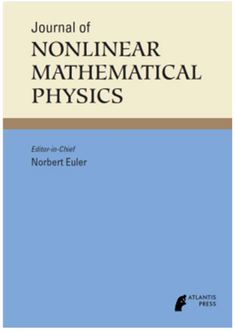

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In this paper we study realizations of infinite-dimensional Witt and Virasoro algebras. We obtain a complete description of realizations of the Witt algebra by Lie vector fields of first-order differential operators over the space $\mathbb{R}^3$. We prove that none of them admits non-trivial central extension, which means that there are no realizations of the Virasoro algebra in $\mathbb{R}^3$. We describe all inequivalent realizations of the direct sum of the Witt algebras by Lie vector fields over $\mathbb{R}^3$. This result enables complete description of all possible (1+1)-dimensional partial differential equations that admit infinite dimensional symmetry algebras isomorphic to the direct sum of Witt algebras. In this way we have constructed a number of new classes of nonlinear partial differential equations admitting infinite-dimensional Witt algebras. So new integrable models which admit infinite symmetry algebra are obtained.

Keywords: Witt algebra; Virasoro algebra; Lie vector field; equivalence transformation; integrable equation.

2000 Mathematics Subject Classification: 37K05,37K30,37K35

1. Introduction

Methods and ideas of the group approach to analyzing differential equations go back to the pioneering works of Sophus Lie [1, 10, 39]. It is nothing short of amazing that two centuries later these methods have not significantly changed in their content. What has changed dramatically is the range of applications of these methods. The Lie group approach is used to analyze linear and nonlinear algebraic, differential and integro-differential equations modeling natural phenomena in many application areas such as physics, chemistry, biology and finance, to name just a few.

A necessary prerequisite for applicability of Lie group methods to a specific model is that the model should possess nontrivial symmetry. The ultimate success of the Lie theory relies heavily upon the fact that symmetry is one of the fundamental properties of nature, which is reflected in the symmetries of the models describing it.

It comes as no surprise that the richer the symmetry properties of a model under study are, the more efficient the application of group methods for its analysis. The best possible scenario is when the model admits infinite symmetries, which in many cases may lead to a linearizing transformation or even general solution of the model in question. Bluman and Kumei analyzed the connection between infinite-dimensional symmetry and linearization of partial differential equations (PDEs) in

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[7,36]. They have established necessary and sufficient conditions of the linearizability of nonlinear PDEs admitting infinite dimensional symmetry algebra and described the method for constructing invertible mappings of nonlinear differential equations into linear ones.

A classical example is the hyperbolic type Liouville equation

$$u_{tx} = \exp(u). \tag{1.1}$$

It admits the infinite-parameter Lie group

$$\tilde{t} = t + f(t), \quad \tilde{x} = x + g(x), \quad \tilde{u} = u - \dot{f}(t) - \dot{g}(x), \tag{1.2}$$

where f and g are arbitrary smooth functions. Note that an alternative term for these kinds of groups is Lie pseudo-group [14,43,44,46,55].

The general solution of Eq. (1.1) can be obtained by the action of the transformation group (1.2) on its particular traveling wave solution of the form $u(t,x) = \varphi(x+t)$ (see, e.g., [16]). An alternative way to solve Eq. (1.1) is reducing it to the linear form $U_{tx} = 0$ by the nonlocal transformation $u = \ln(2U_tU_x/U^2)$ [16].

In most cases application of Lie methods boils down to solving one of the two basic problems [6, 16, 17, 27, 45, 48]. The first one is computing the maximal symmetry group admitted by the model under study. The second (classification) problem is describing all possible models of some prescribed form invariant with respect to a given Lie group or Lie algebra. Typically classification problems split into two sub-problems, (i) describing all inequivalent realizations of the Lie algebra in question, and, (ii) constructing corresponding classes of invariant models (see, [4,18,19,28,37, 54,56] and the references therein).

Lie group classification of PDEs is a very popular topic and there are literally hundreds of publications devoted to applying it to various classes of linear and nonlinear PDEs. However, the overwhelming majority of these tackle the classification with respect to finite-dimensional Lie algebras.

The situation is drastically different for the case of generalized (higher) Lie symmetries, which have played a critical role in the success of the theory of nonlinear integrable systems in $(1+1)$- and $(1+2)$-dimensions (see, e.g. [27,45]). Progress in this area has been complemented by advances in development of the theory of infinite-dimensional Lie algebras such as loop [30,50], Kac-Moody [8,32] and Virasoro [3,20,29,34] algebras.

Virasoro algebra continues to play an increasingly important role in mathematical physics in general [5,24] and in the theory of integrable systems in particular. A number of nonlinear evolution equations in $(1+2)$-dimensions modeling various phenomena in modern physics are invariant with respect to Virasoro algebras. An incomplete list of these models includes Kadomtsev-Petviashvili (KP) [15,26], modified KP, cylindrical KP [38], Davey-Stewartson [13,25], Nizhnik-Novikov-Veselov, stimulated Raman scattering, $(1+2)$-dimensional Sine-Gordon [52] and the KP hierarchy [47] equations.

It is a common belief that nonlinear PDEs admitting symmetry algebras of a Virasoro type are prime candidates for the role of an integrable system. This is why systematic classification of Virasoro algebra realizations could come in handy in constructing new integrable systems by the symmetry approach (see, e.g., [40,41]).

It should be pointed out that there are integrable equations which do not admit Virasoro algebras, for example, breaking soliton and Zakharov-Strachan equations [52].

Classification of realizations of Lie algebras by vector fields of differential operators within the action of a local diffeomorphism group was pioneered by Sophus Lie himself. It is still a very popular and efficient tool for group analysis of nonlinear PDEs. Relatively recent applications of this approach include geometric control theory [31], theory of systems of nonlinear ordinary differential equations possessing superposition principle [53], and an algebraic approach to molecular dynamics [2, 51]. Analysis of realizations of Lie algebras by first-order differential operators is at the core of almost every approach to group classification of PDEs (see, e.g., [1, 10, 17, 22, 23, 33, 35]). Let us emphasize that systematic and exhaustive description of realizations of infinite-dimensional Lie algebras within the context of group classification of nonlinear PDEs is an important problem (see, e.g., [14, 43, 44, 55] and the references therein).

It is straightforward to verify that the Lie algebra of the group (1.2) is a direct sum of two infinite-dimensional Witt algebras. This observation is the starting point of our search for nonlinear generalizations of the wave equation possessing the same (infinite) symmetries. Namely, we are looking for all possible PDEs in two dimensions admitting a direct sum of two Witt algebras. In order to solve this classification problem we first need to construct all inequivalent realizations of the Witt algebra. The next step is to describe all possible inequivalent realizations of the direct sum of two Witt algebras. The remarkable fact is that this problem has a complete and elegant solution and there are only four inequivalent classes of PDEs that admit a direct sum of the Witt algebras. The wave and Liouville equations are particular cases of the so obtained PDEs.

The paper is organized as follows. In Section 2, we give a brief account of necessary facts and definitions. We also describe the algorithmic procedure for classification of inequivalent realizations of the Virasoro algebra in full detail. We construct all inequivalent realizations of the Witt algebra (a.k.a. centerless Virasoro algebra) in Section 3. Section 4 is devoted to analysis of realizations of the Virasoro algebra. We prove that there are no central extensions of the Witt algebra over the space $\mathbb{R}^3$. In Section 5 we construct broad classes of nonlinear PDEs admitting the Witt algebras. All inequivalent realizations of the direct sum of two Witt algebras are obtained in Section 6. This enables us to completely solve the classification problem of second-order PDEs whose invariance algebras contain a direct sum of the Witt algebras. We prove that any such PDE is equivalent to one of the four canonical equations (6.2)–(6.5). The last section, Section 7, contains a brief summary of the main results of the paper and concluding remarks regarding future work.

2. Notations and Definitions

Definition 2.1 ([12,21,29,42]). The Virasoro algebra $\mathfrak{V}$ is the infinite-dimensional Lie algebra with basis elements $\{L_n, C \mid n \in \mathbb{Z}\}$ satisfying the following commutation relations:

$$[L_m,\, L_n] = (m-n)L_{m+n} + \frac{1}{12}m(m^2-1)\delta_{m,-n}C, \quad [L_m, C] = 0$$

for $m, n \in \mathbb{Z}$. Here $[Q,P] = QP - PQ$ is the standard commutator of Lie vector fields P and Q, and the symbol $\delta_{a,b}$ stands for the Kronecker delta

$$\delta_{a,b} = \begin{cases} 1, & a = b, \\ 0, & \text{otherwise}. \end{cases}$$

The operator C commutes with all other basis elements. It is called the central element or the central charge of the Virasoro algebra $\mathfrak{V}$. When the central element C is zero, the algebra $\mathfrak{V}$ reduces to centerless Virasoro algebra, which is usually called the Witt algebra $\mathfrak{W}$ [11, 49]. Consequently, the full Virasoro algebra is a nontrivial one-dimensional central extension of the Witt algebra.

Consider the Virasoro algebra as a linear subspace of the infinite-dimensional Lie algebra $\mathfrak{L}_\infty$ spanned by the basis elements of the form

$$Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u \tag{2.1}$$

over $\mathbb{R}^3$. Applying the transformation

$$\tilde{t} = T(t,x,u), \quad \tilde{x} = X(t,x,u), \quad \tilde{u} = U(t,x,u) \tag{2.2}$$

with $D(T,X,U)/D(t,x,u) \neq 0$ to (2.1), we get

$$\tilde{Q} = (\tau T_t + \xi T_x + \eta T_u)\partial_{\tilde{t}} + (\tau X_t + \xi X_x + \eta X_u)\partial_{\tilde{x}} + (\tau U_t + \xi U_x + \eta U_u)\partial_{\tilde{u}}.$$

Evidently, $\tilde{Q} \in \mathfrak{L}_\infty$. Consequently the set of operators (2.1) is invariant with respect to the transformation (2.2).

The correspondence, $Q \sim \tilde{Q}$, is an equivalence relation on (2.1). It splits the set of differential operators (2.1) into several equivalence classes. Any two elements within the same equivalence class are related by a transformation (2.2), while two elements belonging to different classes cannot be transformed into each other by a transformation of the form (2.2). Thus to describe all possible realizations of the Virasoro algebra, it is sufficient to construct a representative of each equivalence class. The remaining realizations can be obtained by applying transformations (2.2) to the representatives.

Our method for the construction of all inequivalent realizations of the Virasoro algebra is implemented as a three-step process.

The first step is to describe all inequivalent forms of L_0, L_1 and L_{-1} such that the commutation relations

$$[L_0, L_1] = -L_1, \quad [L_0, L_{-1}] = L_{-1}, \quad [L_1, L_{-1}] = 2L_0, \tag{2.3}$$

hold together with $[L_i, C] = 0$, $(i = 0, 1, -1)$. Note that algebra $\langle L_0, L_1, L_{-1}\rangle$ is isomorphic to $sl(2,\mathbb{R})$.

At the second step, we construct all inequivalent realizations of the operators L_2 and L_{-2}, which commute with C and satisfy the relations

$$\begin{gathered} [L_0, L_2] = -2L_2, \quad [L_{-1}, L_2] = -3L_1, \quad [L_1, L_{-2}] = 3L_{-1}, \\ [L_0, L_{-2}] = 2L_{-2}, \quad [L_2, L_{-2}] = 4L_0 + \tfrac{1}{2}C. \end{gathered} \tag{2.4}$$

The third step is to derive the forms of the remaining basis operators of the Virasoro algebra using the recurrence relations

$$L_{n+1} = (1-n)^{-1}[L_1,\ L_n], \quad L_{-n-1} = (n-1)^{-1}[L_{-1},\ L_{-n}]$$

with

$$[L_{n+1}, L_{-n-1}] = 2(n+1)L_0 + \frac{1}{12}n(n+1)(n+2)C, \quad [L_i, C] = 0,$$

where $i = n+1, -n-1$ and $n = 2, 3, 4, \ldots$.

In Section 3 and 4, we use this procedure to obtain all inequivalent realizations of the Witt and Virasoro algebras by Lie vector fields of the general form (2.1).

3. Realizations of the Witt Algebra

We now turn to analysis of realizations of the Witt algebra $\mathfrak{W}$. We note that the algebra $\mathfrak{W}$ is obtained from the Virasoro algebra by putting $C = 0$.

Let vector field L_0 be of the general form (2.1), namely,

$$L_0 = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u,$$

where $\tau^2 + \xi^2 + \eta^2 \neq 0$ since otherwise L_0 is trivial. Transformation (2.2) maps L_0 into

$$\tilde{L}_0 = (\tau T_t + \xi T_x + \eta T_u)\partial_{\tilde{t}} + (\tau X_t + \xi X_x + \eta X_u)\partial_{\tilde{x}} + (\tau U_t + \xi U_x + \eta U_u)\partial_{\tilde{u}}.$$

By choosing solutions of the equations

$$\tau T_t + \xi T_x + \eta T_u = 1, \quad \tau X_t + \xi X_x + \eta X_u = 0, \quad \tau U_t + \xi U_x + \eta U_u = 0$$

as T, X and U, we can reduce L_0 to the form ∂_t. Thus L_0 is equivalent to the canonical operator ∂_t, which generates group of displacements by t. From now on, we drop the tildes.

With L_0 in hand, we proceed to constructing L_1 and L_{-1} which obey the commutation relations (2.3). Taking L_1 in the general form (2.1) and inserting it into $[L_0, L_1] = -L_1$ yield

$$L_1 = \mathrm{e}^{-t} f(x,u)\partial_t + \mathrm{e}^{-t} g(x,u)\partial_x + \mathrm{e}^{-t} h(x,u)\partial_u,$$

where $f,\ g,\ h$ are arbitrary smooth functions of their arguments.

In what follows we use only those equivalence transformations (2.2) which preserve L_0. Applying (2.2) to L_0 gives

$$L_0 \to \tilde{L}_0 = T_t\partial_{\tilde{t}} + X_t\partial_{\tilde{x}} + U_t\partial_{\tilde{u}} = \partial_{\tilde{t}}.$$

Hence, the change of variables

$$\tilde{t} = t + T(x,u),\ \tilde{x} = X(x,u),\ \tilde{u} = U(x,u)$$

is the most general transformation that does not alter the form of L_0. It transforms Lie vector field L_1 into

$$\tilde{L}_1 = \mathrm{e}^{-t}(f + gT_x + hT_u)\partial_{\tilde{t}} + \mathrm{e}^{-t}(gX_x + hX_u)\partial_{\tilde{x}} + \mathrm{e}^{-t}(gU_x + hU_u)\partial_{\tilde{u}}.$$

To further simplify L_1, we need to consider two inequivalent cases $g^2 + h^2 = 0$ and $g^2 + h^2 \neq 0$ separately.

Case 1. If $g^2 + h^2 = 0$, we have $\tilde{L}_1 = \mathrm{e}^{-t} f(x,u)\partial_{\tilde{t}}$. Choosing $\tilde{t} = t - \ln|f(x,u)|$ yields $L_1 = \mathrm{e}^{-t}\partial_t$. Taking into account (2.1) and (2.3) yields $L_{-1} = \mathrm{e}^{t}\partial_t$.

Case 2. Given the condition $g^2 + h^2 \neq 0$, we can select $T(x,u)$, $X(x,u)$ and $U(x,u)$ to satisfy the relation

$$\mathrm{e}^{-T} = f + gT_x + hT_u, \quad gX_x + hX_u = \mathrm{e}^{-T}, \quad gU_x + hU_u = 0.$$

As a consequence, L_1 takes the form $\mathrm{e}^{-t}(\partial_t + \partial_x)$. Taking into account (2.1) and (2.3) we arrive at the formula

$$L_{-1} = \mathrm{e}^{t}[1 - \mathrm{e}^{-2x} f_1(u)]\partial_t + \mathrm{e}^{t}[-1 - \mathrm{e}^{-2x} f_1(u) + \mathrm{e}^{-x} g_1(u)]\partial_x + \mathrm{e}^{t-x} h_1(u)\partial_u,$$

where f_1, g_1, h_1 are arbitrary smooth functions of the variable u.

Applying the transformation

$$\tilde{t} = t, \quad \tilde{x} = x + X(u), \quad \tilde{u} = U(u), \tag{3.1}$$

which keeps L_0 and L_1 invariant, to L_{-1} gives

$$\tilde{L}_{-1} = \mathrm{e}^t[1-\mathrm{e}^{-2x}f_1(u)]\partial_{\tilde{t}} + \mathrm{e}^t[-1-\mathrm{e}^{-2x}f_1(u)+\mathrm{e}^{-x}g_1(u)+\mathrm{e}^{-x}h_1\dot{X}]\partial_{\tilde{x}} + \mathrm{e}^{t-x}h_1(u)\dot{U}\partial_{\tilde{u}}.$$

Consider the cases $f_1(u) \neq 0$ and $f_1(u) = 0$ separately.

Assuming $f_1(u) \neq 0$ we set

$$X(u) = -\ln\sqrt{|f_1(u)|}, \quad \phi(u) = [g_1(u) + h_1(u)\dot{X}(u)]\sqrt{|f_1(u)|}$$

and

$$U(u) = \begin{cases} \int \frac{1}{h_1(u)}\mathrm{d}u, & h_1 \neq 0, \\ \text{arbitrary non-constant function}, & h_1 = 0, \end{cases} \tag{3.2}$$

whence

$$L_{-1} = \mathrm{e}^t(1+\alpha\mathrm{e}^{-2x})\partial_t + \mathrm{e}^t[-1+\alpha\mathrm{e}^{-2x}+\mathrm{e}^{-x}\phi(u)]\partial_x + \beta\mathrm{e}^{t-x}\partial_u$$

with $\alpha = \pm 1$ and $\beta = 0, 1$.

The case $f_1(u) = 0$ gives rise to the realization

$$\tilde{L}_{-1} = \mathrm{e}^t\partial_{\tilde{t}} + \mathrm{e}^t[-1+\mathrm{e}^{-x}g_1(u)+\mathrm{e}^{-x}h_1(u)\dot{X}]\partial_{\tilde{x}} + \mathrm{e}^{t-x}h_1(u)\dot{U}\partial_{\tilde{u}}.$$

Selecting $X = 0$ and U satisfying (3.2) we get

$$L_{-1} = \mathrm{e}^t\partial_t + \mathrm{e}^t[-1+\mathrm{e}^{-x}g_1(u)]\partial_x + \beta\mathrm{e}^{t-x}\partial_u,$$

where $\beta = 0, 1$.

Hence we conclude that the following assertion holds.

Lemma 3.1. *Any realization of the Lie algebra $\langle L_0, L_1, L_{-1}\rangle$ over $\mathbb{R}^3$ is equivalent to one of the following canonical realizations:*

$$\langle \partial_t,\ \mathrm{e}^{-t}\partial_t,\ \mathrm{e}^t\partial_t\rangle, \tag{3.3}$$

$$\langle \partial_t,\ \mathrm{e}^{-t}(\partial_t+\partial_x),\ \mathrm{e}^t(1+\alpha\mathrm{e}^{-2x})\partial_t + \mathrm{e}^t[-1+\phi(u)\mathrm{e}^{-x}+\alpha\mathrm{e}^{-2x}]\partial_x + \beta\mathrm{e}^{t-x}\partial_u\rangle. \tag{3.4}$$

Here $\alpha = 0, \pm 1$, $\beta = 0, 1$ and $\phi(u)$ is an arbitrary smooth function.

To obtain the complete description of all inequivalent Witt algebras, we need to implement the last two steps of the classification procedure outlined in Section 2 and extend (3.3) and (3.4) up to full realizations of the Witt algebra.

The following assertion holds.

Theorem 3.1. *There are at most eleven inequivalent realizations of the Witt algebra $\mathfrak{W}$ over the space $\mathbb{R}^3$. The representatives, $\mathfrak{W}_i$, $(i = 1, 2, \ldots, 11)$, of each equivalence class are listed below.*

$\mathfrak{W}_1:\ \langle \mathrm{e}^{-nt}\partial_t\rangle,$

$\mathfrak{W}_2:\ \langle \mathrm{e}^{-nt}\partial_t + \mathrm{e}^{-nt}[n+\frac{1}{2}n(n-1)\alpha\mathrm{e}^{-x}]\partial_x\rangle,$

$$\mathfrak{W}_3: \ \langle \mathrm{e}^{-nt+(n-1)x}[\mathrm{e}^{2x} - (n+1)\gamma \mathrm{e}^x + \frac{1}{2}n(n+1)\gamma^2](\mathrm{e}^x - \gamma)^{-n-1}\partial_t + \mathrm{e}^{-nt+(n-1)x}[n\mathrm{e}^x - \frac{1}{2}n(n+1)\gamma](\mathrm{e}^x - \gamma)^{-n}\partial_x\rangle,$$

$$\mathfrak{W}_4: \ L_0 = \partial_t,$$

$$L_1 = \mathrm{e}^{-t}\partial_t + \mathrm{e}^{-t}\partial_x,$$

$$L_{-1} = \mathrm{e}^{t}(1+\gamma\mathrm{e}^{-2x})\partial_t + \mathrm{e}^{t}(-1+\gamma\mathrm{e}^{-2x}+\mathrm{e}^{-x}\tilde{\phi})\partial_x,$$

$$L_2 = \mathrm{e}^{-2t}f(x,u)\partial_t + \mathrm{e}^{-2t}g(x,u)\partial_x,$$

$$L_{-2} = \mathrm{e}^{2t}[1+3\gamma\mathrm{e}^{-2x} - \frac{1}{2}\mathrm{e}^{-3x}(6\gamma\tilde{\phi}+\tilde{\phi}^3 \pm (4\gamma+\tilde{\phi}^2)^{3/2})]\partial_t + \mathrm{e}^{2t}[-2+3\mathrm{e}^{-x}\tilde{\phi}+6\gamma\mathrm{e}^{-2x} - \frac{1}{2}\mathrm{e}^{-3x}(6\gamma\tilde{\phi}+\tilde{\phi}^3 \pm (4\gamma+\tilde{\phi}^2)^{3/2})]\partial_x,$$

$$L_{n+1} = (1-n)^{-1}[L_1,\, L_n], \quad L_{-n-1} = (n-1)^{-1}[L_{-1},\, L_{-n}], \quad n \geq 2,$$

$$\mathfrak{W}_5: \ \langle \mathrm{e}^{-nt+(n-1)x}(\mathrm{e}^x \pm n)(\mathrm{e}^x \pm 1)^{-n}\partial_t + n\mathrm{e}^{-nt+(n-1)x}(\mathrm{e}^x \pm 1)^{1-n}\partial_x\rangle,$$

$$\mathfrak{W}_6: \ \langle \mathrm{e}^{-nt}\partial_t + \gamma\mathrm{e}^{-nt}[\mathrm{e}^{nx} - (\mathrm{e}^x-\gamma)^n](\mathrm{e}^x-\gamma)^{1-n}\partial_x\rangle,$$

$$\mathfrak{W}_7: \ L_0 = \partial_t,$$

$$L_1 = \mathrm{e}^{-t}\partial_t + \mathrm{e}^{-t}\partial_x,$$

$$L_{-1} = \mathrm{e}^{t}(1+\gamma\mathrm{e}^{-2x})\partial_t + \mathrm{e}^{t}(-1+\gamma\mathrm{e}^{-2x}+\mathrm{e}^{-x}\tilde{\phi})\partial_x,$$

$$L_2 = \mathrm{e}^{-2t+x}\frac{\mathrm{e}^x - \tilde{\phi}}{\mathrm{e}^{2x} - \mathrm{e}^x\tilde{\phi} - \gamma}\partial_t + \mathrm{e}^{-2t+x}\frac{2\mathrm{e}^x - \tilde{\phi}}{\mathrm{e}^{2x} - \mathrm{e}^x\tilde{\phi} - \gamma}\partial_x,$$

$$L_{-2} = \mathrm{e}^{2t-3x}(\mathrm{e}^{3x}+3\gamma\mathrm{e}^x - \gamma\tilde{\phi})\partial_t + \mathrm{e}^{2t-3x}(2\mathrm{e}^x - \tilde{\phi})(-\mathrm{e}^{2x}+\mathrm{e}^x\tilde{\phi}+\gamma)\partial_x,$$

$$L_{n+1} = (1-n)^{-1}[L_1,\, L_n], \quad L_{-n-1} = (n-1)^{-1}[L_{-1},\, L_{-n}], \quad n \geq 2,$$

$$\mathfrak{W}_8: \ \langle \mathrm{e}^{-nt}\partial_t + \mathrm{e}^{-nt}[n - \mathrm{sgn}(n)\frac{\gamma}{2}\sum_{j=1}^{|n|-1} j(j+1)\mathrm{e}^{-2x}]\partial_x\rangle,$$

$$\mathfrak{W}_9: \ \langle \frac{\mathrm{e}^{-nt+(n-1)x}}{(\mathrm{e}^x-1)^{n+2}}[(-1+\sum_{j=1}^{|n|-1}(2j+1))n + (2n+1)\mathrm{e}^x - (n+2)\mathrm{e}^{2x} + \mathrm{e}^{3x} + \mathrm{sgn}(n)\frac{\tilde{\phi}}{2}\sum_{j=1}^{|n|-1} j(j+1)]\partial_t + \frac{\mathrm{e}^{-nt+(n-1)x}}{(\mathrm{e}^x-1)^{n+1}}[(1-\sum_{j=1}^{|n|-1}(2j+1))n - 2n\mathrm{e}^x + n\mathrm{e}^{2x} - \mathrm{sgn}(n)\frac{\tilde{\phi}}{2}\sum_{j=1}^{|n|-1} j(j+1)]\partial_x\rangle,$$

$$\mathfrak{W}_{10}: \langle \mathrm{e}^{-nt}\partial_t + n\mathrm{e}^{-nt}\partial_x + \frac{\mathrm{sgn}(n)}{2}\sum_{j=1}^{|n|} j(j-1)\mathrm{e}^{-nt-2x}\partial_u\rangle,$$

$$\mathfrak{W}_{11}: \langle \mathrm{e}^{-nt}\partial_t + \mathrm{e}^{-nt}[n + \frac{\alpha n(n-1)}{2}\mathrm{e}^{-x}]\partial_x + \frac{n(n-1)}{2}\mathrm{e}^{-nt-x}\partial_u\rangle.$$

Here $n \in \mathbb{Z}$, $\alpha = 0, \pm 1$, $\gamma = \pm 1$, $\mathrm{sgn}(\cdot)$ is the standard sign function, the symbol $\tilde{\phi}(u)$ stands for either u or an arbitrary real constant c, and besides

$$\begin{aligned} f(x,u) =& \mathrm{e}^x[4\mathrm{e}^{4x} - 10\mathrm{e}^{3x}\tilde{\phi} - 36\gamma\mathrm{e}^{2x} + 2\mathrm{e}^x(31\gamma\tilde{\phi} + 6\tilde{\phi}^3 \pm 6(4\gamma + \tilde{\phi}^2)^{3/2}) \\ & - 64\gamma^2 - 54\gamma\tilde{\phi}^2 - 9\tilde{\phi}^4 \mp 9\tilde{\phi}(4\gamma + \tilde{\phi}^2)^{3/2}]r^{-1} \\ g(x,u) =& \mathrm{e}^x[8\mathrm{e}^{4x} - 16\mathrm{e}^{3x}\tilde{\phi} - 2\mathrm{e}^{2x}(44\gamma + 5\tilde{\phi}^2) + 2\mathrm{e}^x(44\gamma\tilde{\phi} + 9\tilde{\phi}^3 \pm 9(4\gamma + \tilde{\phi}^2)^{3/2}) \\ & - 64\gamma^2 - 54\gamma\tilde{\phi}^2 - 9\tilde{\phi}^4 \mp 9\tilde{\phi}(4\gamma + \tilde{\phi}^2)^{3/2}]r^{-1}, \\ r =& 4\mathrm{e}^{5x} - 10\mathrm{e}^{4x}\tilde{\phi} - 40\gamma\mathrm{e}^{3x} + 10\mathrm{e}^{2x}(6\gamma\tilde{\phi} + \tilde{\phi}^3 \pm (4\gamma + \tilde{\phi}^2)^{3/2}) - 10\mathrm{e}^x(6\gamma^2 + 6\gamma\tilde{\phi}^2 \\ & + \tilde{\phi}^4 \pm \tilde{\phi}(4\gamma + \tilde{\phi}^2)^{3/2}) + 30\gamma^2\tilde{\phi} + 20\gamma\tilde{\phi}^3 + 3\tilde{\phi}^5 \pm (2\gamma + 3\tilde{\phi}^2)(4\gamma + \tilde{\phi}^2)^{3/2}. \end{aligned}$$

Proof. To prove the theorem we need to construct all possible inequivalent extensions of the realizations (3.3) and (3.4).

Case 1. Inserting (3.3) into (2.4) we have

$$L_2 = \mathrm{e}^{-2t}\partial_t, \qquad L_{-2} = \mathrm{e}^{2t}\partial_t.$$

The remaining basis elements of the corresponding Witt algebra are readily obtained through recursion, which yields $L_n = \mathrm{e}^{-nt}\partial_t$, $n \in \mathbb{Z}$. Thus the realization $\mathfrak{W}_1$ is obtained.

Case 2. Turn now to realization (3.4). Inserting L_0, L_1, L_{-1} into the commutation relations $[L_0, L_{-2}] = 2L_{-2}$ and $[L_1, L_{-2}] = 3L_{-1}$ and solving the obtained PDEs give rise to the following form of L_{-2}

$$\begin{aligned} L_{-2} = & \mathrm{e}^{2t}[1 + 3\alpha\mathrm{e}^{-2x} + \psi_1(u)\mathrm{e}^{-3x}]\partial_t + \mathrm{e}^{2t}[-2 + 3\phi(u)\mathrm{e}^{-x} + \psi_2(u)\mathrm{e}^{-2x} \\ & + \psi_1(u)\mathrm{e}^{-3x}]\partial_x + \mathrm{e}^{2t}[3\beta\mathrm{e}^{-x} + \psi_3(u)\mathrm{e}^{-2x}]\partial_u, \end{aligned}$$

where ψ_1, ψ_2, ψ_3 are arbitrary smooth functions of u.

Utilizing commutation relations $[L_0, L_2] = -2L_2$ and $[L_{-1}, L_2] = -3L_1$ we arrive at

$$L_2 = \mathrm{e}^{-2t}f(x,u)\partial_t + \mathrm{e}^{-2t}g(x,u)\partial_x + \mathrm{e}^{-2t}h(x,u)\partial_u$$

with f, g, h satisfying the following system of PDEs:

$$-3(\alpha\mathrm{e}^{-2x} + 1)f + 2\alpha\mathrm{e}^{-2x}g + (\phi\mathrm{e}^{-x} + \alpha\mathrm{e}^{-2x} - 1)f_x + \beta\mathrm{e}^{-x}f_u + 3 = 0, \tag{3.5a}$$

$$(1 - \phi\mathrm{e}^{-x} - \alpha\mathrm{e}^{-2x})f + (\phi\mathrm{e}^{-x} - 2)g - \phi_u\mathrm{e}^{-x}h + (\phi\mathrm{e}^{-x} + \alpha\mathrm{e}^{-2x})g_x + \beta\mathrm{e}^{-x}g_u + 3 = 0, \tag{3.5b}$$

$$\beta\mathrm{e}^{-x}f - \beta\mathrm{e}^{-x}g + 2(1 + \alpha\mathrm{e}^{-2x})h - (\phi\mathrm{e}^{-x} + \alpha\mathrm{e}^{-2x} - 1)h_x - \beta\mathrm{e}^{-x}h_u = 0. \tag{3.5c}$$

Inserting above L_2 and L_{-2} into the commutation relation $[L_2, L_{-2}] = 4L_0$ gives three more PDEs

$$\begin{aligned} & 4(\psi_1\mathrm{e}^{-3x} + 3\alpha\mathrm{e}^{-2x} + 1)f - 3\mathrm{e}^{-2x}(\psi_1\mathrm{e}^{-x} + 2\alpha)g + \mathrm{e}^{-3x}\dot{\psi}_1 h \\ & - (\psi_1\mathrm{e}^{-3x} + \psi_2\mathrm{e}^{-2x} + 3\phi\mathrm{e}^{-x} - 2)f_x - \mathrm{e}^{-x}(\psi_3\mathrm{e}^{-x} + 3\beta)f_u - 4 = 0, \end{aligned}$$

$$2(\psi_1 e^{-3x}+\psi_2 e^{-2x}+3\phi e^{-x}-2)f-(\psi_1 e^{-3x}-2(3\alpha-\psi_2)e^{-2x}+3\phi e^{-x}-2)g \quad (3.6)$$
$$+e^{-x}(\dot\psi_1 e^{-2x}+\dot\psi_2 e^{-x}+3\dot\phi)h-(\psi_1 e^{-3x}+\psi_2 e^{-2x}+3\phi e^{-x}-2)g_x-e^{-x}(\psi_3 e^{-x}+3\beta)g_u=0,$$
$$2e^{-x}(\psi_3 e^{-x}+3\beta)f-e^{-x}(2\psi_3 e^{-x}+3\beta)g+(2\psi_1 e^{-3x}+(6\alpha+\dot\psi_3)e^{-2x}+2)h$$
$$-(\psi_1 e^{-3x}+\psi_2 e^{-2x}+3\phi e^{-x}-2)h_x-e^{-x}(\psi_3 e^{-x}+3\beta)h_u=0.$$

To determine the forms of L_2 and L_{-2}, we have to solve Eqs. (3.5) and (3.6). It is straightforward to verify that the relation

$$\Delta=e^{-t-4x}[\beta e^{3x}+\psi_3 e^{2x}+(\beta\psi_2-\phi\psi_3-3\alpha\beta)e^{x}+\beta\psi_1-\alpha\psi_3]\neq 0$$

is the necessary and sufficient condition for (3.5) and (3.6) to have a unique solution in terms of f_x, f_u, g_x, g_u, h_x and h_u. For this reason, we need to differentiate between the cases $\Delta=0$ and $\Delta\neq 0$.

Case 2.1. Let $\Delta=0$ or, equivalently, $\beta=\psi_3=0$. Eqs. (3.5) and (3.6) do not contain derivatives of the functions $f,\ g,\ h$ with respect to u.

Solving (3.5) and (3.6) with respect to the derivatives f_x, g_x, h_x we obtain two expressions for each of them. Equating the right-hand sides of equations containing h_x yields

$$he^{x}\frac{e^{4x}-2\phi e^{3x}-\psi_2 e^{2x}-2\psi_1 e^{x}+3\alpha^2+\phi\psi_1-\alpha\psi_2}{(e^{2x}-\phi e^{x}-\alpha)(2e^{3x}-3\phi e^{2x}-\psi_2 e^{x}-\psi_1)}=0.$$

Hence $h=0$.

Analyzing compatibility conditions for equations containing f_x and g_x we derive two more equations, which are linear in f and g. These equations form a system of two linear equations with non-vanishing determinant. Consequently, the system in question has a unique solution for f and g. Computing the derivatives of the expressions for f and g with respect to x and comparing the results with the previously obtained formulas for f_x and g_x, we arrive at the equations

$$(\psi_2-6\alpha)(\phi^3+\phi\psi_2+2\psi_1)e^{11x}+F_{10}[x,u]=0, \quad (3.7)$$

and

$$[10\phi^3\psi_1-3\alpha\phi^2(3\psi_2-8\alpha)+3\phi\psi_1(2\alpha+3\psi_2)+2(5\psi_1^2-4\alpha(2\alpha^2-3\psi_2+\psi_2^2))]e^{10x}$$
$$+F_9[x,u]=0. \quad (3.8)$$

Hereafter $F_n[x,u]$ with $n\in\mathbb{N}$ denotes an nth degree polynomial in $\exp(x)$.

To determine admissible forms of f and g we need to construct the most general ϕ and ψ_i $(i=1,2,3)$ satisfying Eqs. (3.7) and (3.8).

If (3.7) holds, then at least one of the equations $\psi_2=6\alpha$ and $\psi_1=-(\phi^3+\phi\psi_2)/2$ is identically satisfied.

Case 2.1.1. Given $\psi_2=6\alpha$, Eqs. (3.7) and (3.8) hold if and only if

$$16\alpha^3+3\alpha^2\phi^2-6\alpha\phi\psi_1-\phi^3\psi_1-\psi_1^2=0,$$

whence $\psi_1=[-6\alpha\phi-\phi^3\pm(4\alpha+\phi^2)^{\frac{3}{2}}]/2$. Choice of plus or minus in this formula leads to different realizations. So we need to consider those cases separately.

Case 2.1.1.1. Let $\psi_1 = [-6\alpha\phi - \phi^3 - (4\alpha + \phi^2)^{\frac{3}{2}}]/2$. If $\alpha = 0$, then we have either $\psi_1 = 0$ or $\psi_1 = -\phi^3$.

The case $\alpha = \psi_1 = 0$ gives $L_{-1} = \mathrm{e}^t\partial_t + \mathrm{e}^t(-1 + \mathrm{e}^{-x}\phi)\partial_x$. Making equivalence transformation $\tilde{x} = x + X(u)$, we can reduce ϕ to one of the three forms $a = 0, \pm 1$ and get

$$f = 1, \qquad g = 2 + a\mathrm{e}^{-x}.$$

Utilizing the recurrence relations we derive the remaining basis elements and obtain the realization $\mathfrak{W}_2$.

Provided $\alpha = 0$ and $\psi_1 = -\phi^3$, we reduce the function ϕ to the form $b = 0, \pm 1$ by applying the equivalence transformation $\tilde{x} = x + X(u)$. The case $b \neq 0$ gives rise to the following f and g:

$$f = \frac{\mathrm{e}^x(\mathrm{e}^{2x} - 3b\mathrm{e}^x + 3b^2)}{(\mathrm{e}^x - b)^3}, \qquad g = \frac{\mathrm{e}^x(2\mathrm{e}^x - 3b)}{(\mathrm{e}^x - b)^2}.$$

Hence the realization $\mathfrak{W}_3$ is obtained. Note that the case $b = 0$ leads to the particular case of $\mathfrak{W}_2$.

Assuming $\alpha = \pm 1$, we get $\psi_1 = [-6\alpha\phi - \phi^3 - (4\alpha + \phi^2)^{\frac{3}{2}}]/2$, which yields $\mathfrak{W}_4$.

Case 2.1.1.2. Let $\psi_1 = [-6\alpha\phi - \phi^3 + (4\alpha + \phi^2)^{\frac{3}{2}}])/2$. Analysis similar to the one applied to the Case 2.1.1.1 gives the realizations $\mathfrak{W}_2$ and $\mathfrak{W}_4$.

This completes analysis of the Case 2.1.1.

Case 2.1.2. If $\psi_1 = -(\phi^3 + \phi\psi_2)/2$, then Eq. (3.7) takes the form

$$(4\alpha + \phi^2)[\psi_2 - (4\alpha - 5\phi^2)/4][\psi_2 - (2\alpha - \phi^2)]\mathrm{e}^{10x} + F_9[x, u] = 0.$$

Constructing the general solution of this equation boils down to analysis of the following three subcases.

Case 2.1.2.1. Given the relation $\psi_2 = (4\alpha - 5\phi^2)/4$, Eqs. (3.7) and (3.8) hold if and only if

$$4\alpha + \phi^2 = 0.$$

Consequently $\alpha \leq 0$.

In the case when $\alpha < 0$ we choose $\alpha = -b^2$ so that $b = \pm(-\alpha)^{\frac{1}{2}}$ and $\phi = 2b$. Solving (3.7) and (3.8), we immediately get the expression for f

$$f = \frac{\mathrm{e}^x(\mathrm{e}^x - 2b)}{(\mathrm{e}^x - b)^2}, \qquad g = \frac{2\mathrm{e}^x}{\mathrm{e}^x - b},$$

which leads to $\mathfrak{W}_5$.

If $\alpha = 0$ and $\phi = \psi_1 = \psi_2 = 0$, then the realization $\mathfrak{W}_2$ with $\alpha = 0$ is obtained.

Case 2.1.2.2. Let $\psi_2 = 2\alpha - \phi^2$. Provided $\alpha = 0$, we can choose ϕ to be $b = \pm 1$ without any loss of generality. We drop the case $b = 0$ since it has already been considered. Taking into account the above relations we get from Eqs. (3.7) and (3.8)

$$f = 1, \quad g = \frac{2\mathrm{e}^x - b}{\mathrm{e}^x - b}.$$

The realization $\mathfrak{W}_6$ is obtained.

Provided $\alpha \neq 0$, we can apply an equivalence transformation to get $\alpha = \pm 1$, whence

$$f = \frac{e^x(e^x - \phi)}{e^{2x} - e^x\phi - b}, \qquad g = \frac{e^x(2e^x - \phi)}{e^{2x} - e^x\phi - b}$$

with $b = \pm 1$. Since ϕ can be reduced to the form $\tilde{u}$ by the equivalence transformation $\tilde{u} = \phi$ with $\dot{\phi} \neq 0$, we thus get the realization $\mathfrak{W}_7$.

Case 2.1.2.3. If $4\alpha + \phi^2 = 0$ then using Eqs. (3.7) and (3.8) we get $\alpha \leq 0$, whence $\alpha = 0, -1$.

Given the relation $\alpha = 0$, we can reduce ϕ to the form $a = 0, \pm 1$. Inserting these expressions into (3.7), (3.8) yields $f = 1$ and $g = 2 - ae^{-x}$. The realization $\mathfrak{W}_8$ is obtained.

In the case when $\alpha = -1$, we have

$$f = \frac{e^x(e^{3x} - 4e^{2x} + 5e^x + 4 + \psi_2)}{(e^x - 1)^4}, \quad g = \frac{e^x(2e^{2x} - 4e^x - 4 - \psi_2)}{(e^x - 1)^3}.$$

And furthermore, the function ψ_2 is reduced to the form $\tilde{u}$ by equivalence transformation $\tilde{u} = \psi_2$, provided ψ_2 is a non-constant function. As a result, we get $\mathfrak{W}_9$.

This completes analysis of Case 2.1. In summary we conclude that the case $\Delta = 0$ leads to the realizations $\mathfrak{W}_i$, $i = 2, 3, \ldots, 9$.

Case 2.2. Suppose now that $\Delta \neq 0$, or equivalently, $\beta^2 + \psi_3^2 \neq 0$. Given this condition we can solve Eqs. (3.5) and (3.6) with respect to f_x, f_u, g_x, g_u, h_x and h_u. The system obtained is overdetermined, so we need to analyze its compatibility.

The compatibility conditions

$$f_{xu} - f_{ux} = 0, \quad g_{xu} - g_{ux} = 0, \quad h_{xu} - h_{ux} = 0$$

can be rearranged into the system of three linear equations for the functions f, g and h

$$a_1 f + a_2 g + a_3 h + d_1 = 0,$$

$$b_1 f + b_2 g + b_3 h + d_2 = 0,$$

$$c_1 f + c_2 g + c_3 h + d_3 = 0.$$

Here a_i, b_i, c_i, d_i, $(i = 1, 2, 3)$ are functions of t, x, ϕ, ψ_1, ψ_2 and ψ_3.

It is straightforward to verify that the above system has a unique solution f, g, h when $\beta^2 + \psi_3^2 \neq 0$. We have solved the system in question using *Mathematica* and obtained very cumbersome formulas. For brevity, we do not exhibit them here. Inserting the expressions for f, g, h into Eq. (3.5a) yields

$$\alpha\beta^6 e^{42x} + F_{41}[x, u] = 0.$$

Consequently, we have either $\beta = 0$ or $\alpha = 0$.

Case 2.2.1. If $\beta = 0$, then Eq. (3.5a) takes the form

$$\alpha\psi_3^6 e^{36x} + F_{35}[x, u] = 0,$$

which gives $\alpha = 0$ and $\psi_3 \neq 0$ (since otherwise $\Delta = 0$). Now we can rewrite Eq. (3.6) as follows

$$\psi_1 \psi_3^6 e^{36x} + F_{35}[x,u] = 0,$$
$$(15\phi^2 + 2\psi_2)\psi_3^6 e^{37x} + F_{36}[x,u] = 0,$$
$$(57\phi^2 - 2\psi_2)\psi_3^7 e^{35x} + F_{34}[x,u] = 0.$$

Hence we conclude that $\phi = \psi_1 = \psi_2 = 0$. Inserting these formulas into the initial Eqs. (3.5) and (3.6) and solving the obtained system yield

$$f = 1,\ g = 2,\ h = -e^{-2x}\psi_3.$$

The function ψ_3 can be reduced to -1 by the equivalence transformation $\tilde{u} = -\int 1/\psi_3 du$. As a result, we get the realization $\mathfrak{W}_{10}$.

Case 2.2.2. Provided $\alpha = 0$, Eq. (3.5c) takes the form

$$\beta^5(4\beta\phi\psi_3 - 6\psi_3^2 + \beta^2\dot{\psi}_3)e^{41x} + 30\beta^5\phi\psi_3^2 e^{40x} + F_{39}[x,u] = 0.$$

Since the case $\alpha = \beta = 0$ has already been analyzed in Case 2.2.1, we can restrict our considerations to the cases $\psi_3 = 0$, $\beta = 1$ and $\phi = 0$, $\beta = 1$ without any loss of generality.

If $\psi_3 = 0$ and $\beta = 1$, then it follows from (3.5c) and (3.6) that $\psi_1 = \psi_2 = 0$. Thus

$$f = 1,\ g = 2 + e^{-x}\phi,\ h = e^{-x}.$$

Furthermore, the function ϕ can be reduced to $0, 1$ or -1 by the equivalence transformations $\tilde{x} = x + X(u)$ and $\tilde{u} = U(u)$. Hence we get the realization $\mathfrak{W}_{11}$.

In the case when $\phi = 0$ and $\beta = 1$, Eqs. (3.5) and (3.6) are incompatible. This completes analysis of the Case 2.2.

We check by direct computation that the realizations $\mathfrak{W}_i$ $(i = 1, 2, \ldots, 11)$ cannot be mapped into one another by any equivalence transformation. Consequently, they are inequivalent. □

While proving Theorem 3.1, we have also obtained the exhaustive description of the Witt algebras over the spaces $\mathbb{R}^1$ and $\mathbb{R}^2$.

Theorem 3.2. $\mathfrak{W}_1$ *is the only inequivalent realization of the Witt algebra over the space* $\mathbb{R}$.

Theorem 3.3. *The realizations* $\mathfrak{W}_1$–$\mathfrak{W}_9$ *with* $\tilde{\phi} = c \in \mathbb{R}$ *exhaust the list of inequivalent realizations of the Witt algebra over the space* $\mathbb{R}^2$.

4. Realizations of the Virasoro Algebra

After obtaining the full list of inequivalent realizations of the Witt algebra we can proceed to classification of realizations of the Virasoro algebra $\mathfrak{V}$. To this end we need to extend the inequivalent realizations of the Witt algebra listed in Theorem 3.1 by all possible nonzero central elements C. In this section we prove that realizations of the Virasoro algebra with nonzero central element over the space $\mathbb{R}^3$ do not exist. We describe the major elements of the proof skipping cumbersome intermediate calculations.

Let us begin by constructing all possible central extensions of the subalgebra $\langle L_0, L_1, L_{-1}\rangle$. According to Lemma 3.1 it suffices to consider the algebras (3.3) and (3.4) only.

Case 1. Given the realization (3.3) we have

$$L_0 = \partial_t,\ L_1 = \mathrm{e}^{-t}\partial_t,\ L_{-1} = \mathrm{e}^{t}\partial_t.$$

Choosing the basis element C in the general form (2.1) and inserting it into the commutation relations $[L_i, C] = 0$, $(i = 0, 1, -1)$ yield

$$C = \xi(x,u)\partial_x + \eta(x,u)\partial_u, \quad \xi^2 + \eta^2 \neq 0.$$

Applying the transformation

$$\tilde{t} = t, \quad \tilde{x} = X(x,u), \quad \tilde{u} = U(x,u),$$

which preserves L_0, L_1 and L_{-1}, to the central element C we get

$$C \to \widetilde{C} = (\xi X_x + \eta X_u)\partial_{\tilde{x}} + (\xi U_x + \eta U_u)\partial_{\tilde{u}}.$$

Choosing solutions of the equations

$$\xi X_x + \eta X_u = 0, \quad \xi U_x + \eta U_u = 1$$

as X and U yields $C = \partial_u$.

We now turn to the basis element L_2. Using the commutation relations $[L_0, L_2] = -2L_2$, $[L_{-1}, L_2] = -3L_1$ and $[L_2, C] = 0$ yields $L_2 = \mathrm{e}^{-2t}\partial_t$.

Inserting the basis element L_{-2} of the general form (2.1) into (2.4), which involve L_{-2}, results in incompatible over-determined system of PDEs for the functions τ, ξ and η. Hence the realization (3.3) cannot be extended to a realization of the Virasoro algebra with nonzero central element.

Case 2. Consider now the algebra (3.4). Since C commutes with L_0 and L_1, we have

$$C = \mathrm{e}^{-x}f(u)\partial_t + [g(u) + \mathrm{e}^{-x}f(u)]\partial_x + h(u)\partial_u,$$

where f, g and h are arbitrary smooth functions.

Acting by transformation (3.1), which does not alter L_0 and L_1, on C gives

$$\widetilde{C} = \mathrm{e}^{-x}f(u)\partial_{\tilde{t}} + [g(u) + \mathrm{e}^{-x}f(u) + h(u)\dot{X}(u)]\partial_{\tilde{x}} + h(u)\dot{U}(u)\partial_{\tilde{u}}.$$

To further simplify $\widetilde{C}$, we need to analyze the cases $f(u) \neq 0$ and $f(u) = 0$ separately.

If $f(u) \neq 0$, then choosing $X(u) = -\ln|f(u)|$ we get $\widetilde{C} = \mathrm{e}^{-\tilde{x}}\partial_{\tilde{t}} + [\mathrm{e}^{-\tilde{x}} + \beta(g + h\dot{X})]\partial_{\tilde{x}} + \beta h\dot{U}\partial_{\tilde{u}}$, where $\beta = \pm 1$.

Provided $h = 0$ and $\dot{g} \neq 0$, we can make the equivalence transformation $\tilde{u} = g(u)$ and thus get $C_1 = \mathrm{e}^{-x}\partial_t + (\mathrm{e}^{-x} + u)\partial_x$. The case $h = \dot{g} = 0$ leads to $C_2 = \mathrm{e}^{-x}\partial_t + (\mathrm{e}^{-x} + \lambda)\partial_x$, where λ is an arbitrary constant.

Next, if the condition $h \neq 0$ holds, we choose the functions X and U satisfying $g + h\dot{X} = 0$ and $h\dot{U} = 1/\beta$ thus getting $C_3 = \mathrm{e}^{-x}\partial_t + \mathrm{e}^{-x}\partial_x + \partial_u$.

Given $f(u) = 0$ we have $\widetilde{C} = (g + h\dot{X})\partial_{\tilde{x}} + h\dot{U}\partial_{\tilde{u}}$.

If $h \neq 0$, we can reduce $\widetilde{C}$ to the form $C_4 = \partial_u$ by a suitable choice of X and U. The case when h vanishes yields $\widetilde{C} = g\partial_{\tilde{x}}$, which can be simplified to one of the following inequivalent forms, $C_5 = u\partial_x$ or $C_6 = \partial_x$.

Therefore there exist six inequivalent nonzero central element C commuting with basis elements $L_0 = \partial_t$ and $L_1 = \mathrm{e}^{-t}\partial_t + \mathrm{e}^{-t}\partial_x$.

The next step is extending the realizations $\langle L_0, L_1, C_i \rangle$, $(i = 1, 2, \ldots, 6)$ up to realizations of the full Virasoro algebra. Here we present the calculation details for the case $i = 1$ only. The remaining five cases are handled in a similar fashion.

We begin by constructing all possible realizations of L_{-1}. Taking into account (2.3), we have

$$L_{-1} = \frac{e^{t-2x}(u^2 e^{2x} - 1)}{u^2}\partial_t - \frac{e^{t-2x}(ue^x + 1)^2}{u^2}\partial_x.$$

With L_{-1} in hand we can proceed to constructing L_2. Using the commutation relations (2.4) yields

$$L_2 = \frac{ue^x(ue^x + 2)}{e^{2t}(ue^x + 1)^2}\partial_t + \frac{2ue^x}{e^{2t}(ue^x + 1)}\partial_x.$$

Inserting the obtained expressions into the commutation relations for L_{-2}, we arrive at an incompatible system of PDEs. Hence, the algebra $\langle L_0, L_1, C_1 \rangle$ cannot be extended to a realization of the full Virasoro algebra.

The same statement is true for the remaining C_i, $(i = 2, 3, \ldots, 6)$.

Theorem 4.1. *There are no realizations of the Virasoro algebra with nonzero central element C over the space* $\mathbb{R}^n$, $(n = 1, 2, 3)$.

5. PDEs Invariant under the Witt Algebras

In this section, we construct a number of new nonlinear (1+1)-dimensional second-order PDEs whose invariance algebra contains the Witt algebra and is consequently infinite-dimensional.

Constructing invariant equations is a straightforward application of Lie's infinitesimal method (see, e.g., [45, 48]). Below we give a brief description of the method and present an example of derivation of PDE invariant under $\mathfrak{W}_1$.

A second-order PDE of the form

$$F(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0$$

is invariant with respect to the Witt algebra $\langle L_n \rangle$ if and only if the condition

$$\mathrm{pr}^{(2)} L_n(F)|_{F=0} = 0$$

holds for any $n \in \mathbb{N}$, where $\mathrm{pr}^{(2)} L_n$ is the second-order prolongation of the vector field L_n, that is

$$\mathrm{pr}^{(2)} L_n = L_n + \eta^t \partial_{u_t} + \eta^x \partial_{u_x} + \eta^{tt} \partial_{u_{tt}} + \eta^{tx} \partial_{u_{tx}} + \eta^{xx} \partial_{u_{xx}}$$

with

$$\eta^t = D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),$$

$$\eta^x = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),$$

$$\eta^{tt} = D_t(\eta^t) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi),$$

$$\eta^{tx} = D_x(\eta^t) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi),$$

$$\eta^{xx} = D_x(\eta^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi).$$

Here the symbols D_t and D_x stand for the total differentiation operators with respect to t and x respectively, namely,

$$D_t = \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{xt}\partial_{u_x} + \cdots,$$

$$D_x = \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \cdots.$$

Let us apply the above method to derivation of PDEs admitting the realization $\mathfrak{W}_1$. The second-order prolongation of the basis elements of $\mathfrak{W}_1$ reads

$$\mathrm{pr}^{(2)}L_n = \mathrm{e}^{-nt}\partial_t + n\mathrm{e}^{-nt}u_t\partial_{u_t} + (2n\mathrm{e}^{-nt}u_{tt} - n^2\mathrm{e}^{-nt}u_t)\partial_{u_{tt}} + n\mathrm{e}^{-nt}u_{tx}\partial_{u_{tx}}. \tag{5.1}$$

The next step is computing the complete set of functionally-independent second-order differential invariants $I_m(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx})$ $(m = 1,2,\ldots,7)$ of the infinite set of first-order differential operators L_n. Integrating the characteristic equations

$$\frac{\mathrm{d}t}{\mathrm{e}^{-nt}} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}u}{0} = \frac{\mathrm{d}u_t}{n\mathrm{e}^{-nt}u_t} = \frac{\mathrm{d}u_x}{0} = \frac{\mathrm{d}u_{tt}}{2n\mathrm{e}^{-nt}u_{tt} - n^2\mathrm{e}^{-nt}u_t} = \frac{\mathrm{d}u_{tx}}{n\mathrm{e}^{-nt}u_{tx}} = \frac{\mathrm{d}u_{xx}}{0},$$

which correspond to the operator L_n with arbitrary n, gives the following invariants

$$I_1 = x,\ I_2 = u,\ I_3 = u_x,\ I_4 = u_{xx},\ I_5 = \frac{u_{tx}}{u_t},\ I_6 = \mathrm{e}^{-nt}u_t,\ I_7 = \mathrm{e}^{-2nt}u_{tt} - n\mathrm{e}^{-2nt}u_t.$$

Thus the most general L_n-invariant equation is of the form

$$F(I_1, I_2, \ldots, I_7) = 0.$$

Since this equation should be invariant under every basis element of the Witt algebra $\mathfrak{W}_1$, it should be independent of n. To meet this requirement the function F has to be independent of I_6 and I_7. Consequently, the most general $\mathfrak{W}_1$ invariant second-order PDE is

$$F(I_1, I_2, I_3, I_4, I_5) = 0,$$

or, equivalently,

$$F\left(x, u, u_x, u_{xx}, \frac{u_{tx}}{u_t}\right) = 0.$$

Furthermore, we have succeeded in constructing the general forms of PDEs invariant under $\mathfrak{W}_2$, $\mathfrak{W}_6$, $\mathfrak{W}_8$ and $\mathfrak{W}_{10}$. The corresponding invariant equations are listed in Table 1, where F is an arbitrary smooth real-valued function.

6. The Direct Sums of the Witt Algebras

This section is devoted to the classification of realizations of the direct sum of the Witt algebras in $\mathbb{R}^3$. We obtain a complete description of inequivalent realizations of the direct sum of two Witt algebras.

In view of Theorem 3.1, it suffices to consider realizations of the form

$$\mathfrak{W}_i \oplus \langle \tilde{L}_n,\ n \in \mathbb{Z} \rangle, \qquad i = 1,2,\ldots,11,$$

where $\mathfrak{W}_i$ are given in Theorem 3.1 and $\tilde{L}_n$, $n \in \mathbb{Z}$ are basis elements of the Witt algebra commuting with the realization $\mathfrak{W}_i$.

Table 1. Second-order PDEs admitting the Witt algebra

Symmetry algebra	Invariant equation
$\mathfrak{W}_1$	$F(x,u,u_x,u_{xx},\frac{u_{tx}}{u_t})=0$
$\mathfrak{W}_2$	$F(u,u_x,u_{xx},\mathrm{e}^{-x}(u_tu_{xx}-u_xu_{tx}))=0, \quad \alpha=0$ $F(u,\frac{u_{xx}-u_x}{u_x^2},\frac{u_tu_x-u_tu_{xx}+u_xu_{tx}+u_x^2}{\mathrm{e}^xu_x}-2\alpha u_x)=0, \ \ \alpha=\pm1$
$\mathfrak{W}_6$	$F(u,\frac{\mathrm{e}^x(u_x+u_{xx})-\gamma(u_{xx}+u_{tx})}{u_x(\mathrm{e}^xu_x-\gamma(u_t+u_x))})=0$
$\mathfrak{W}_8$	$F(u,\frac{u_{xx}-2u_x}{u_x^2})=0$
$\mathfrak{W}_{10}$	$F(u_x+2u,u_{xx}-4u)=0$

Consider first realizations of direct sum of the Witt algebras $\mathfrak{W}_1\oplus\langle\tilde{L}_n\rangle$. Taking into account that $\tilde{L}_n$ commutes with basis elements of $\mathfrak{W}_1$, we conclude that

$$\tilde{L}_n=f_n(x,u)\partial_x+g_n(x,u)\partial_u. \tag{6.1}$$

Here f_n and g_n are arbitrary smooth functions.

Now we can utilize the results of classification of inequivalent realizations of the Witt algebra over $\mathbb{R}^2$ and get the final form of the basis elements $\tilde{L}_n$ from Theorem 3.3 by replacing t, x with x, u respectively.

The remaining realizations $\mathfrak{W}_i$ $(i=2,\ldots,11)$ are analyzed analogously. We skip intermediate computations and present the final result.

Theorem 6.1. *Any realization of the direct sum of two Witt algebras in $\mathbb{R}^3$ is equivalent to one of the following ten inequivalent realizations:*

$$\mathfrak{D}_1: \quad \langle\mathrm{e}^{-mt}\partial_t\rangle\oplus\langle\mathrm{e}^{-nx}\partial_x\rangle,$$

$$\mathfrak{D}_2: \quad \langle\mathrm{e}^{-mt}\partial_t\rangle\oplus\langle\mathrm{e}^{-nx}\partial_x+n\mathrm{e}^{-nx}\partial_u\rangle,$$

$$\mathfrak{D}_3: \quad \langle\mathrm{e}^{-mt}\partial_t+m\mathrm{e}^{-mt}\partial_x\rangle\oplus\langle n\mathrm{e}^{-nu}\partial_x+\mathrm{e}^{-nu}\partial_u\rangle,$$

$$\mathfrak{D}_4: \quad \langle\mathrm{e}^{-mt}\partial_t\rangle\oplus\langle\mathrm{e}^{-nx}\partial_x+\gamma\mathrm{e}^{-nx}[\mathrm{e}^{nu}-(\mathrm{e}^u-\gamma)^n](\mathrm{e}^u-\gamma)^{1-n}\partial_u\rangle,$$

$$\mathfrak{D}_5: \quad \langle\mathrm{e}^{-mt}\partial_t\rangle\oplus\langle\mathrm{e}^{-nx}\partial_x+\mathrm{e}^{-nx}[n-\mathrm{sgn}(n)\frac{\gamma}{2}\sum_{j=1}^{|n|-1}j(j+1)\mathrm{e}^{-2u}]\partial_u\rangle,$$

$$\mathfrak{D}_6: \quad \langle\mathrm{e}^{-mt}\partial_t\rangle\oplus\langle\mathrm{e}^{-nx+(n-1)u}(\mathrm{e}^u\pm n)(\mathrm{e}^u\pm1)^{-n}\partial_x+n\mathrm{e}^{-nx+(n-1)u}(\mathrm{e}^u\pm1)^{1-n}\partial_u\rangle,$$

$$\begin{aligned}\mathfrak{D}_7: \quad &\langle\mathrm{e}^{-mt}\partial_t\rangle\oplus\langle\mathrm{e}^{-nx+(n-1)u}[\mathrm{e}^{2u}-(n+1)\gamma\mathrm{e}^u+\frac{1}{2}n(n+1)](\mathrm{e}^u-\gamma)^{-n-1}\partial_x\\ &\quad+\mathrm{e}^{-nx+(n-1)u}[n\mathrm{e}^u-\frac{1}{2}n(n+1)\gamma](\mathrm{e}^u-\gamma)^{-n}\partial_u\rangle,\end{aligned}$$

$$\mathfrak{D}_8: \quad \langle\mathrm{e}^{-mt}\partial_t\rangle\oplus\langle J_1\partial_x+J_2\partial_u\rangle,$$

$$\mathfrak{D}_9: \quad \langle\mathrm{e}^{-mt}\partial_t\rangle\oplus\widetilde{\mathfrak{W}}_4,$$

$$\mathfrak{D}_{10}: \quad \langle\mathrm{e}^{-mt}\partial_t\rangle\oplus\widetilde{\mathfrak{W}}_7.$$

Here

$$J_1 = \frac{e^{-nx+(n-1)u}}{(e^u-1)^{n+2}}[(-1+\sum_{j=1}^{|n|-1}(2j+1))n+(2n+1)e^u-(n+2)e^{2u}+e^{3u}+\mathrm{sgn}(n)\frac{c}{2}\sum_{j=1}^{|n|-1}j(j+1)],$$

$$J_2 = \frac{e^{-nx+(n-1)u}}{(e^u-1)^{n+1}}[(1-\sum_{j=1}^{|n|-1}(2j+1))n-2ne^u+ne^{2u}-\mathrm{sgn}(n)\frac{c}{2}\sum_{j=1}^{|n|-1}j(j+1)],$$

n, $m\in\mathbb{Z}$, $c\in\mathbb{R}$ and the symbols $\widetilde{\mathfrak{W}}_4$ and $\widetilde{\mathfrak{W}}_7$ stand for the realizations obtained from $\mathfrak{W}_4$ and $\mathfrak{W}_7$ listed in Theorem 3.3 by replacing (t,x) with (x,u).

When the paper has been submitted for publication, one of the referees drew the authors' attention to the paper by Medolaghi [44]. Medolaghi had constructed fourteen types of infinite-parameter groups of point transformations over $\mathbb{R}^3$ and obtained corresponding invariant PDEs.

The majority of the realizations listed in Theorems 3.1–3.3 can be obtained from those given in in Sec. 4 of [44]. For example, the algebra $\langle\xi(t)\partial_t\rangle$ reduces to $\mathfrak{W}_1$ under $\xi(t)=e^{-nt}$. Additionally, the realization $\mathfrak{W}_2$ with $\alpha=0$ can be derived from $\langle\xi(t)\partial_t+\dot{\xi}(t)\partial_x\rangle$ given in [44] by choosing $\xi(t)=e^{-nt}$ and $x\to -x$. The realizations $\mathfrak{W}_4$, $\mathfrak{W}_7$ and $\mathfrak{W}_9$ are, to the best of our knowledge, new.

Medolaghi had not considered the problem of classifying PDEs invariant under sums of infinite-dimensional algebras. However, some of the realizations listed in Theorem 6.1 can be represented as the direct sums of infinite-dimensional algebras given in Sec. 4 of the paper [44]. For example, the realizations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ can be obtained from $\langle\xi_1(t)\partial_t\rangle\oplus\langle\xi_2(x)\partial_x\rangle$ and $\langle\xi_1(t)\partial_t\rangle\oplus\langle\xi_2(x)\partial_x+\dot{\xi}_2(x)\partial_u\rangle$ by choosing $\xi_1(t)=e^{-mt}$ and $\xi_2(x)=e^{-nx}$, respectively. The realization $\mathfrak{D}_3$ is a direct sum of $\langle\xi_1(t)\partial_t+\dot{\xi}_1(t)\partial_x\rangle$ and $\langle\xi_2(u)\partial_u+\dot{\xi}_2(u)\partial_x\rangle$ with $\xi_1(t)=e^{-mt}$, $\xi_2(u)=e^{-nu}$, $x\to -x$. The realizations $\mathfrak{D}_7$–$\mathfrak{D}_{10}$ are new.

Analysis of the second-order PDEs invariant under the direct sum of the Witt algebras shows that the determining equations are incompatible for the realizations $\mathfrak{D}_4$, $\mathfrak{D}_5$, $\mathfrak{D}_7$–$\mathfrak{D}_{10}$. The remaining four realizations give rise to four classes of equations admitting symmetry algebras which are direct sums of two Witt algebras.

$$\mathfrak{D}_1:\quad F\left(u,\frac{u_{tx}}{u_tu_x}\right)=0, \tag{6.2}$$

$$\mathfrak{D}_2:\quad F\left(\frac{u_{tx}}{u_t}e^{-u}\right)=0, \tag{6.3}$$

$$\mathfrak{D}_3:\quad F\left(\frac{u_tu_{xx}-u_xu_{tx}}{u_x^3}e^{-x}\right)=0, \tag{6.4}$$

$$\mathfrak{D}_6:\quad F\left(\frac{u_{tx}(1-u_x\pm e^u)+u_t(u_{xx}-u_x^2+u_x)}{u_t[e^{2u}+(u_x-1)(u_x-1\mp 2e^u)]}\right)=0. \tag{6.5}$$

Here F is an arbitrary smooth real-valued function.

Let us reiterate, *any* second-order PDE, in two independent variables, which is invariant under the direct sum of two Witt algebras, is equivalent to one of the equations (6.2)–(6.5).

We refer to these equations as integrable, since they admit infinite-dimensional symmetry algebras involving two arbitrary functions. This is similar to the concept of integrable solitonic equations, since an overwhelming majority of these equations admit infinite generalized symmetry [27].

One more analogy comes from the theory of Hamitonian systems, which are completely integrable provided a sufficient number of integrals are known [45].

Applying the results by Bluman and Kumei [7, 36] to an integrable PDE one can, in theory, linearize these PDEs. This fact justifies usage of the term *integrable*. Note that integrability in this sense is closely related to the concept of C-integrability by Calogero [9]. Nonlinear PDE is called C-integrable if it can be linearized by a local or nonlocal change of variables.

Eq. (6.2) can be rewritten in the equivalent form

$$u_{tx} = f(u)u_t u_x.$$

By making the change of variable $u = R(U)$, where $R(U)$ is an arbitrary solution of the ordinary differential equation

$$f(R)R'^2 - R'' = 0,$$

we reduce the above nonlinear PDE to the linear wave equation $U_{tx} = 0$.

Without any loss of generality we can rewrite (6.3) as follows

$$u_{tx} = \lambda \mathrm{e}^u u_t, \quad \lambda \in \mathbb{R}.$$

Integrating it with respect to t gives

$$u_x = \lambda \mathrm{e}^u + \frac{g''(x)}{g'(x)},$$

where g is an arbitrary smooth non-constant function of x. We rewrite the above equation in the form

$$(u - \ln g'(x))_x = \lambda \mathrm{e}^{(u-\ln g'(x))}\mathrm{e}^{\ln g'(x)}.$$

Integrating this equation yields the general solution

$$u(t,x) = \ln \frac{g'(x)}{h(t) - \lambda g(x)}$$

of the initial nonlinear PDE (6.3).

Eq. (6.4) is equivalent to the equation

$$u_t u_{xx} - u_x u_{tx} = \lambda \mathrm{e}^x u_x^3, \quad \lambda \in \mathbb{R}.$$

The hodograph transformation $x \to u, u \to x$ followed by re-scaling $t \to \lambda t$ turns it into the Liouville equation (1.1), which is known to be integrable.

To the best of our knowledge, Eq. (6.5) is a new integrable nonlinear equation. The simplest PDE from the class of equations (6.5) is of the form

$$u_{tx}(1 - u_x \pm \mathrm{e}^u) + u_t(u_{xx} - u_x^2 + u_x) = 0.$$

7. Concluding Remarks

In this paper, we have classified all possible inequivalent realizations of the Witt and Virasoro algebras by Lie vector fields over the space $\mathbb{R}^n$ with $n = 1,2,3$. The complete lists of these realizations are given in Theorems 3.1–3.3, 4.1 and 6.1.

The main results can be briefly summarized as follows:

- There exists only one inequivalent realization of the Witt algebra in $\mathbb{R}$.
- There are nine inequivalent realizations of the Witt algebra in $\mathbb{R}^2$.
- There exist eleven inequivalent realizations of the Witt algebra in $\mathbb{R}^3$ space.
- There are no realizations of the Virasoro algebra with nonzero central element over the space $\mathbb{R}^n$ with $n \leq 3$.
- There exist ten inequivalent realizations of the direct sum of two Witt algebras in $\mathbb{R}^3$.

As an application, we construct a number of nonlinear PDEs which are invariant under various realizations of the Witt algebra. This enables constructing broad classes of new nonlinear equations whose symmetry algebra involves, at least, one arbitrary function of one variable.

We have constructed all second-order PDEs in two independent variables whose invariance algebras contain a direct sum of the Witt algebras. As a result we get four canonical invariant equations (6.2)–(6.5). Each of these admits infinite-dimensional algebra involving two arbitrary functions. The massless wave and Liouville equations in (1+1)-dimensions are typical examples of such PDEs. They are particular cases of Eqs. (6.2)–(6.4), which are well-known, while model (6.5) is new.

Since the Virasoro algebra is a subalgebra of the Kac-Moody-Virasoro algebra, the results obtained here can be directly applied to classify the integrable KP type equations in $(1+2)$ dimensions. The starting point would be describing inequivalent realizations of the Kac-Moody-Virasoro algebras by differential operators over $\mathbb{R}^4$. This problem is in progress now and will be reported elsewhere.

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