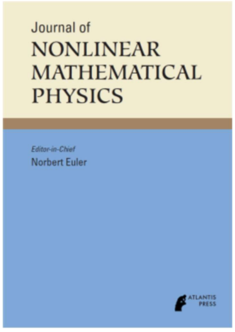

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Askold M. Perelomov

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Remarks on the mass spectrum of two-dimensional Toda lattice of E_8 type

Askold M. Perelomov
Institute of Theoretical and Experimental Physics, 117259 Moscow, Russia
aperelomo@gmail.com

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A simple procedure for obtaining the mass spectrum of 2-dimensional Toda lattice of E_8 type is given.

1. Introduction. Basics

Let us recall several definitions; for more details, see the book [10].

Let $\mathfrak{g}$ be a simple Lie algebra of rank l, let R_+ (resp. R_-) be the set of its positive (resp. negative) roots, and $\{\alpha_1, \dots, \alpha_l\}$ be the set of simple roots. Let W be the Weyl group of the root system R acting in the space $V = \mathbb{R}^l$, let $(\cdot, \cdot)$ be the non-degenerate W-invariant bilinear form in V, $\delta = \sum_{1 \le j \le l} n_j \alpha_j$ be the highest root, $\alpha_0 = -\delta$, and $h = 1 + \sum_{1 \le j \le l} n_j$ be the Coxeter number.

The 2-*dimensional Toda lattice* is 2-dimensional relativistic field theory describing l interacting scalar fields. The 2-*dimensional Toda lattice* is related with $\mathfrak{sl}(n)$.

In the paper [8], the 2-dimensional Toda lattice was generalized for the case of any simple finite-dimensional Lie algebra $\mathfrak{g}$; it was shown that the generalized construction has remarkable integrability properties. This is a relativistic system with Lagrangian

$$L = \frac{1}{2}\, \partial_\mu \, \partial^\mu \, \phi - U(\phi), \quad \text{where } \mu = 0,\, 1 \text{ and } \phi = \phi(x_0, x_1) \text{ is an } l\text{-dimensional vector.}$$

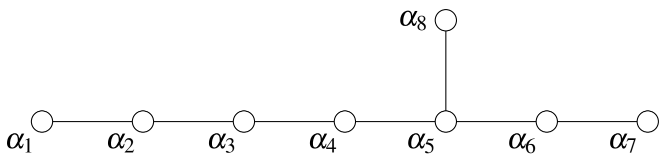
The potential $U(\phi)$ is constructed using the set of simple roots $\{\alpha_j\}_{j=0}^{l}$ of the simple Lie algebra $\mathfrak{g}$ of rank l:

$$U(\phi) = \sum_{0 \le j \le l} \exp(2\, \alpha_j,\, \phi).$$

In [8], the mass spectrum of scalar fields was found for all simple Lie algebras, except for the most complicated case $\mathfrak{g} = E_8$. For this algebra only numerical result was given.

In this note I describe two simple methods for obtaining the mass spectrum in the E_8 case. Note that both methods work also for any other finite-dimensional simple Lie algebra.

The numbering of simple roots of the Lie algebra E_8 is given on the Dynkin diagram:



The Dynkin diagram for the Lie algebra E_8.

For this numbering, the highest root δ has the form

$$\delta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8.$$

Observe that in 1989 A.B. Zamolodchikov discovered, using conformal theory, that this system appears also in the Ising model with nonzero magnetic field and explicitly calculated the mass spectrum, see [13]. The four mass ratios are equal to the "golden ratio"

$$r = \frac{\sqrt{5}+1}{2} = 2\cos\left(\frac{\pi}{5}\right) = 1.6180339887...$$

This remarkable property is related to the fact that the Coxeter number $h = 30$ of the Lie algebra E_8 is divisible by 5.

In 2010, Zamolodchikov's theory was experimentally confirmed for 1-dimensional Ising ferromagnet (cobalt niobate) near its critical point [2].

2. Method 1

As it was shown in papers [1, 5] the masses of particles are proportional to the components of a special eigenvector of the matrix $\mathrm{A} = 2\mathrm{I} - \mathrm{C}$, where C is the Cartan matrix of $\mathfrak{g}$. This eigenvector is called the *Perron–Frobenius vector*, see [6, 12]. For $\mathfrak{g} = E_8$, we have

$$\mathrm{A} = \begin{pmatrix} 0&1&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&1&0&1&0&0&0\\ 0&0&0&1&0&1&0&1\\ 0&0&0&0&1&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0 \end{pmatrix}$$

The characteristic equation of this matrix is

$$x^8 - 7x^6 + 14x^4 - 8x^2 + 1 = 0,$$

and its roots are

$$x_j = 2\cos(a_j\,\theta),\ \text{where } \theta = \frac{\pi}{h}, \text{ and } h = 30 \text{ is the Coxeter number;}$$

the numbers $a_j \in \{1, 7, 11, 13, 17, 19, 23, 29\}$ for $1 \le j \le 8$ are called the *exponents* of E_8. Note that they have no common divisors with the Coxeter number.

Note also that $x_5 = -x_4$, $x_6 = -x_3$, $x_7 = -x_2$, $x_8 = -x_1$. Let us give the expressions of the x_j in terms of radicals (these expressions might be used in calculations):

$$x_1 = \frac{1}{2}\sqrt{7+\sqrt{5}+\sqrt{30+6\sqrt{5}}}, \quad x_2 = \frac{1}{2}\sqrt{7+\sqrt{5}-\sqrt{30+6\sqrt{5}}},$$
$$x_3 = \frac{1}{2}\sqrt{7-\sqrt{5}+\sqrt{30-6\sqrt{5}}}, \quad x_4 = \frac{1}{2}\sqrt{7-\sqrt{5}-\sqrt{30-6\sqrt{5}}}.$$

The matrix A has nonnegative elements and according to the Perron–Frobenius theorem [6, 12] it has a unique eigenvector (the Perron-Frobenius eigenvectors)

$$u = (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8)$$

all coordinates of which are positive. This eigenvector corresponds to the maximal eigenvalue $\lambda = 2\cos(\theta)$ and we have

$$u\mathrm{A} = \lambda\, u\,,$$

or, in more details,

$$\begin{array}{llll} u_2 = \lambda\, u_1, & u_1 + u_3 = \lambda\ u_2, & u_2 + u_4 = \lambda\ u_3, & u_3 + u_5 = \lambda\, u_4, \\ u_4 + u_6 + u_8 = \lambda\, u_5, & u_5 + u_7 = \lambda\, u_6, & u_6 = \lambda\, u_7, & u_5 = \lambda\, u_8\,. \end{array}$$

Solving the system of these equations, and fixing $u_1 = 2\sin(\theta)$, we obtain:

$$u = \left(2\sin(\theta), 2\sin(2\,\theta), 2\sin(3\,\theta), 2\sin(4\,\theta), 2\sin(5\,\theta), \frac{\sin(2\,\theta)}{\sin(3\,\theta)}, \frac{\sin(\theta)}{\sin(3\,\theta)}, \frac{\sin(\theta)}{\sin(2\,\theta)}\right), \tag{2.1}$$

or, approximately,

$$u = (0.2091\,; 0.4158\,; 0.6180\,; 0.8135\,; 1\,; 0.6728\,; 0.3383\,; 0.5028).$$

Note that from eq. (2.1) it follows that (recall that $\theta = \frac{\pi}{30}$)

$$\frac{u_7}{u_1} = r,\ \ \frac{u_6}{u_2} = r,\ \ \frac{u_5}{u_3} = r,\ \ \frac{u_4}{u_8} = r,\ \text{ where } r = \frac{1+\sqrt{5}}{2} = 2\cos\left(\frac{\pi}{5}\right). \tag{2.2}$$

This is a very nice solution, because these expressions for u_j can be written immediately just by looking at the Dynkin diagram of E_8.

Observe that for any simple Lie algebra the eigenvector corresponding to the maximal eigenvalue can also be written just by looking at the corresponding Dynkin diagram.

Let me also give expressions for some trigonometric quantities in terms of radicals (and use this occasion to correct a typo in the definition of H_3 on p. 382 of [9], where $\varepsilon = 2\cos\left(\frac{\pi}{3}\right)$ should be $\varepsilon = 2\cos\left(\frac{\pi}{5}\right)$)

$$\begin{array}{ll} 2\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{2} = r, & 2\sin\left(\frac{\pi}{5}\right) = \sqrt{\frac{5-\sqrt{5}}{2}}\,, \\ 2\cos\left(\frac{\pi}{10}\right) = \sqrt{\frac{5+\sqrt{5}}{2}} & 2\sin\left(\frac{\pi}{10}\right) = \sqrt{\frac{3-\sqrt{5}}{2}}\,, \\ 2\cos\left(\frac{\pi}{15}\right) = \frac{1}{2}\sqrt{9+\sqrt{5}+2\sqrt{3}\sqrt{\frac{5-\sqrt{5}}{2}}}\,, & 2\sin\left(\frac{\pi}{15}\right) = \frac{1}{2}\sqrt{7-\sqrt{5}-2\sqrt{3}\sqrt{\frac{5-\sqrt{5}}{2}}}\,, \\ 2\cos\left(\frac{\pi}{30}\right) = \frac{1}{2}\sqrt{7+\sqrt{5}+2\sqrt{3}\sqrt{\frac{5+\sqrt{5}}{2}}}\,, & 2\sin\left(\frac{\pi}{30}\right) = \frac{1}{2}\sqrt{9-\sqrt{5}-2\sqrt{3}\sqrt{\frac{5+\sqrt{5}}{2}}}\,. \end{array}$$

3. Method 2

In the paper [8] it was shown that the squares of masses are eigenvalues of the 8×8 matrix whose elements are

$$B_{a,b} = \sum_{0 \le j \le l} n_j \alpha_j^a \alpha_j^b, \text{ where } n_0 = 1,$$

and where quantities α_j^a are coordinates of the vector α_j, and the n_j for $j > 0$ are coordinates of the vector $\delta = \sum_{1 \le j \le l} n_j \alpha_j$.

For the Lie algebra E_8, the characteristic polynomial P of this matrix is

$$P = x^8 - 60x^7 + 1440x^6 - 18000x^5 + 127440x^4 - 518400x^3 + 1166400x^2 - 1296000x + 518400.$$

In the paper [1], it was observed that $P = P_1 P_2$, where

$$P_1 = x^4 - 30x^3 + 240x^2 - 720x + 720, \quad P_2 = x^4 - 30x^3 + 300x^2 - 1080x + 720.$$

It is easy to check that the roots of polynomial P_1 (resp. P_2) are

$$m_1^2, m_3^2, m_4^2, m_6^2 \text{ (resp. } m_2^2, m_5^2, m_7^2, m_8^2).$$

Note that

$$u_2 u_5 u_7 u_8 = u_1 u_3 u_4 u_6, \text{ and } m_j^2 = M u_j^2. \tag{3.1}$$

The quantity $M = 2\sqrt{3}\,\frac{\sin(6\theta)}{\sin(\theta)}$ can be found from the equation

$$M^4 (u_2 u_5 u_7 u_8)^2 = 720.$$

So, formula (3.1) gives a relation between methods 1 and 2.

Let me give also the explicit expression for quantities m_j^2 in terms of radicals:

$$\begin{aligned}
m_5^2 &= \tfrac{1}{2}\sqrt{15 + 3\sqrt{5} + \sqrt{6}\sqrt{25 + 11\sqrt{5}}}, & m_4^2 &= \tfrac{1}{2}\sqrt{15 + 3\sqrt{5} + \sqrt{6}\sqrt{5 - \sqrt{5}}},\\
m_7^2 &= \tfrac{1}{2}\sqrt{15 + 3\sqrt{5} - \sqrt{6}\sqrt{25 + 11\sqrt{5}}}, & m_6^2 &= \tfrac{1}{2}\sqrt{15 + 3\sqrt{5} - \sqrt{6}\sqrt{5 - \sqrt{5}}},\\
m_8^2 &= \tfrac{1}{2}\sqrt{15 - 3\sqrt{5} + \sqrt{6}\sqrt{25 - 11\sqrt{5}}}, & m_3^2 &= \tfrac{1}{2}\sqrt{15 - 3\sqrt{5} + \sqrt{6}\sqrt{5 + \sqrt{5}}},\\
m_2^2 &= \tfrac{1}{2}\sqrt{15 - 3\sqrt{5} - \sqrt{6}\sqrt{25 - 11\sqrt{5}}}, & m_1^2 &= \tfrac{1}{2}\sqrt{15 - 3\sqrt{5} - \sqrt{6}\sqrt{5 + \sqrt{5}}}.
\end{aligned}$$

4. Conclusion

The remarkable property of the system under consideration is that the four mass ratios in (2.2) are equal to the “golden ratio”.

This is one more phenomenon of many in which the golden ratio appears. The golden ratio has a very long history, see e.g., the book [4, Ch. 11]. The first book on this topic, “Divina Proportione”, illustrated by Leonardo da Vinci, was published by Italian mathematician Luca Paccioli in 1509 [11].

Concluding, I would like to give here a quotation of the outstanding astronomer and mathematician Johannes Kepler [7]: “Geometry has two treasures: one of them is the Pythagorean theorem, and the other is dividing the segment in average and extreme respect ... The first can be compared to the measure of gold; the second is more like a gem”.

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