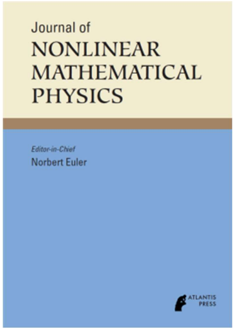

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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Journal of Nonlinear Mathematical Physics, Vol. 27, No. 1 (2020) 95–105

Interrelations of discrete Painlevé equations through limiting procedures

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Received 4 June 2019

Accepted 8 July 2019

We study the discrete Painlevé equations associated to the $E_7^{(1)}$ affine Weyl group which can be obtained by the implementation of a special limits of $E_8^{(1)}$-associated equations. This study is motivated by the existence of two $E_7^{(1)}$-associated discrete both having a double ternary dependence in their coefficients and which have not been related before. We show here that two equations correspond to two different limits of a $E_8^{(1)}$-associated discrete Painlevé equation. Applying the same limiting procedures to other $E_8^{(1)}$-associated equations we obtained several $E_7^{(1)}$-related equations most of which have not been previously derived.

Keywords: discrete Painlevé equations, affine Weyl groups, limiting procedures, canonical forms

PACS numbers: 02.30.Ik, 05.45.Yv

1. Introduction

Integrable systems are reputed for the numerous interrelations which establish links among them. Miura and/or Bäcklund transformations are omnipresent in the integrability domain. Once a relation of this type is established between two equations, they cease to be two different ones but are, in fact, the same system under two different guises. Thus establishing relations between known integrable systems is of paramount importance. Painlevé equations, be they continuous or discrete, possess the same rich structure of relations.

This paper focuses on a particular class of discrete Painlevé equations [1], multiplicative ones which are associated to the affine Weyl group [2] $E_7^{(1)}$. The main bulk of discrete Painlevé equations with $E_7^{(1)}$ symmetry are obtained by deautonomisation of a QRT mapping [3] belonging to one of three families characterised by distinct canonical forms [4] of the A_1 QRT matrix. The first two and best known families, referred to as VI and VI′ respectively, correspond to the matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1-q^2 & 0 \\ 0 & 0 & q^2 \end{pmatrix}, \tag{1.1}$$

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and

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -z^2 - 1/z^2 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.2}$$

(The third family, recently discovered [5], will not be the object of the present study.) The canonical form of the mappings of the VI and VI′ families are

$$\frac{(x_{n+1}x_n - q^2)(x_n x_{n-1} - q^2)}{(x_{n+1}x_n - 1)(x_n x_{n-1} - 1)} = R(x_n) \tag{1.3}$$

and

$$\left(\frac{x_n - z^2 x_{n+1}}{z^2 x_n - x_{n+1}}\right)\left(\frac{x_n - z^2 x_{n-1}}{z^2 x_n - x_{n-1}}\right) = R'(x_n) \tag{1.4}$$

respectively. The transformation from a form VI to a from VI′ is straightforward. Separating even and odd indices and thus splitting (1.3) into two equations, we replace the variable x_m for m even by q^2/x_m. We obtain thus an equation of the form (1.4), but written as a QRT-asymmetric one, where moreover we have $q = z^2$. The same procedure, starting from (1.4) and replacing x_m for m even by z^2/x_m leads to a QRT-asymmetric equation of the form (1.3). Given this observation one would expect all the discrete Painlevé equations in form VI, studied in [6], and in form VI′, studied in [7], to be intimately related. An obvious criterion as to which equations may be related is the periodic dependence of their various parameters: if two equations in VI and VI′ forms have the same periodic dependence one would expect them to be somehow related. If we take, for instance, equations (2.16) of [6] and (27) of [7], which both have a quaternary periodic dependence in their parameters, we find that the transformation we outlined above allows to transform one into the other. However a difficulty does exist. It stems from two equations, one in form VI and the other in form VI′ which have the same double ternary dependence and which cannot be related through the transformation introduced above. This paper originated from our aspiration to resolve this minor enigma.

2. A short digression

Since in this paper we will be heavily using the QRT formalism [3] for the writing of mappings in symmetric and asymmetric form it is useful to summarise here the conventions that govern it and their consequences for discrete Painlevé equations.

The starting point is a QRT-symmetric mapping relating the variable w over three adjacent points, i.e. $w_{n+1} = F(w_n, w_{n-1})$. Typically a discrete Painlevé equation involves a variable introducing the secular dependence for which we can simply assume the form $s_n = \alpha n + \beta$. Moreover a discrete Painlevé equation equation can have parameters with a periodic dependence of the independent variable. To this end we have introduced two useful periodic functions ϕ_k and χ_{2k} obeying the periodicity relations $\phi_k(n) = \phi_k(n+k)$ (but where the constant solution of this relation is excluded) and $\chi_{2k}(n) = -\chi_{2k}(n+k)$. Practically this gives the following forms for ϕ_k and χ_{2k}:

$$\phi_k(n) = \sum_{\ell=1}^{k-1} \delta_\ell^{(k)} \exp\left(\frac{2i\pi\ell n}{k}\right), \quad \text{and} \quad \chi_{2k}(n) = \sum_{\ell=1}^{k} \eta_\ell^{(k)} \exp\left(\frac{i\pi(2\ell-1)n}{k}\right).$$

When dealing with multiplicative equations the expressions above must be understood as defining the logarithms of the periodic functions and thus the ϕ and χ we shall encounter in section 2, 3 and 4 are the exponentials of the right-hand sides of the above expressions.

Starting from a QRT-symmetric mapping one constructs an asymmetric one by separating the evolution for even and for odd values of the indices. One has thus two mappings one giving w_{2n+1} as a function of w_{2n} and w_{2n-1} and one giving w_{2n+2} as a function of w_{2n+1} and w_{2n}. The QRT convention is to introduce two distinct dependent variables corresponding to indices of different parity. One has thus $x_m = w_{2n}$ and $y_m = w_{2n+1}$. Similarly for the secular dependence one introduces $z_m = s_{2n}$ and $\zeta_m = s_{2n+1}$. Given this notation one remarks that $\zeta_m = z_{m+1/2}$.

Let us illustrate this by producing the asymmetric form of the symmetric mapping

$$\frac{(w_{n+1}+w_n-s_n-s_{n+1})(w_n+w_{n-1}-s_n-s_{n-1})}{(w_{n+1}+w_n)(w_n+w_{n-1})} = R(w_n). \tag{2.1}$$

Using the conventions explained in the previous paragraph we find for $n = 2m$, $n = 2m+1$ respectively

$$\frac{(y_m+x_m-z_m-\zeta_m)(x_m+y_{m-1}-z_m-\zeta_{m-1})}{(y_m+x_m)(x_m+y_{m-1})} = R(x_m), \tag{2.2a}$$

$$\frac{(x_{m+1}+y_m-\zeta_m-z_{m+1})(y_m+x_m-z_m-\zeta_m)}{(x_{m+1}+y_m)(y_m+x_m)} = R(y_m). \tag{2.2b}$$

Up to this point the transcription is straightforward. However in the case of discrete Painlevé equations one must also be able to take care of the periodic functions. The periodic function $\phi_2(n)$ is simply equal to $\gamma(-1)^n$. Thus whenever the equation involves this function the latter will enter as γ in the even-index equation (2.2a) in the example above) and as $-\gamma$ in the odd-index one (2.2b) in the example). The periodic function $\chi_2(n)$ is identical to $\phi_2(n)$. For the function ϕ_3 we consider the four points $\phi_3(2n-1)$, $\phi_3(2n)$, $\phi_3(2n+1)$, and $\phi_3(2n+2)$. Using the property $\phi_3(n) = \phi_3(n+3)$ we can rewrite these four quantities as $\phi_3(2n-4)$, $\phi_3(2n)$, $\phi_3(2n-2)$, and $\phi_3(2n+2)$ and the transcription becomes now easy: $\varphi_3(m-2)$, $\varphi_3(m)$, $\varphi_3(m-1)$, and $\varphi_3(m+1)$. In fact, the periodicity property can be used for all ϕs of odd periods. While $\phi_k(2n)$ and $\phi_k(2n+2)$ are simply $\phi_k(m)$ and $\phi_k(m+1)$, when $k = 2p+1$ we have $\phi_k(2n+1) = \phi_k(2n+1-k) = \phi_k(2n-2p)$ which can be written as $\varphi_k(m-p)$. The case of even k is more complicated. We start with the remark that the ϕ_k for even k can be decomposed in terms of a ϕ of lower index and a χ based on the identity $\phi_{2p}(n) = \phi_p(n) + \chi_{2p}(n)$. Since we already discussed the case of ϕ_2 it suffices to study the transcription of the first few χs. The function $\chi_4(n)$ is represented in the QRT-asymmetric form by two different functions $\chi_2(m)$ and $\hat{\chi}_2(m)$ for the even and odd-index equation respectively. A short analysis shows that $\chi_6(n)$ is transcribed as $\phi_3(m)+c$ in the even-index equation and $\phi_3(m-1)-c$ in the odd-index one. The function $\chi_8(n)$ becomes two different functions $\chi_4(m)$ and $\hat{\chi}_4(m)$ for the even- and odd-index equation respectively. The pattern becomes now clear. For $\chi_{2(2p+1)}$ we must introduce $\phi_{2p+1}(m)+c$ in the even-index equation and $\phi_{2p+1}(m-p)-c$ in the odd-index one. The χ_{4p} on the other hand become two different functions χ_{2p} and $\hat{\chi}_{2p}$.

Using the procedure described above one can transcribe any QRT-symmetric discrete Painlevé equation into a QRT-asymmetric one in a straightforward way.

3. The case of the two $E_7^{(1)}$-associated equations

The case that motivated the present study is that of two discrete Painlevé equations which have a double ternary periodicity, i.e. one involving two different periodic functions $\phi_3(n)$ and $\hat{\phi}_3$. This is a feature appearing in equations associated to $E_7^{(1)}$ and which does not exist for any of the "lower" groups. The first equation was obtained in [6] under the form

$$\frac{(w_n w_{n+1} - s_n s_{n+1})(w_n w_{n-1} - s_n s_{n-1})}{(w_n w_{n+1} - 1)(w_n w_{n-1} - 1)} = \frac{(w_n - a_n s_n^2)(w_n - b_n s_n^2)}{(w_n - c)(w_n - 1/c)}, \tag{3.1}$$

where s_n has a secular dependence of the form $\log s_n = \alpha n + \beta$ plus a periodic dependence involving the two different period-3 functions $\phi_3(n)$ and $\hat{\phi}_3(n)$. The two quantities a_n and b_n obey the relation $ab = 1$ at the autonomous limit and are such that each of the products as^2 and bs^2 involve one of the $\phi_3(n)$, $\hat{\phi}_3(n)$. Finally c is a pure constant. The second equation was obtained in [7] and has the form

$$\frac{(w_{n+1} s_n s_{n+1} - w_n)(w_n - w_{n-1} s_n s_{n-1})}{(w_{n+1} - s_n s_{n+1} w_n)(w_n s_n s_{n-1} - w_{n-1})} = \frac{(w_n - d_n s_{n-1} s_n s_{n+1})(w_n - f_n s_n)}{(s_{n-1} s_n s_{n+1} w_n - g_n)(s_n w_n - h_n)}. \tag{3.2}$$

Here s_n has a secular dependence of the form $\log s_n = \alpha n + \beta$ and a periodic dependence involving a period-3 function $\phi_3(n)$. The functions d_n and g_n have the same ternary periodicity involving a function $\hat{\phi}_3(n)$ as well as an even-odd dependence, which is also present in f_n, h_n. Neglecting the period-3 dependence we have simply $d_n f_n = 1$ and $h_n g_n = 1$ and moreover $g_{n+1} = d_n$.

Before addressing the question of whether (3.1) and (3.2) are related in full generality by the simple transformation presented in the previous paragraph, we can simply ask here whether it is possible to transform (3.1) into (3.2), and vice versa, in the case where the ternary dependence is neglected and we keep just the secular one. We start from (3.2) and write it in asymmetric, in the QRT sense, form as

$$\frac{(y_m z_m \zeta_m - x_m)(x_m - y_{m-1} z_m \zeta_{m-1})}{(y_m - z_m \zeta_m x_m)(x_m z_m \zeta_{m-1} - y_{m-1})} = \frac{(x_m - d_m \zeta_{m-1} \zeta_m z_m)(x_m - z_m/d_m)}{(\zeta_{m-1} \zeta_m z_m x_m - g_m)(z_m x_m - 1/g_m)}, \tag{3.3a}$$

$$\frac{(y_m - z_m \zeta_m x_m)(x_{m+1} \zeta_m z_{m+1} - y_m)}{(y_m z_m \zeta_m - x_m)(x_{m+1} - y_m \zeta_m z_{m+1})} = \frac{(y_m - g_m z_{m+1} \zeta_m z_m)(y_m - \zeta_m/g_m)}{(z_{m+1} \zeta_m z_m y_m - d_m)(\zeta_m y_m - 1/d_m)}, \tag{3.3b}$$

where $\zeta(m) = z(m+1/2)$. Similarly starting from (3.1) we obtain the QRT-asymmetric form

$$\frac{(x_m y_m - p_m \rho_m)(x_m y_{m-1} - p_m \rho_{m-1})}{(x_m y_m - 1)(x_m y_{m-1} - 1)} = \frac{(x_m - a p_n^2)(x_m - p_n^2/a)}{(x_m - c)(x_m - 1/c)}, \tag{3.4a}$$

$$\frac{(x_m y_m - p_m \rho_m)(x_{m+1} y_m - p_{m+1} \rho_m)}{(x_m y_m - 1)(x_{m+1} y_m - 1)} = \frac{(y_m - a \rho_m^2)(y_m - \rho_m^2/a)}{(y_m - c)(y_m - 1/c)}, \tag{3.4b}$$

where $\rho(m) = p(m+1/2)$. We start with equation (3.3), perform the transformation $x_n = X_n/z_n$, $y_n = \zeta_n/Y_n$ and use the relation $p_n = z_n^2$. We transform thus the form-VI$'$ equation (3.3) into

$$\frac{(X_m Y_m - p_m \rho_m)(X_m Y_{m-1} - p_m \rho_{m-1})}{(X_m Y_m - 1)(X_m Y_{m-1} - 1)} = \frac{(X_m - d_m p_m^2)(X_m - p_m/d_m)}{(X_m - g_m/p_m)(X_m - 1/g_m)}, \tag{3.5a}$$

$$\frac{(X_m Y_m - p_m \rho_m)(X_{m+1} Y_m - p_{m+1} \rho_m)}{(X_m Y_m - 1)(X_{m+1} Y_m - 1)} = \frac{(Y_m - \rho_m/d_m)(Y_m - d_m \rho_m^2)}{(Y_m - g_m)(Y_m - 1/(g_m \rho_m))}. \tag{3.5b}$$

A simple glance at the right-hand sides of (3.5) suffices to convince oneself that the terms do not match those of (3.4). Thus even when one considers just the secular dependence one cannot relate the two form-VI and form-VI′ equations.

Does the result above mean that there is no relation whatsoever between equations (3.1) and (3.2)? As we shall show in the next section this is not the case: the two equations have a common ancestor, namely an equation with $E_8^{(1)}$ symmetry from which they can be obtained through two different limiting procedures.

4. From $E_8^{(1)}$ to $E_7^{(1)}$ through limiting procedures

Our starting point will be the multiplicative $E_8^{(1)}$-associated equation which has the form

$$\frac{(x_{n+1}z_{n+1}z_n - x_n)(x_{n-1}z_{n-1}z_n - x_n) - (z_{n+1}^2 z_n^2 - 1)(z_{n-1}^2 z_n^2 - 1)}{(x_{n+1} - z_{n+1}z_n x_n)(x_{n-1} - z_{n-1}z_n x_n) - (z_{n+1}^2 z_n^2 - 1)(z_{n-1}^2 z_n^2 - 1)/(z_{n+1}z_n^2 z_{n-1})} = R(x_n). \tag{4.1}$$

Taking the limit $x \to \infty$ while keeping z finite leads to the equation

$$\frac{(x_{n+1}z_n z_{n+1} - x_n)(x_n - x_{n-1}z_n z_{n-1})}{(x_{n+1} - z_n z_{n+1}x_n)(x_n z_n z_{n-1} - x_{n-1})} = S(x_n), \tag{4.2}$$

where the right-hand side S must now be understood as the limit of the right-hand side of (4.1) where we have taken x as well as certain of the parameters which appear in $R(x_n)$ to infinity. We remark readily that the equation obtained is of the form VI′.

In practical calculations, using the explicit form of $R(x)$ may turn out to be cumbersome and it is preferable in this case to resort to the ancillary representation we have introduced in [8]. As we showed there, we can introduce the ancillary dependent variable ξ by

$$x_n = \xi_n + \frac{1}{\xi_n} \tag{4.3}$$

and use it to rewrite the right-hand side of (4.1) as

$$R(x_n) = z_{n+1}z_n^2 z_{n-1}\frac{\xi_n P(\xi_n) - \xi_n^{-1}Q(\xi_n)}{\xi_n Q(\xi_n) - \xi_n^{-1}P(\xi_n)}. \tag{4.4}$$

The $P(\xi)$ and $Q(\xi)$ are given by

$$P(\xi_n) = \prod_{i=1}^{8}(\xi_n - A_n^i) \quad \text{and} \quad Q(\xi_n) = \prod_{i=1}^{8}(A_n^i \xi_n - 1), \tag{4.5}$$

where the A_n^i are eight quantities which, in principle, depend on the independent variable n. As we had shown in [8] they obey the relation

$$\prod_{i=1}^{8} A_n^i = z_{n+1}^2 z_n^4 2 z_{n-1}^2. \tag{4.6}$$

Before proceeding to the calculations aiming at relating (3.1) and (3.2) we introduce another auxiliary variable $q_n = q_0\lambda^n$ which will be used in order to represent the secular dependence of the various parameters we are going to work with.

In our exhaustive investigation of symmetric $E_8^{(1)}$-associated equations we have, among others, derived a single equation, 4.3.1 in [9], which has a double ternary dependence, just like equations (3.1) and (3.2). The equation was obtained in additive form but its transcription to multiplicative form is straightforward. In the latter case its parameters are:

$$\begin{aligned} z_n &= q_n\phi_3(n) \\ A_n^1 &= q_n^3\phi_2(n)\hat{\phi}_3(n)c \\ A_n^2 &= q_n^3\phi_2(n)/(\hat{\phi}_3(n)c) \\ A_n^3 &= q_n\phi_3(n)b/\phi_2(n) \\ A_n^4 &= q_n\phi_3(n)/(\phi_2(n)b) \end{aligned} \tag{4.7}$$

where $\phi_2(n) = \gamma^{(-1)^n}$. The right-hand side of this equation is a ratio of two quadratic polynomials which means that it was obtained after two simplifications obtained by assuming $A_n^7A_n^8 = 1$ and $A_n^5A_n^6 = 1$. It has the form

$$R(x_n) = z_{n+1}z_n^2z_{n-1}\frac{x_n^2 - x_n\sigma_1 - \sigma_4 + \sigma_2 - 1}{x_n^2\sigma_4 - x_n\sigma_3 - \sigma_4 + \sigma_2 - 1}, \tag{4.8}$$

where σ_1 is the sum of the four A^i, σ_2, σ_3 the sum of all the products of two and three A^i respectively and σ_4 the product of all four A^i. As we have seen, in order to obtain an equation of form VI$'$ we must take $x_n \to \infty$. Clearly, if we wish that the right-hand side remain quadratic, some of the A^i's must also go to infinity with some going to zero so as to preserve (4.6). Given the form of the A^i in (4.7) this can be done in two different ways. The first possibility is to take both c and b to infinity (while keeping the ratio c/b finite and free). We find in this case the right-hand side

$$R(x_n) = \frac{(x_n - \phi_2(n)\hat{\phi}_3(n)q_n^3c/b)(x_n - \phi_3(n)q_n/\phi_2(n))}{(x_nq_n^3 - \hat{\phi}_3(n)c/(b\phi_2(n)))(x_nq_n\phi_3(n) - \phi_2(n))}. \tag{4.9}$$

Thus is exactly the right-hand side of (3.2) (calling x what is called there w) where $s_n = q_n\phi_3(n)$ (as in equation 4.3.1), $d_n = \phi_2(n)\hat{\phi}_3(n)c/b$, $g_n = \hat{\phi}_3(n)c/(b\phi_2(n))$, $f_n = \phi_3(n)/\phi_2(n)$ and $h_n = \phi_2(n)$. Thus by taking the limit $x_n \to \infty$ together with b and c we obtained precisely equation (3.2). However another possibility of limit does exist. We can balance the limit $x_n \to \infty$ by taking the γ of the $\phi_2(n)$ to infinity. In that case it is more convenient to split the equation according to the parity of the indices and write it in QRT-asymmetric form.

$$\begin{aligned} A_{2n}^1 &= q_{2n}^3\gamma\hat{\phi}_3(2n)c \\ A_{2n}^2 &= q_{2n}^3\gamma/(\hat{\phi}_3(2n)c) \\ A_{2n}^3 &= q_{2n}\phi_3(2n)b/\gamma \\ A_{2n}^4 &= q_{2n}\phi_3(2n)/(\gamma b) \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} A_{2n+1}^1 &= q_{2n+1}^3\hat{\phi}_3(2n+1)c/\gamma \\ A_{2n+1}^2 &= q_{2n+1}^3/(\gamma\hat{\phi}_3(2n+1)c) \\ A_{2n+1}^3 &= q_{2n+1}\gamma\phi_3(2n+1)b \\ A_{2n+1}^4 &= q_{2n+1}\gamma\phi_3(2n+1)/b \end{aligned} \tag{4.11}$$

Taking the limit $\gamma \to \infty$ we obtain for the even-index equation. In order to use the QRT notations we introduce $s_m = z_{2m}$ and $\zeta_m = z_{2m+1}$. Since $z_n = q_n\phi_3(n)$ we can define $\varphi_3(m) = \phi_3(2n)$ and rewrite $s_m = q_{2m}\varphi_3(m)$ which allows us to write $\zeta_m = q_{2m+1}\phi_3(2m+1) = q_{2m+1}\phi_3(2m-2) = q_{2m+1}\varphi_3 = (m-1)$. The right-hand side is

$$R_{2n} = \frac{1}{\zeta_{m-1}s_m^2\zeta_m}\frac{(x_m - c\hat{\varphi}_3(m)\zeta_{m-1}s_m\zeta_m)(x_m - \zeta_{m-1}s_m\zeta_m/(c\hat{\varphi}_3(m))}{(x_m - b/s_m)(x_m - 1/(bs_m))}, \tag{4.12a}$$

where $\hat{\varphi}_3(m) = \hat{\phi}_3(2m)$. Similarly for the odd-index right-hand side we find

$$R_{2n+1} = \frac{1}{s_m\zeta_m^2 s_{m+1}}\frac{(y_m - b\zeta_m)(y_m - \zeta_m/b)}{(y_m - c\hat{\varphi}_3(m-1)/(s_m\zeta_m s_{m+1}))(y_m - 1/(s_m\zeta_m s_{m+1}c\hat{\varphi}_3(m-1)))}, \tag{4.12b}$$

where $\hat{\phi}_3(2m+1) = \hat{\phi}_3(2m-2)$ and thus equal to $\hat{\varphi}_3(m-1)$. Since the resulting equation is in VI′ form we must transform it to a form VI by introducing the transformation $x_m = X_m/s_m$, $y_m = \zeta_m/Y_m$. We obtain thus the system

$$\frac{(X_mY_m - s_m^2\zeta_m^2)(X_mY_{m-1} - s_m^2\zeta_{m-1}^2)}{(X_mY_m - 1)(X_mY_{m-1} - 1)} = \frac{(X_m - c\hat{\varphi}_3(m)\zeta_{m-1}s_m^2\zeta_m)(X_m - \zeta_{m-1}s_m^2\zeta_m/(c\hat{\varphi}_3(m)))}{(X_m - b)(X_m - 1/b)}, \tag{4.13a}$$

$$\frac{(X_mY_m - s_m^2\zeta_m^2)(X_{m+1}Y_m - s_{m+1}^2\zeta_m^2)}{(X_mY_m - 1)(X_{m+1}Y_m - 1)} = \frac{(Y_m - c\hat{\varphi}_3(m-1)s_m\zeta_m^2 s_{m+1})(Y_m - s_m\zeta_m^2 s_{m+1}/(c\hat{\varphi}_3(m-1)))}{(Y_m - b)(Y_m - 1/b)}. \tag{4.13b}$$

This is just equation (3.1), written in form (3.4).

Thus, by taking two different limits of the same $E_8^{(1)}$-associated equation we obtained the two $E_7^{(1)}$-related equations with double ternary dependence.

5. More limits

Having introduced the two possible ways of taking limits in $E_8^{(1)}$-associated equations in order to reduce them to $E_7^{(1)}$-associated ones, one can ask the question whether the case of the two equations analysed in the previous section is unique or more similar cases do exist. It turns out that the latter is indeed the case.

We start from the equations derived in [9] and identified the equations where at least one constant and one $\phi_2(n)$ were present in their parameters. Four such equations are obtained, the ones labeled 4.2.1, 4.2.5, 5.2.1 and 5.2.5 in [9]. In what follows we shall present their different limits. (As pointed out in the previous section, the results of [9] concerned additive equations but their transcription to multiplicative ones is straightforward).

a) Equation 4.2.1 The parameters of this equation are

$$\begin{aligned} z_n &= q_n\phi_5(n+1)^2\phi_5(n-1)^2/\phi_5(n) \\ A_n^1 &= q_n^3\phi_2(n)/\phi_5(n) \\ A_n^2 &= q_n^2\hat{\phi}_2(n)\phi_5(n)c \\ A_n^3 &= q_n^2\phi_5(n)/(\hat{\phi}_2(n)c) \\ A_n^4 &= q_n\phi_5(n+1)^2\phi_5(n-1)^2/(\phi_5(n)\phi_2(n)). \end{aligned} \tag{5.1}$$

Note that two different period-2 functions appear here $\phi_2(n) = \gamma^{(-1)^n}$ and $\hat{\phi}_2(n) = \delta^{(-1)^n}$. In order to balance the limit $x_n \to \infty$ we must take the γ of the $\phi_2(n)$ to infinity as well as either of the c or the δ of $\hat{\phi}_2(n)$. It turns out that the two latter choices are perfectly equivalent leading to the same final equation.

Once the equation is VI′ form is obtained it is interesting to convert it into a form VI, using the transformations we outlined in the Introduction. The end result is a QRT-symmetric equation of the form

$$\frac{(X_nX_{n+1} - q_n^2q_{n+1}^2\phi_5^2(n+2)\phi_5^2(n-1)/\phi_5^2(n-2))(X_nX_{n-1} - q_n^2q_{n-1}^2\phi_5^2(n+1)\phi_5^2(n-2)/\phi_5^2(n+2))}{(X_nY_n-1)(X_nY_{n-1}-1)}$$
$$= \frac{(X_n - \phi_5^2(n+1)\phi_5^2(n-1)q_n^4/\phi_5^2(n))(X_n - c\phi_5^2(n+1)\phi_5^2(n-1)\varphi_2(n)q_n^3)}{(X_n - c\phi_5^2(n+1)\phi_5^2(n-1)\varphi_2(n)/(q_n\phi_5^2(n)))(X_n-1)}, \tag{5.2}$$

where $\varphi_2(n)$ must be understood as the limit of $\hat{\phi}_2(n)/\phi_2(n)$ when both γ and δ go to infinity. This is an $\mathrm{E}_7^{(1)}$-associated equation which, at least to the authors knowledge, has not been previously derived.

b) Equation 4.2.5 Before giving the parameters of this equation it is convenient to introduce the variable Z_n, a variable simply related to z_n by $Z_n = z_nz_{n+1}$. We can now write the parameters as

$$\begin{aligned} Z_n &= q_nq_{n+1}\phi_6(n) \\ A_n^1 &= q_n^2q_{n-1}\phi_2(n)\phi_6(n+2)c/\phi_6(n+3) \\ A_n^2 &= q_n^2q_{n+1}\phi_2(n)\phi_6(n)\phi_6(n-2)\phi_6(n-3)/c \\ A_n^3 &= q_{n-1}\phi_6(n)\phi_6(n+1)\phi_6(n-1)/(\phi_2(n)\phi_6(n-2)c) \\ A_n^4 &= q_{n+1}\phi_6(n-1)c/(\phi_2(n)\phi_6(n+2)\phi_6(n+1)). \end{aligned} \tag{5.3}$$

We remark that, here, it is possible to balance the limit $x_n \to \infty$ either by taking c to infinity or by taking the γ of the $\phi_2(n)$ to infinity. Taking c to infinity results to an equation in a QRT-symmetric equation of form VI′. Its precise expression is

$$q_{n+1}q_n^2q_{n-1}\phi_6(n)\phi_6(n-1)\frac{(x_{n+1}q_nq_{n+1}\phi_6(n) - x_n)(x_n - x_{n-1}q_nq_{n-1}\phi_6(n-1))}{(x_{n+1} - q_nq_{n+1}\phi_6(n)nx_n)(x_nq_nq_{n-1}\phi_6(n-1) - x_{n-1})}$$
$$\frac{(x_n - q_nq_{n+1}\phi_6(n-2)\phi_6(n-3)\phi_2(n))(x_n - q_{n-1}\phi_6(n)\phi_6(n+1)\phi_6(n-1)/(\phi_2(n)\phi_6(n-2)))}{(x_n - \phi_6(n-3)/(\phi_6(n+2)\phi_2(n)q_n^2q_{n-1}))(x_n - \phi_2(n)\phi_6(n+1)\phi_6(n+2)/(\phi_6(n-1)q_{n+1}))}. \tag{5.4}$$

When we take γ to infinity we obtain a QRT-asymmetric equation in VI′ form. However it is possible to cast it is form VI in which case the equation becomes QRT-symmetric. We are not going to give the intermediate calculations (which are straightforward but rather tedious) and present directly the form of the resulting equation:

$$\frac{(X_{n+1}X_n - Z_n)(X_nX_{n-1} - Z_{n-1})}{(X_{n+1}X_n - 1)(X_nX_{n-1} - 1)} = \frac{(X_n - A_n)(X_n - B_n)}{(X_n - C_n)(X_n - D_n)}. \tag{5.5}$$

The various quantities appearing in (5.5) are given by the expressions: $\log Z_n alphan + \beta - 2\gamma(-1)^n + \phi_3(n+1) + \chi_6(n) + \chi_6(n-1)$, $\log A_n = \alpha n + \beta + (-1)^n(\gamma n + \delta) + \phi_3(n) + \chi_6(n)$ and $B_n = Z_nZ_{n+4}/A_{n+5}, C_n = Z_{n+2}/A_{n+3}, D_n = A_{n+2}/Z_{n+1}$. (Note that in these expressions we have used parameters and functions without making any attempt to link them to those of the initial equation, lest the result become inextricably complicated.)

Both equations (5.4) and (5.5) are new, derived here for the first time (with, the usual caveat, that the novelty statement concerns the present authors and their knowledge of discrete Painlevé equations).

c) Equation 5.2.1 Here we have six A_n^i with expressions

$$\begin{aligned}
z_n &= q_n\phi_3(n)\\
A_n^1 &= q_n^2\phi_2(n)a/\phi_3(n)\\
A_n^2 &= q_n^2/(\phi_2(n)\phi_3(n)a)\\
A_n^3 &= q_n\phi_3(n)\hat{\phi}_2(n)b\\
A_n^4 &= q_n\phi_3(n)\hat{\phi}_2(n)/b\\
A_n^5 &= q_n\phi_3(n)c/\hat{\phi}_2(n)\\
A_n^6 &= q_n\phi_3(n)/(\hat{\phi}_2(n)c)
\end{aligned}\tag{5.6}$$

Clearly there are four different ways one can balance the limit $x_n \to \infty$: taking all three a, b, c to infinity or the γ of $\phi_2(n)$ and b, c or the δ of $\hat{\phi}_2(n)$ and a or both γ and δ to infinity. All four limits lead to the same equation which is given in a QRT-symmetric VI$'$ form in the case $(a,b,c) \to \infty$ or can be converted to such a form through a simple gauge choice. Without entering into further details we present the form obtained in the straightforward case when all three a, b, c go to infinity. We find

$$\begin{aligned}
&\frac{(x_{n+1}z_nz_{n+1}-x_n)(x_n-x_{n-1}z_nz_{n-1})}{(x_{n+1}-z_nz_{n+1}x_n)(x_nz_nz_{n-1}-x_{n-1})} = \\
&\frac{(x_n-z_{n+1}z_{n-1}/\phi_2(n)(x_n-z_n/(c\hat{\phi}_2(n)))(x_n-cz_n/\hat{\phi}_2(n))}{(x_n-1/(z_{n+1}z_{n-1}\phi_2(n))(x_n-b/(z_n\hat{\phi}_2(n)))(x_n-1/(bz_n\hat{\phi}_2(n))},
\end{aligned}\tag{5.7}$$

where b, c must be understood as the ratios b/a, c/a when all three parameters go to infinity. Equation (5.7) is a discrete Painlevé equation already derived in [7].

d) Equation 5.2.5 Here again we have six A_n^i. Their expressions are

$$\begin{aligned}
Z_n &= q_n^3q_{n+1}^3\phi_5(n)\phi_5(n+1)\\
A_n^1 &= q_n^5\phi_5(n)\phi_5(n+1)^2\phi_5(n+2)^2a/\phi_2(n)\\
A_n^2 &= q_n^5q_{n-1}^2\phi_2(n)/(\phi_5(n)a)\\
A_n^3 &= q_nq_{n+1}^4\phi_5(n)\phi_5(n-1)^2\phi_5(n-2)^2a/\phi_2(n)\\
A_n^4 &= q_{n-2}\phi_5(n)/(\phi_5(n+2)^2\phi_5(n-2)^2\phi_2(n)a)\\
A_n^5 &= q_n^3\phi_2(n)\phi_5(n)b\\
A_n^6 &= q_n^3\phi_2(n)\phi_5(n)/b
\end{aligned}\tag{5.8}$$

We remark that the limit $x_n \to \infty$ can be balanced either by taking both a, b to infinity or by taking to infinity the γ of $\phi_2(n)$. In the first case we obtain a QRT-symmetric discrete Painlevé equation in form VI′:

$$Z_nZ_{n-1}\frac{(x_{n+1}Z_n-x_n)(x_n-x_{n-1}Z_{n-1})}{(x_{n+1}-Z_nx_n)(x_nZ_{n-1}-x_{n-1})}=$$
$$\frac{(x_n-\phi_2(n)q_n^5q_{n-1}^2/\phi_5(n)(x_n-b\phi_2(n)\phi_5(n)q_n^3)(x_n-\phi_5(n)q_{n-2}/(\phi_2(n)\phi_5(n+2)^2\phi_5(n-2)^2))}{(x_n-b/(\phi_2(n)\ phi_5(n)q_n^3))(x_n-\phi_2(n)/(\Phi(n)q_n^5))(x_n-\phi_2(n)\Phi(n)/(q_nq_{n+1}^4))}, \tag{5.9}$$

where $\Phi(n)=\phi_5(n)\phi_5(n+1)^2\phi_5(n+2)^2$. Again, this is an equation which does not appear among those previously derived. Next we turn to the limit obtained when we take the γ of $\phi_2(n)$ to infinity. In this case the equation obtained has a QRT-asymmetric form. However by transforming it into a VI form we find that the resulting discrete Painlevé equation can be cast into a QRT-symmetric form. Without entering into the tedious (but straightforward) details we give the final form of the equation:

$$\frac{(X_{n+1}X_n-Z_n^2)(X_nX_{n-1}-Z_{n-1}^2)}{(X_{n+1}X_n-1)(X_nX_{n-1}-1)}=$$
$$\frac{(X_n-aq_n^4q_{n+1}^4\phi_5(n)/\Phi(n))(X_n-aq_n^8\phi_5(n)\Phi(n))(X_n-q_n^2q_{n-1}^2\phi_5(n-1)^2\phi_5(n)^4\phi_5(n+1)^2/a)}{(X_n-b)(X_n-1/b)(X_n-a\phi_5(n)^2/(q_n^2q_{n-1}^2))}. \tag{5.10}$$

Again, this is a new, to the authors' knowledge, discrete Painlevé equation.

6. Conclusions

This paper was motivated by the existence of two $E_7^{(1)}$-associated discrete Painlevé equations, known under different canonical forms, both having a double ternary dependence in their coefficients. The usual conversion of form-VI to form-VI′ did not suffice in order to link the two equations and thus their relation was somewhat puzzling. In this paper we have shown that these two equations correspond to two different limits of a $E_8^{(1)}$-associated discrete Painlevé equation the coefficients of which do have a double ternary dependence.

Going from discrete Painlevé equations with $E_8^{(1)}$ symmetry to ones with $E_7^{(1)}$ can be obtained by taking the dependent variable to infinity. In order to balance this and obtain some meaningful equation necessitates taking also some of the parameters to infinity. In every case the independent variable must remain finite. In the case at hand the first of the two limits was a straightforward one, corresponding to taking the limit of a constant appearing explicitly in the equation to infinity. The second limit was somewhat more tricky since it consisted in taking the value of the parameter appearing in the period-2 function to infinity. Since we have $\phi_2(n)=\gamma^{(-1)^n}$, this means that we have to split the equation into even- and odd-index parts, i.e. write it in QRT-asymmetric form, and then apply the limit.

Having produced the complete list of QRT-symmetric $E_8^{(1)}$-associated discrete Painlevé equations, we asked ourselves whether there existed other cases where two different limits could be implemented and whether these two limits yielded different $E_7^{(1)}$-associated discrete Painlevé equations or the same one written under two different, but equivalent, forms. It turned out that both situations materialised: two of the equations led each to a single one at the limit (out of two and four possibilities respectively) the remaining ones yielding each two distinct equations. In all cases we were able to cast the equations into a QRT-symmetric form, either a VI form or a VI′ one. With one exception, (5.7), all the equations thus obtained had not been derived before (at least as far as the present authors can confirm). In this sense the method introduced here turned out to be a particularly fruitful one.

In our approach we took particular care of balancing the equations so as to lose just one parameter and moreover to be able to do this in at least two different ways. Thus we limited ourselves to the cases where that was possible. Of course it is also possible to implement the limit for $E_8^{(1)}$ to $E_7^{(1)}$-associated equations in just a single way, and, in the light of the present results, derive new discrete Painlevé equations. What is potentially even more interesting is to consider the limits where one can go from a discrete Painlevé equation with $E_8^{(1)}$ symmetry to one associated to $E_6^{(1)}$ or even lower groups. Given the limiting procedure all these equations would have a canonical VI′ form. From our study in [10] we know that such equations do exist and deriving all of them in a systematic approach could be an interesting extension of the present work.

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