)-1.
\end{aligned}$$

By (3.1), we know that $V_n=B(I_F+M_nL_n)^{-1}(b\otimes d_n)$ with some vector function d_n, i.e. $V_n=b\otimes v_n$ for $v_n=B(I_F+M_nL_n)^{-1}d_n$. In particular, also $(I_F+V_n)^{-1}(V_n-V_{n-1})=b\otimes\big((I_F+V_n)^{-1}(v_n-v_{n-1})\big)$ is one-dimensional. Using that $\delta(I+T)=1+\tau(T)$ holds for one-dimensional operators T, we conclude

$$\left(\frac{\tau_{n+1}}{\tau_n}\right)\Big/\left(\frac{\tau_n}{\tau_{n-1}}\right)-1=\tau\Big((I_F+V_{n-1})^{-1}(V_n-V_{n-1})\Big). \tag{C.5}$$

Comparing (C.5) with (C.4), we see that it is sufficient to show

$$B^{-1}\frac{\partial}{\partial y}V_n=(I_F+V_{n-1})^{-1}(V_n-V_{n-1}).$$

To this end we fall back on some identities from the proof of Theorem 2.1, together with the following identity derived[h] from Appendix D,

$$\frac{\partial}{\partial y}T_2^+(n)=S_2^-(n)S_1^+(n). \tag{C.6}$$

Namely, we get

$$\begin{aligned}B^{-1}\frac{\partial}{\partial y}V_n &= B^{-1}\frac{\partial}{\partial y}(I_F+V_n)\\ &\overset{(2.4c)}{=} \frac{\partial}{\partial y}T_2^+(n)\\ &\overset{(C.6)}{=} S_2^-(n)S_1^+(n)\\ &\overset{(2.7)}{=} (I_F+V_{n-1})^{-1}(V_n-V_{n-1}).\end{aligned}$$

This completes the proof. □

Remark C.2. If Ω is dense in $\mathbb{Z}\times\mathbb{R}^2$, Proposition C.1 yields that τ_n satisfies (C.1) everywhere by continuity. This will be our way to verify (C.1) in applications.

D. A toolbox of operator identities

The operator identities collected in the subsequent lemmata origin from and extend results in [30, Chapter 3]. It is worth mentioning that they also are crucial in the operator treatment of continuous integrable systems like the KP equation.

We start with some purely algebraic identities.

Lemma D.1. *Let G_1, G_2 be Banach spaces, and $T_1\in\mathscr{L}(G_2,G_1)$, $T_2\in\mathscr{L}(G_1,G_2)$ bounded linear operators such that $I_{G_1}+T_1T_2$ (and hence also $I_{G_2}+T_2T_1$) is invertible. Then the following identities hold*

(a) $T_2(I_{G_1}+T_1T_2)^{-1}=(I_{G_2}+T_2T_1)^{-1}T_2$,

(b) $T_2(I_{G_1}+T_1T_2)^{-1}T_1=I_{G_2}-(I_{G_2}+T_2T_1)^{-1}$.

---

[h]Use $L_1(n,x,y)=M_n(x,y)$, $L_2(n,x,y)=L_n(x,y)$ and $A_1=B$, $A_2=A$, and

$$f_1^{[a]}(z)=f_2^{[b]}(z)=z,\;\; f_2^{[a]}(z)=f_1^{[b]}(z)=-z^{-1}.$$

Then $T_2^+(n)=-T_1^{[b]}$, and Lemma D.5 b) implies $\frac{\partial}{\partial y}T_1^{[b]}=-S_1^{[a]}S_2^{[b]}$, with the identifications $S_1^{[a]}=S_2^-(n)$ and $S_2^{[b]}=S_1^+(n)$.

Proof. Assume that $I_{G_1}+T_1T_2$ is invertible. Then

$$\begin{aligned}
&\Big(I_{G_2}-T_2(I_{G_1}+T_1T_2)^{-1}T_1\Big)(I_{G_2}+T_2T_1)\\
&=I_{G_2}(I_{G_2}+T_2T_1)-T_2(I_{G_1}+T_1T_2)^{-1}T_1(I_{G_2}+T_2T_1)\\
&=I_{G_2}+T_2T_1-T_2(I_{G_1}+T_1T_2)^{-1}(I_{G_1}+T_1T_2)T_1\\
&=I_{G_2}+T_2T_1-T_2T_1\\
&=I_{G_2}\,.
\end{aligned}$$

The same holds if one computes the product in reversed order. Hence also $I_{G_2}+T_2T_1$ is invertible and

$$(I_{G_2}+T_2T_1)^{-1}=I_{G_2}-T_2(I_{G_1}+T_1T_2)^{-1}T_1\,,$$

which gives (b).

Note that the key in the above argument was the identity

$$(I_{G_1}+T_1T_2)T_1=T_1(I_{G_2}+T_2T_1).$$

To verify (a), we start from the analogous identity $(I_{G_2}+T_2T_1)T_2=T_2(I_{G_1}+T_1T_2)$ and multiply by $(I_{G_1}+T_1T_2)^{-1}$ from the right, and by $(I_{G_2}+T_2T_1)^{-1}$ from the left. □

The next lemma treats products of operators of a particular form which play a key role in establishing general solution formulas for non-commutative integrable systems such as (1.4) or the nc KP equation.

Lemma D.2. *Let G_1, G_2 be Banach spaces, and $L_1\in\mathscr{L}(G_2,G_1)$, $L_2\in\mathscr{L}(G_1,G_2)$ such that $I_{G_1}+L_1L_2$ (and hence also $I_{G_2}+L_2L_1$) is invertible.*

Let $A_1\in\mathscr{L}(G_1)$, $A_2\in\mathscr{L}(G_2)$ be invertible operators and $f_1^{[\alpha]}(z),f_2^{[\alpha]}(z)$, $\alpha\in\{a,b\}$, be complex functions such that the spectrum of A_1 (resp. A_2) is contained in the domain where $f_1^{[\alpha]}$ (resp. $f_2^{[\alpha]}$) are holomorphic, and set

$$f_1^{[c]}:=\ f_1^{[a]}f_1^{[b]},\qquad f_2^{[c]}:=-f_2^{[a]}f_2^{[b]}.$$

Define

$$\begin{aligned}
S_1^{[\alpha]}&=(I_{G_1}+L_1L_2)^{-1}\Big(f_1^{[\alpha]}(A_1)L_1+L_1f_2^{[\alpha]}(A_2)\Big),\\
T_1^{[\alpha]}&=(I_{G_1}+L_1L_2)^{-1}\Big(f_1^{[\alpha]}(A_1)-L_1f_2^{[\alpha]}(A_2)L_2\Big),
\end{aligned}$$

for $\alpha\in\{a,b,c\}$, and analogously $S_2^{[\alpha]}$, $T_2^{[\alpha]}$ built in the same way but with the role of the lower indices interchanged. Then the following identities between products of these operator-functions hold

(a) $\quad T_1^{[a]}S_1^{[b]}-S_1^{[a]}T_2^{[b]}\ =\ S_1^{[c]}$,

(b) $\quad T_1^{[a]}T_1^{[b]}+S_1^{[a]}S_2^{[b]}\ =\ T_1^{[c]}$.

Proof. To improve readability, we write $f_j^{[\alpha]}$ instead of $f_j^{[\alpha]}(A_j)$. On application of Lemma D.1 on all boxed terms, we obtain

$$\begin{aligned}
&\big(f_1^{[a]} - L_1 f_2^{[a]} L_2 \big) S_1^{[b]} \\
&= \big(f_1^{[a]} - L_1 f_2^{[a]} L_2 \big) (I_{G_1} + L_1 L_2)^{-1} \big(f_1^{[b]} L_1 + L_1 f_2^{[b]} \big) \\
&= f_1^{[a]} \boxed{(I_{G_1} + L_1 L_2)^{-1}} f_1^{[b]} L_1 + f_1^{[a]} \boxed{(I_{G_1} + L_1 L_2)^{-1} L_1} f_2^{[b]} \\
&\qquad - L_1 f_2^{[a]} \boxed{L_2 (I_{G_1} + L_1 L_2)^{-1}} f_1^{[b]} L_1 - L_1 f_2^{[a]} \boxed{L_2 (I_{G_1} + L_1 L_2)^{-1} L_1} f_2^{[b]} \\
&= f_1^{[a]} \Big(I_{G_1} - L_1 (I_{G_2} + L_2 L_1)^{-1} L_2 \Big) f_1^{[b]} L_1 + f_1^{[a]} L_1 (I_{G_2} + L_2 L_1)^{-1} f_2^{[b]} \\
&\qquad - L_1 f_2^{[a]} (I_{G_2} + L_2 L_1)^{-1} L_2 f_1^{[b]} L_1 - L_1 f_2^{[a]} \big(I_{G_2} - (I_{G_2} + L_2 L_1)^{-1} \big) f_2^{[b]} \\
&= (f_1^{[a]} f_1^{[b]} L_1 - L_1 f_2^{[a]} f_2^{[b]}) + \big(f_1^{[a]} L_1 + L_1 f_2^{[a]} \big) (I_{G_2} + L_2 L_1)^{-1} \big(f_2^{[b]} - L_2 f_1^{[b]} L_1 \big) \\
&= (f_1^{[c]} L_1 + L_1 f_2^{[c]}) + \big(f_1^{[a]} L_1 + L_1 f_2^{[a]} \big) T_2^{[b]},
\end{aligned}$$

yielding the identity in (a) after multiplying with $(I_{G_1} + L_1 L_2)^{-1}$ from the left, and, similarly,

$$\begin{aligned}
&\big(f_1^{[a]} - L_1 f_2^{[a]} L_2 \big) T_1^{[b]} \\
&= \big(f_1^{[a]} - L_1 f_2^{[a]} L_2 \big) (I_{G_1} + L_1 L_2)^{-1} \big(f_1^{[b]} - L_1 f_2^{[b]} L_2 \big) \\
&= f_1^{[a]} \boxed{(I_{G_1} + L_1 L_2)^{-1}} f_1^{[b]} - f_1^{[a]} \boxed{(I_{G_1} + L_1 L_2)^{-1} L_1} f_2^{[b]} L_2 \\
&\qquad - L_1 f_2^{[a]} \boxed{L_2 (I_{G_1} + L_1 L_2)^{-1}} f_1^{[b]} + L_1 f_2^{[a]} \boxed{L_2 (I_{G_1} + L_1 L_2)^{-1} L_1} f_2^{[b]} L_2 \\
&= f_1^{[a]} \Big(I_{G_1} - L_1 (I_{G_2} + L_2 L_1)^{-1} L_2 \Big) f_1^{[b]} - f_1^{[a]} L_1 (I_{G_2} + L_2 L_1)^{-1} f_2^{[b]} L_2 \\
&\qquad - L_1 f_2^{[a]} (I_{G_2} + L_2 L_1)^{-1} L_2 f_1^{[b]} + L_1 f_2^{[a]} \big(I_{G_2} - (I_{G_2} + L_2 L_1)^{-1} \big) f_2^{[b]} L_2 \\
&= (f_1^{[a]} f_1^{[b]} + L_1 f_2^{[a]} f_2^{[b]} L_2) - \big(f_1^{[a]} L_1 + L_1 f_2^{[a]} \big) (I_{G_2} + L_2 L_1)^{-1} \big(f_2^{[b]} L_2 + L_2 f_1^{[b]} \big) \\
&= (f_1^{[c]} - L_1 f_2^{[c]} L_2) - \big(f_1^{[a]} L_1 + L_1 f_2^{[a]} \big) S_2^{[b]}
\end{aligned}$$

applying (b), again after multiplying with $(I_{G_1} + L_1 L_2)^{-1}$ from the left. $\square$

Corollary D.3. *If $f_j^{[b]} = \sigma / f_j^{[a]}$ with $\sigma \in \{\pm 1\}$, we have $S_1^{[c]} = 0$, and Lemma D.2 in particular yields*

$$T_1^{[a]} S_1^{[b]} = S_1^{[a]} T_2^{[b]}.$$

Next we turn to identities involving operator functions depending on a (scalar) variable ξ. Let us first recall the derivation rule for inverses.

Lemma D.4. *Let G be a Banach space and T be an $\mathscr{L}(G)$-valued function depending $\mathscr{C}^1$-smoothly on some variable ξ. If $T(\xi)$ is invertible for all ξ, then $T^{-1}(\xi)$ is $\mathscr{C}^1$-smooth and*

$$\frac{d}{d\xi} T^{-1}(\xi) = -T^{-1}(\xi) T'(\xi) T^{-1}(\xi).$$

Now we look at derivation rules for operator functions with a similar structure as in Lemma D.2.

Lemma D.5. *Let G_1, G_2 be Banach spaces, and L_1, L_2 are operator-valued functions depending $\mathscr{C}^1$-smoothly on some variable ξ with $L_1(\xi) \in \mathscr{L}(G_2, G_1)$, $L_2(\xi) \in \mathscr{L}(G_1, G_2)$ such that $I_{G_1} + L_1L_2$ (and hence also $I_{G_2} + L_2L_1$) is invertible.*

Let $f_j^{[a]}(z)$, $f_j^{[b]}(z)$, $j = 1, 2$, be complex functions, and let $A_j \in \mathscr{L}(G_j)$ be invertible operators such that the spectrum of A_j is contained in the domain where $f_j^{[a]}$, $f_j^{[b]}$ are holomorphic.

Assume that the following base equations are satisfied

$$\frac{d}{d\xi} L_j = f_j^{[a]}(A_j) L_j,$$

and define

$$\begin{aligned} S_1^{[\alpha]} &= (I_{G_1} + L_1L_2)^{-1} \Big(f_1^{[\alpha]}(A_1) L_1 + L_1 f_2^{[\alpha]}(A_2) \Big), \\ T_1^{[\alpha]} &= (I_{G_1} + L_1L_2)^{-1} \Big(f_1^{[\alpha]}(A_1) - L_1 f_2^{[\alpha]}(A_2) L_2 \Big), \end{aligned}$$

for $\alpha \in \{a, b\}$, and analogously $S_2^{[\alpha]}$, $T_2^{[\alpha]}$ build in the same way but with the role of the lower indices interchanged.

Then the following derivation rules hold:

(a) $\quad \dfrac{d}{d\xi} S_1^{[b]} = T_1^{[a]} S_1^{[b]},$

(b) $\quad \dfrac{d}{d\xi} T_1^{[b]} = -S_1^{[a]} S_2^{[b]}.$

Remark D.6. Note that if $f_j^{[a]} = f_j^{[b]}$, we have $S_j^{[a]} = S_j^{[b]}$, and$T_j^{[a]} = T_j^{[b]}$ for the operator functions in the above lemma. Hence we also obtain derivation rules for $S_j^{[a]}$, $T_j^{[a]}$ as a particular case.

Proof. To improve readability, we write $f_j^{[\alpha]}$ instead of $f_j^{[\alpha]}(A_j)$.

Observe first that, by the non-commutative product rule, and on use of the base equations, we have

$$\frac{d}{d\xi}(L_1L_2) = \left(\frac{d}{d\xi} L_1 \right) L_2 + L_1 \left(\frac{d}{d\xi} L_2 \right) = f_1^{[a]} L_1 L_2 + L_1 f_2^{[a]} L_2 = \big(f_1^{[a]} L_1 + L_1 f_2^{[a]} \big) L_2.$$

Hence, by Lemma D.4,

$$\begin{aligned} \frac{d}{d\xi} S_1^{[b]} &= -(I_{G_1} + L_1L_2)^{-1} \Big(\frac{d}{d\xi}(L_1L_2) \Big) (I_{G_1} + L_1L_2)^{-1} \big(f_1^{[b]} L_1 + L_1 f_2^{[b]} \big) \\ &\qquad + (I_{G_1} + L_1L_2)^{-1} \frac{d}{d\xi} \big(f_1^{[b]} L_1 + L_1 f_2^{[b]} \big) \\ &= -(I_{G_1} + L_1L_2)^{-1} (f_1^{[a]} L_1 + L_1 f_2^{[a]}) L_2 (I_{G_1} + L_1L_2)^{-1} (f_1^{[b]} L_1 + L_1 f_2^{[b]}) \\ &\qquad + (I_{G_1} + L_1L_2)^{-1} \big(f_1^{[b]} f_1^{[a]} L_1 + f_1^{[a]} L_1 f_2^{[b]} \big) \\ &= (I_{G_1} + L_1L_2)^{-1} \big(-(f_1^{[a]} L_1 + L_1 f_2^{[a]}) L_2 + f_1^{[a]} (I_{G_1} + L_1L_2) \big) S_1^{[b]} \\ &= (I_{G_1} + L_1L_2)^{-1} \big(f_1^{[a]} - L_1 f_2^{[a]} L_2 \big) S_1^{[b]} \end{aligned}$$

$$= T_1^{[a]} S_1^{[b]}.$$

Second, using Lemma D.1(a) for the boxed identity, we compute

$$\begin{aligned}
\frac{d}{d\xi} T_1^{[b]} &= -(I_{G_1} + L_1 L_2)^{-1} \Big(\frac{d}{d\xi}(L_1 L_2) \Big) (I_{G_1} + L_1 L_2)^{-1} \big(f_1^{[b]} - L_1 f_2^{[b]} L_2 \big) \\
&\quad + (I_{G_1} + L_1 L_2)^{-1} \frac{d}{d\xi} \big(f_1^{[b]} - L_1 f_2^{[b]} L_2 \big) \\
&= -(I_{G_1} + L_1 L_2)^{-1} \big(f_1^{[a]} L_1 + L_1 f_2^{[a]} \big) L_2 (I_{G_1} + L_1 L_2)^{-1} \big(f_1^{[b]} - L_1 f_2^{[b]} L_2 \big) \\
&\quad + (I_{G_1} + L_1 L_2)^{-1} \Big(- f_1^{[a]} L_1 f_2^{[b]} L_2 - L_1 f_2^{[b]} f_2^{[a]} L_2 \Big) \\
&= -S_1^{[a]} \Big(\boxed{L_2 (I_{G_1} + L_1 L_2)^{-1}} \big(f_1^{[b]} - L_1 f_2^{[b]} L_2 \big) + f_2^{[b]} L_2 \Big) \\
&= -S_1^{[a]} \Big(\boxed{(I_{G_2} + L_2 L_1)^{-1} L_2} \big(f_1^{[b]} - L_1 f_2^{[b]} L_2 \big) + f_2^{[b]} L_2 \Big) \\
&= -S_1^{[a]} \, (I_{G_2} + L_2 L_1)^{-1} \Big(L_2 (f_1^{[b]} - L_1 f_2^{[b]} L_2) + (I_{G_2} + L_2 L_1) f_2^{[b]} L_2 \Big) \\
&= -S_1^{[a]} \, (I_{G_2} + L_2 L_1)^{-1} \big(f_2^{[b]} L_2 + L_2 f_1^{[b]} \big) \\
&= -S_1^{[a]} S_2^{[b]},
\end{aligned}$$

which concludes the proof. □

Acknowledgement

The authors would like to thank the referees for careful reading and valuable remarks.

References

[1] H. Aden. *Elementary Operators and Solutions of the Korteweg-de Vries Equation.* Thesis, Jena 1996.

[2] H. Aden and B. Carl. On realizations of solutions of the KdV equation by determinants on operator ideals. *J. Math. Phys.*, **37** (1996), 1833–1857.

[3] G. Biondini and S. Chakravarty. Soliton solutions of the Kadomtsev-Petviashvili II equation. *J. Math. Phys.*, **47** (2006), 033541, 26 pp.

[4] G. Biondini and D. Wang. On the soliton solutions of the two-dimensional Toda lattice. *J. Phys. A*, **43** (2010), 434007, 20 pp.

[5] H. Blohm. Solution of nonlinear equations by trace methods. *Nonlinearity*, **13** (2000), 1925–1964.

[6] A. Cauchy. *Exercises d'Analyse et de Physique Mathématique*. Tome 2. Bachelier, Paris 1841.

[7] B. Carl and S.-Z. Huang. On realizations of solutions of the KdV equation by the C_0-semigroup method. *Amer. J. Math.*, **122** (2000), 403–438.

[8] B. Carl and C. Schiebold. Nonlinear equations in soliton physics and operator ideals. *Nonlinearity*, **12** (1999), 333–364.

[9] B. Carl and C. Schiebold. Ein direkter Ansatz zur Untersuchung von Solitonengleichungen. *Jahresber. Deutsch. Math.-Verein.*, **102** (2000), 102–148.

[10] G. Darboux. *Leçon sur la Théorie des Surfaces 2*. Gauthier-Villars, Paris 1915.

[11] A. Eschmeier. Tensor products and elementary operators. *J. reine und angew. Mathematik*, **390** (1988), 47–66.

[12] A. Grothendieck. *Produits Tensoriels Topologiques et Espaces Nucléaires*. Memoirs of the American Mathematical Society Vol. 16, Amer. Math. Soc., Providence, Rhode Island, 1955.

[13] R. Hirota. *The Direct Method in Soliton Theory*. Cambridge University Press, Cambridge 2004.

[14] R. Hirota, M. Ito and F. Kako. Two-dimensional Toda lattice equations. *Progr. Theoret. Phys. Suppl.*, **94** (1988), 42–58.
[15] R. Hirota, Y. Ohta and J. Satsuma. Solutions of the Kadomtsev-Petviashvili equation and ths two-dimensional Toda equations. *J. Phys. Soc. Jpn.*, **57** (1988) 1901–1904.
[16] R. Hirota, Y. Ohta and J. Satsuma. Wronskian structures of solitons for soliton equations. *Prog. Theor. Phys. Suppl.*, **94** (1988), 59–72.
[17] Y. Kodama. *KP Solitons and the Grassmannians. Combinatorics and Geometry of Two-Dimensional Wave Patterns*. Springer Briefs in Mathematical Physics Vol. 22, Springer, Singapore, 2017.
[18] S. Li, G. Biondini and C. Schiebold. On the degenerate soliton solutions of the focusing nonlinear Schrödinger equation. *J. Math. Phys.*, **58** (2017), 033507, 27 pp.
[19] K. Maruno and G. Biondini. Resonance and web structure in discrete soliton systems: the two-dimensional Toda lattice and its fully discrete and ultra-discrete analogues. *J. Phys. A*, **36** (2004), 11819–11839.
[20] A.V. Mikhailov. Integrability of a two-dimensional generalization of the Toda chain. *JETP Lett.*, **30** (1979), 443–448.
[21] J. Miles Resonantly interacting solitary waves. *J. Fluid Mech.*, **79** (1976), 171–179.
[22] F.W. Nijhoff, J. Atkinson and J. Hietarinta. Soliton solutions for ABS lattice equations: I Cauchy matrix approach. *J. Phys. A*, **42** (2009), 404005, 34 pp.
[23] S. Novikov, S.V. Manakov, L.P. Pitaevskii and V.E. Zakharov. *Theory of Solitons. The Inverse Scattering Transform.* Plenum, New York 1984.
[24] A. Pietsch. *Eigenvalues and s-Numbers*. Cambridge Studies in Advanced Mathematics Vol. 13, Cambridge University Press, Cambridge, 1987.
[25] G. Pólya and G. Szegö. *Aufgaben und Lehrsätze zur Analysis II*. Springer, Berlin 1976.
[26] E. Saksman and H.-O. Tylli. Multiplications and elementary operators in the Banach space setting. In: *Methods in Banach Space Theory: Proceedings of the V Conference on Banach Spaces*, Cáceres, Spain, 13-18 September 2004, edited by J.M.F. Castillo and W.B. Johnson (London Mathematical Society Lecture Notes Series, **337**, Cambridge University Press 2006), pp. 253–292.
[27] C. Schiebold. An operator theoretic approach to the Toda lattice equation. *Physica D*, **122** (1998), 37–61.
[28] C. Schiebold. Solitons of the sine-Gordon equation coming in clusters. *Revista Mat. Complut.*, **15** (2002), 265–325.
[29] C. Schiebold. On Negatons of the Toda Lattice. *J. Nonlin. Math. Phys.*, **10** (2003), 181–193.
[30] C. Schiebold. *Integrable Systems and Operator Equations*. Habilitation Thesis, Jena 2004.
[31] C. Schiebold. From the non-abelian to the scalar two-dimensional Toda lattice. *Glasgow Math. J.*, **47** (2005), 177–189.
[32] C. Schiebold. Explicit solution formulas for the matrix-KP. *Glasgow Math. J.*, **51** (2009), 147–155.
[33] C. Schiebold. Noncommutative AKNS systems and multisoliton solutions to the matrix sine-Gordon equation. *Discrete and Continuous Dynamical Systems, Supplement* 2009 (2009), 678–690.
[34] C. Schiebold. Cauchy-type determinants and integrable systems. *Linear Algebra Appl.*, **433** (2010), 447–475.
[35] C. Schiebold. Asymptotics for the multiple poles solutions of the Nonlinear Schrödinger equation. *Nonlinearity*, **30** (2017), 2930–2981.
[36] C. Schiebold. Matrix solutions for equations of the AKNS system. In: *Nonlinear Systems and Their Remarkable Mathematical Structures*, edited by N. Euler (CRC Press, Boca Raton, FL, USA 2018), Chapter B. 5, pp. 256–293.
[37] K. Takasaki. Toda hierarchies and their applictions. *J. Phys. A*, **51** (2018), 203001, 35 pp.
[38] M.C. White. Analytic multivalued functions and spectral trace. *Math. Ann.*, **304** (1996), 669–683.
[39] D.D. Xu, D.J. Zhang and S.L. Zhao. The Sylvester equation and integrable equations: I. The Korteweg-de Vries system and sine-Gordon equation. *J. Nonlin. Math. Phys.*, **21** (2014), 382–406.
[40] D.J. Zhang and S.L. Zhao. Solutions to ABS lattice equations via generalized Cauchy matrix approach. *Stud. Appl. Math.*, **131** (2013), 72–103.