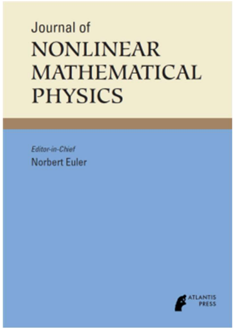

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LETTER TO THE EDITOR

Connection between the ideals generated by traces and by supertraces in the superalgebras of observables of Calogero models

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If G is a finite Coxeter group, then symplectic reflection algebra $H := H_{1,\eta}(G)$ has Lie algebra $\mathfrak{sl}_2$ of inner derivations and can be decomposed under spin: $H = H_0 \oplus H_{1/2} \oplus H_1 \oplus H_{3/2} \oplus \dots$ We show that if the ideals $\mathscr{I}_i$ $(i = 1,2)$ of all the vectors from the kernel of degenerate bilinear forms $B_i(x,y) := sp_i(x \cdot y)$, where sp_i are (super)traces on H, do exist, then $\mathscr{I}_1 = \mathscr{I}_2$ if and only if $\mathscr{I}_1 \bigcap H_0 = \mathscr{I}_2 \bigcap H_0$.

1. Preliminaries and notation

Let $\mathscr{A}$ be an associative superalgebra with parity π. All expressions of linear algebra are given for homogenous elements only and are supposed to be extended to inhomogeneous elements via linearity.

Definition 1.1. A linear function str on $\mathscr{A}$ is called a *supertrace* if

$$str(f \cdot g) = (-1)^{\pi(f)\pi(g)} str(g \cdot f) \text{ for all } f, g \in \mathscr{A}.$$

Definition 1.2. A linear function tr on $\mathscr{A}$ is called a *trace* if

$$tr(f \cdot g) = tr(g \cdot f) \text{ for all } f, g \in \mathscr{A}.$$

We will use the notation "sp" and the term "(super)trace" to denote both cases, traces and supertraces, simultaneously.

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2. The superalgebra of observables

Let $V = \mathbb{R}^N$ be endowed with a positive definite symmetric bilinear form $(\cdot,\cdot)$. For any nonzero $\vec{v} \in V$, define the *reflections* $r_{\vec{v}}$ as follows:

$$r_{\vec{v}}: \quad \vec{x} \mapsto \vec{x} - 2\frac{(\vec{x},\vec{v})}{(\vec{v},\vec{v})}\vec{v} \qquad \text{for any } \vec{x} \in V. \tag{2.1}$$

A finite set of non-zero vectors $\mathscr{R} \subset V$ is said to be a *root system* and any vector $\vec{v} \in \mathscr{R}$ is called a *root* if the following conditions hold:

i) $\mathscr{R}$ is $r_{\vec{w}}$-invariant for any $\vec{w} \in \mathscr{R}$,

ii) if $\vec{v}_1, \vec{v}_2 \in \mathscr{R}$ are proportional to each other, then either $\vec{v}_1 = \vec{v}_2$ or $\vec{v}_1 = -\vec{v}_2$.

The Coxeter group $G \subset O(N,\mathbb{R}) \subset End(V)$ generated by all reflections $r_{\vec{v}}$ with $\vec{v} \in \mathscr{R}$ is finite.

We do not apply any conditions on the scalar products of the roots because we want to consider both crystallographic and non-crystallographic root systems, e.g., $I_2(n)$ (see Theorem 4.1).

Let η be a complex-valued G-invariant function on $\mathscr{R}$, i.e., $\eta(\vec{v}) = \eta(\vec{w})$ if $r_{\vec{v}}$ and $r_{\vec{w}}$ belong to one conjugacy class of G.

We consider here the Symplectic Reflection (Super)algebra over complex numbers (see [6]) $H := H_{1,\eta}(G)$ and call it the *superalgebra of observables of Calogero model based on root system* $\mathscr{R}$. [a]

This algebra consists of noncommuting polynomials in $2N$ indeterminates a_i^α, where $\alpha = 0,1$ and $i = 1, ..., N$, with coefficients in $\mathbb{C}[G]$ satisfying the relations (see [6] Eq. (1.15))[b]

$$[a_i^\alpha, a_j^\beta] = \varepsilon^{\alpha\beta}\left(\delta_{ij} + \sum_{\vec{v}\in\mathscr{R}} \eta(\vec{v})\frac{v_i v_j}{(\vec{v},\vec{v})} r_{\vec{v}}\right), \tag{2.2}$$

and

$$r_{\vec{v}} a_i^\alpha = \sum_{j=1}^N \left(\delta_{ij} - 2\frac{v_i v_j}{(\vec{v},\vec{v})}\right) a_j^\alpha r_{\vec{v}}. \tag{2.3}$$

Here $\varepsilon^{\alpha\beta}$ is the antisymmetric tensor such that $\varepsilon^{01} = 1$, and v_i $(i = 1,...,N)$ are the coordinates of the vector $\vec{v}$. The commutation relations (2.2), (2.3) suggest to define the *parity* π by setting:

$$\pi(a_i^\alpha) = 1 \text{ for any } \alpha,\ i; \qquad \pi(r_{\vec{v}}) = 0 \text{ for any } \vec{v} \in \mathscr{R}. \tag{2.4}$$

and we can consider the algebra H as a superalgebra as well.

[a]This algebra has a faithful representation via Dunkl differential-difference operators D_i, see [5], acting on the space of G-invariant smooth functions on V, namely $\hat{a}_i^\alpha = \frac{1}{\sqrt{2}}(x_i + (-1)^\alpha D_i)$, see [1, 14]. The Hamiltonian of the Calogero model based on the root system [2–4, 13] is the operator $\hat{T}^{01}$ defined in (3.2) (see [1]). The wave functions are obtained in this model via the standard Fock procedure with the Fock vacuum $|0\rangle$ such that $\hat{a}_i^0|0\rangle$=0 for all i by acting on $|0\rangle$ with G-invariant polynomials of the $\hat{a}_i^1$.

[b]The sign and coefficient of the sum in the rhs of Eq. (2.2) is chosen for obtaining the Calogero model in the form [1], Eq. (1), Eq. (5), Eq. (9), Eq. (10) when $\mathscr{R}$ is of type A_{N-1}.

3. $\mathfrak{sl}_2$

Observe an important property of the superalgebra H: the Lie (super)algebra of its inner derivations contains the Lie subalgebra $\mathfrak{sl}_2$ generated by operators

$$D^{\alpha\beta}: \quad f \mapsto D^{\alpha\beta} f = [T^{\alpha\beta}, f], \tag{3.1}$$

where $\alpha, \beta = 0, 1$, and $f \in H$, and polynomials $T^{\alpha\beta}$ are defined as follows:

$$T^{\alpha\beta} := \frac{1}{2} \sum_{i=1}^{N} \left(a_i^\alpha a_i^\beta + a_i^\beta a_i^\alpha \right). \tag{3.2}$$

These operators satisfy the following relations:

$$[D^{\alpha\beta}, D^{\gamma\delta}] = \varepsilon^{\alpha\gamma} D^{\beta\delta} + \varepsilon^{\alpha\delta} D^{\beta\gamma} + \varepsilon^{\beta\gamma} D^{\alpha\delta} + \varepsilon^{\beta\delta} D^{\alpha\gamma}, \tag{3.3}$$

since

$$[T^{\alpha\beta}, T^{\gamma\delta}] = \varepsilon^{\alpha\gamma} T^{\beta\delta} + \varepsilon^{\alpha\delta} T^{\beta\gamma} + \varepsilon^{\beta\gamma} T^{\alpha\delta} + \varepsilon^{\beta\delta} T^{\alpha\gamma}.$$

It follows from Eq. (3.3) that the operators D^{00}, D^{11} and $D^{01} = D^{10}$ constitute an $\mathfrak{sl}_2$-triple:

$$[D^{01}, D^{11}] = 2D^{11}, \qquad [D^{01}, D^{00}] = -2D^{00}, \qquad [D^{11}, D^{00}] = -4D^{01}.$$

The polynomials $T^{\alpha\beta}$ commute with $\mathbb{C}[G]$, i.e., $[T^{\alpha\beta}, r_{\vec{v}}] = 0$, and act on the a_i^α as on vectors of the irreducible 2-dimensional $\mathfrak{sl}_2$-modules:

$$D^{\alpha\beta} a_i^\gamma = \left[T^{\alpha\beta}, a_i^\gamma\right] = \varepsilon^{\alpha\gamma} a_i^\beta + \varepsilon^{\beta\gamma} a_i^\alpha, \qquad \text{where } i = 1, \dots, N. \tag{3.4}$$

We will denote this $\mathfrak{sl}_2$ thus realized by the symbol $SL2$.

The subalgebra

$$H_0 := \{ f \in H \mid D^{\alpha\beta} f = 0 \text{ for any } \alpha, \beta \} \subset H \tag{3.5}$$

is called the *subalgebra of singlets*.

Introduce also the subspaces $H_s := \bigoplus_{i_s=1}^{\infty} H_s^{i_s}$, which is the direct sum of all irreducible $SL2$-modules $H_s^{i_s}$ of spin s, for $s = 0, 1/2, 1, \dots$. It is clear that H_0 is the defined above subalgebra of singlets.

The (super)algebra H can be decomposed in the following way

$$H = H_0 \oplus H_{rest}, \quad \text{where} \quad H_{rest} := H_{1/2} \oplus H_1 \oplus H_{3/2} \oplus \dots.$$

Then each element $f \in H$ can be represented in the form $f = f_0 + f_{rest}$, where $f_0 \in H_0$ and $f_{rest} \in H_{rest}$.

Note, that since $SL2$ is generated by inner derivations and $T^{\alpha\beta}$ are even elements, each two-sided ideal $\mathscr{I} \subset H$ can be decomposed in an analogous way: $\mathscr{I} = \mathscr{I}_0 \oplus \mathscr{I}_{1/2} \oplus \dots$.

Since $T^{\alpha\beta}$ are even elements of the superalgebra H, we have $\mathsf{sp}(D^{\alpha\beta} f) = 0$ for any (super)trace sp on H, and hence the following proposition takes place[c]:

Proposition 3.1. $\mathsf{sp}(f) = \mathsf{sp}(f_0)$ *for any* $f \in H$ *and any (super)trace* sp *on* H.

[c]This elementary fact is known for a long time, see, eg, [12].

Proof. If $s \neq 0$, then the elements of the form $D^{\alpha\beta} f$, where $\alpha, \beta = 0, 1$, and $f \in H_s^{i_s}$, $f \neq 0$, span the irreducible $SL2$-module $H_s^{i_s}$. This implies $\mathsf{sp} f = 0$ for any (super)trace on H and any $f \in H_{rest}$. □

4. The (super)traces on H

It is shown in [9, 10, 12] that the algebra H has a multitude of independent (super)traces. For the list of dimensions of the spaces of the (super)traces on $H_{1,\eta}(M)$ for all finite Coxeter groups M, see [8]. In particular, there is an m-dimensional space of traces and an $(m+1)$-dimensional space of supertraces on $H_{1,\eta}(I_2(2m+1))$.

Every (super)trace $\mathsf{sp}(\cdot)$ on any associative (super)algebra $\mathscr{A}$ generates the following bilinear form on $\mathscr{A}$:

$$B_{\mathsf{sp}}(f,g) = \mathsf{sp}(f \cdot g) \text{ for any } f,g \in \mathscr{A}. \tag{4.1}$$

It is obvious that if such a bilinear form B_{sp} is degenerate, then the kernel of this form (i.e., the set of all vectors $f \in \mathscr{A}$ such that $B_{\mathsf{sp}}(f,g) = 0$ for any $g \in \mathscr{A}$) is the two-sided ideal $\mathscr{I}^{\mathsf{sp}} \subset \mathscr{A}$.

The ideals of this sort are found, for example, in [11, Theorem 9.1] (generalizing the results of [15, 16] and [7] for the two- and three-particle Calogero models).

Theorem 9.1 from [11] may be shortened to the following theorem:

Theorem 4.1. *Let $m \in \mathbb{Z}$, where $m \geqslant 1$ and $n = 2m+1$. Then*

1) *The associative algebra $H_{1,\eta}(I_2(n))$ has nonzero traces tr_η such that the symmetric invariant bilinear form $B_{tr_\eta}(x,y) = tr_\eta(x \cdot y)$ is degenerate if and only if $\eta = \frac{z}{n}$, where $z \in \mathbb{Z} \setminus n\mathbb{Z}$. For each such z, all nonzero degenerate traces on $H_{1,z/n}(I_2(n))$ are proportional to each other.*
2) *The associative superalgebra $H_{1,\eta}(I_2(n))$ has nonzero supertraces str_η such that the supersymmetric invariant bilinear form $B_{str_\eta}(x,y) = str_\eta(x \cdot y)$ is degenerate if $\eta = \frac{z}{n}$, where $z \in \mathbb{Z} \setminus n\mathbb{Z}$. For each such z, all nonzero degenerate supertraces on $H_{1,z/n}(I_2(n))$ are proportional to each other.*
3) *The associative superalgebra $H_{1,\eta}(I_2(n))$ has nonzero supertraces str_η such that the supersymmetric invariant bilinear form $B_{str_\eta}(x,y) = str_\eta(x \cdot y)$ is degenerate if $\eta = z + \frac{1}{2}$, where $z \in \mathbb{Z}$. For each such z, all nonzero degenerate supertraces on $H_{1,z+1/2}(I_2(n))$ are proportional to each other.*
4) *For all other values of η, all nonzero traces and supertraces are nondegenerate.*

Theorem 4.1 implies that if $z \in \mathbb{Z} \setminus n\mathbb{Z}$, then there exists the degenerate trace tr_z generating the ideal $\mathscr{I}^{tr_z}$ consisting of the kernel of the degenerate form $B_{tr_z}(f,g) = tr_z(f \cdot g)$, and simultaneously the degenerate supertrace str_z generating the ideal $\mathscr{I}^{str_z}$ consisting of the kernel of the degenerate form $B_{str_z}(f,g) = str_z(f \cdot g)$.

A question arises: is it true that $\mathscr{I}^{tr_z} = \mathscr{I}^{str_z}$?

Answer to this and other similar questions can be considerably simplified by considering only the singlet parts of these ideals.

The following theorem justifies this method:

Theorem 4.2. *Let sp_1 and sp_2 be degenerate (super)traces on H. They generate the two-sided ideals $\mathscr{I}_1$ and $\mathscr{I}_2$ consisting of the kernels of bilinear forms $B_1(f,g) = \mathsf{sp}_1(f \cdot g)$ and $B_2(f,g) = \mathsf{sp}_2(f \cdot g)$, respectively.*

Then $\mathscr{I}_1 = \mathscr{I}_2$ *if and only if* $\mathscr{I}_1 \bigcap H_0 = \mathscr{I}_2 \bigcap H_0$.

Proof. It suffices to prove that if $\mathscr{I}_1 \bigcap H_0 = \mathscr{I}_2 \bigcap H_0$, then $\mathscr{I}_1 = \mathscr{I}_2$.

Consider any non-zero element $f \in \mathscr{I}_1$. For any $g \in H$, we have $\mathsf{sp}_1(f \cdot g) = 0$, $f \cdot g \in \mathscr{I}_1$ and $(f \cdot g)_0 \in \mathscr{I}_1$. So $(f \cdot g)_0 \in \mathscr{I}_1 \bigcap H_0$. Due to hypotheses of this Theorem, $(f \cdot g)_0 \in \mathscr{I}_2 \bigcap H_0$, and hence $\mathsf{sp}_2((f \cdot g)_0) = 0$. Proposition 3.1 gives $\mathsf{sp}_2(f \cdot g) = \mathsf{sp}_2((f \cdot g)_0)$ which implies $\mathsf{sp}_2(f \cdot g) = 0$.

Therefore, $f \in \mathscr{I}_2$. $\square$

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