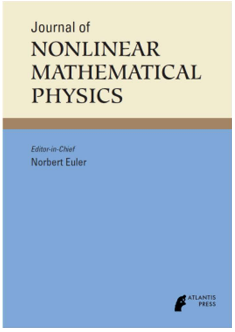

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 ISSN (Print): 1402-9251
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

Solving the constrained modified KP hierarchy by gauge transformations

Huizhan Chen, Lumin Geng, Na Li, Jipeng Cheng

To cite this article: Huizhan Chen, Lumin Geng, Na Li, Jipeng Cheng (2019) Solving the constrained modified KP hierarchy by gauge transformations, Journal of Nonlinear Mathematical Physics 26:1, 54–68, DOI: https://doi.org/10.1080/14029251.2019.1544787

To link to this article: https://doi.org/10.1080/14029251.2019.1544787

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 26, No. 1 (2019) 54–68

Solving the constrained modified KP hierarchy by gauge transformations

Huizhan Chen, Lumin Geng, Na Li and Jipeng Cheng*
School of Mathematics, China University of Mining and Technology,
Xuzhou, Jiangsu 221116, P. R. China
chengjp@cumt.edu.cn

Received 17 February 2018

Accepted 14 June 2018

In this paper, we mainly investigate two kinds of gauge transformations for the constrained modified KP hierarchy in Kupershmidt-Kiso version. The corresponding gauge transformations are required to keep not only the Lax equation but also the Lax operator. For this, by selecting the special generating eigenfunction and adjoint eigenfunction, the elementary gauge transformation operators of modified KP hierarchy $T_D(\Phi) = (\Phi^{-1})_x^{-1}\partial\Phi^{-1}$ and $T_I(\Psi) = \Psi^{-1}\partial^{-1}\Psi_x$, become the ones in the constrained case. Finally, the corresponding successive applications of T_D and T_I on the eigenfunction Φ and the adjoint eigenfunction Ψ are discussed.

Keywords: The constrained mKP hierarchy; gauge transformations; successive applications.

2010 MSC: 35Q53, 37K10, 37K40

1. Introduction

In the early 1980s, the modified Kadomtsev-Petviashvili (mKP) hierarchy [15, 17] is introduced as the nonlinear differential equations satisfied by the tau functions. There are many versions of the mKP hierarchy [6, 8, 10, 16, 18–20, 24, 28, 29], and their common features are to convert the relationship between KdV and mKdV into the KP situation. Here, we will only consider the Kupershmidt-Kiso version [6, 18–20, 24]. The mKP hierarchy in Kupershmidt-Kiso version is the particular case of the coupled modified KP hierarchy [29], which is proved that there are two tau functions τ_0 and τ_1. Compared with the KP hierarchy [9, 11] only owning a single tau function, it is more difficult to study the mKP hierarchy with two tau functions. The existence of τ_1 and τ_0 makes the mKP hierarchy in Kupershmidt-Kiso version becomes a relatively separate system, just like the KP hierarchy [9, 11].

Gauge transformation [3–7, 12–14, 21–25, 27] is one kind of powerful methods to construct the solutions of the integrable systems. By now, the gauge transformations of many integrable hierarchies have been studied. For example, the KP and mKP hierarchies [3, 4, 6, 13, 24, 25], the BKP and CKP hierarchies [14, 27], the discrete KP and modified discrete KP hierarchies [22, 23], the q-KP and modified q-KP hierarchies [5, 7, 12, 21] and so on. For the mKP hierarchy, there are two types of elementary gauge transformation operators [6, 24]: differential type $T_D(\Phi) = (\Phi^{-1})_x^{-1}\partial\Phi^{-1}$ and integral type $T_I(\Psi) = \Psi^{-1}\partial^{-1}\Psi_x$, which commute with each other and thus are more applicable.

In this article, we focus on the gauge transformations of the constrained mKP (cmKP) hierarchy. The corresponding gauge transformations are required to keep not only the Lax equation but also

*Corresponding author.

the Lax operator [1–3,25,30]. For this, by selecting the special generating eigenfunction and adjoint eigenfunction, the elementary gauge transformation operators of the modified KP hierarchy $T_D(\Phi)$ and $T_I(\Psi)$, become the ones in the constrained case. In fact, the Lax operator has $m+1$ components after the gauge transformation, compared with m components at initial. Thus one of the components must be 0 to keep the original form of the Lax operator. Special generating functions of T_D and T_I must be chosen to satisfy this condition. Finally, the corresponding successive applications of T_D and T_I on the eigenfunction Φ and the adjoint eigenfunction Ψ are discussed.

This paper is organized in the following way. In Section 2, some basic facts about the mKP hierarchy are introduced. In Section 3, the gauge transformations of the constrained mKP hierarchy are studied. Then in Section 4, we discuss the successive applications of the gauge transformation operators T_D and T_I. At last in Section 5, some conclusions and discussions are given.

2. The mKP hierarchy

The mKP hierarchy in Kupershmidt-Kiso version [18–20,24] is defined as the following Lax equation

$$\partial_{t_n} L = \left[(L^{n/l})_{\geqslant 1}, L\right], \quad n = 1,2,3,\ldots \tag{2.1}$$

with the Lax operator L given by the pseudo-differential operator below[a]

$$L = \partial^l + u_{l-1}\partial^{l-1} + u_{l-2}\partial^{l-2} + \cdots. \tag{2.2}$$

Here $\partial = \partial_x$ and $u_i = u_i(t_1 = x, t_2, \ldots)$. The algebraic multiplication of ∂^i with the multiplication operator f is given by the usual Leibnitz rule

$$\partial^i f = \sum_{j\geqslant 0} \binom{i}{j} f^{(j)} \partial^{i-j}, \quad i \in \mathbb{Z}, \tag{2.3}$$

where $f^{(j)} = \frac{\partial^j f}{\partial x^j}$. For $A = \sum_i a_i\partial^i$, $A_{\geqslant k} = \sum_{i\geqslant k} a_i\partial^i$ and $A_{<k} = \sum_{i<k} a_i\partial^i$. The name of the mKP hierarchy comes from the fact that (2.1) contains the mKP equation

$$4u_{tx} = (u_{xxx} - 6u^2u_x)_x + 3u_{yy} + 6u_xu_y + 6u_{xx}\int u_y dx. \tag{2.4}$$

In this paper, for any (pseudo-) differential operator A and a function f, the symbol $A(f)$ will indicate the action of A on f, whereas the symbol Af or $A\cdot f$ will denote the operator product of A and f, and $*$ stands for the conjugate operation: $(AB)^* = B^*A^*$, $\partial^* = -\partial$, $f^* = f$.

Similar to the case of the KP hierarchy [9,11], the Lax operator L for the mKP hierarchy can be expressed in terms of the dressing operator Z,

$$L = Z\partial^l Z^{-1}, \tag{2.5}$$

where Z is given by

$$Z = z_0 + z_1\partial^{-1} + z_2\partial^{-2} + \cdots \quad (z_0^{-1} \text{ exists}). \tag{2.6}$$

[a]Usually, the parameter l should be 1. Here, we introduce the parameter l to express the constraint (see (23)) on the Lax operator in a convenient way. When $l \neq 1$, the corresponding description of the modified KP hierarchy is equivalent to the usual case.

Then the Lax equation (2.1) is equivalent to

$$\partial_{t_n} Z = -\left(L^{n/l}\right)_{\leqslant 0} Z = -\left(Z\partial^n Z^{-1}\right)_{\leqslant 0} Z. \tag{2.7}$$

Define the wave function and the adjoint wave function [6, 29] of the mKP hierarchy in the following way:

$$w(t,\lambda) = Z\left(e^{\xi(t,\lambda)}\right) = \widehat{w}(t,\lambda)e^{\xi(t,\lambda)}, \tag{2.8}$$

$$w^*(t,\lambda) = (Z^{-1}\partial^{-1})^*\left(e^{-\xi(t,\lambda)}\right) = \widehat{w}^*(t,\lambda)\lambda^{-1}e^{-\xi(t,\lambda)}, \tag{2.9}$$

with

$$\xi(t,\lambda) = x\lambda + t_2\lambda^2 + t_3\lambda^3 + \cdots, \tag{2.10}$$

$$\widehat{w}(t,\lambda) = z_0 + z_1\lambda^{-1} + z_2\lambda^{-2} + \cdots, \tag{2.11}$$

$$\widehat{w}^*(t,\lambda) = z_0^{-1} + z_1^*\lambda^{-1} + z_2^*\lambda^{-2} + \cdots. \tag{2.12}$$

Then $w(t,\lambda)$ and $w^*(t,\lambda)$ satisfy the bilinear identity [29] below

$$\text{res}_\lambda w(t',\lambda)w^*(t,\lambda) = 1,$$

which is equivalent to the mKP hierarchy. Here $\text{res}_\lambda \sum_i a_i\lambda^i = a_{-1}$.

It is proved in [29] that there exist two tau functions τ_1 and τ_0 for the mKP hierarchy in Kupershmidt-Kiso version such that

$$w(t,\lambda) = \frac{\tau_0(t-[\lambda^{-1}])}{\tau_1(t)}e^{\xi(t,\lambda)}, \tag{2.13}$$

$$w^*(t,\lambda) = \frac{\tau_1(t+[\lambda^{-1}])}{\tau_0(t)}\lambda^{-1}e^{-\xi(t,\lambda)}. \tag{2.14}$$

By comparing (2.11) with (2.13), one can find

$$z_0 = \frac{\tau_0(t)}{\tau_1(t)}, \quad \frac{z_1}{z_0} = -\partial_x \ln\tau_0(t). \tag{2.15}$$

The eigenfunction Φ and the adjoint eigenfunction Ψ of the mKP hierarchy [26] are defined in the identities below,

$$\Phi_{t_n} = \left(L^{n/l}\right)_{\geqslant 1}(\Phi), \quad \Psi_{t_n} = -\partial^{-1}\left(L^{n/l}\right)^*_{\geqslant 1}\partial(\Psi). \tag{2.16}$$

Note that the definition of the adjoint eigenfunction Ψ here is different from the one in [6]. For L to be the Lax operator (2.2) of the mKP hierarchy, let

$$L^{(1)} = TLT^{-1}. \tag{2.17}$$

If the following Lax equation

$$\partial_{t_n} L^{(1)} = \left[\left(L^{(1)}\right)^{n/l}_{\geqslant 1}, L^{(1)}\right]$$

still holds, then T is called the gauge transformation operator of mKP hierarchy.

Lemma 2.1 ([6]). *If the pseudo-differential operator T satisfies*

$$\left(TL^{n/l}T^{-1}\right)_{\geqslant 1} = T\left(L^{n/l}\right)_{\geqslant 1}T^{-1} + T_{t_n}T^{-1}, \tag{2.18}$$

then T is a gauge transformation operator of the mKP hierarchy.

It is proved in [6], there are two kinds of gauge transformation operators which can commute with each other. The specific forms are as follows:

- Differential type

$$T_D(\Phi) = (\Phi^{-1})_x^{-1}\partial\Phi^{-1}, \tag{2.19}$$

- Integral type

$$T_I(\Psi) = \Psi^{-1}\partial^{-1}\Psi_x \tag{2.20}$$

where Φ is eigenfunction and Ψ is adjoint eigenfunction. It is important to note that the adjoint eigenfunction Ψ here is different from the one in [6], so the form of $T_I(\Psi)$ here is different.

Further, one can obtain the following lemma.

Lemma 2.2 ([6]). *Under the gauge transformation operator $T_D(\Phi)$ and $T_I(\Psi)$, the objects in the mKP hierarchy are transformed in the way shown in Table I.*

Table I. gauge transformations $T_D(\Phi)$ and $T_I(\Psi)$

$L_{mKP} \to L^{(1)}_{mKP}$	$Z^{(1)} =$	$\Phi^{(1)}_1 =$	$\Psi^{(1)}_1 =$	$\tau^{(1)}_0 =$	$\tau^{(1)}_1 =$
$T_D(\Phi)$	$T_D(\Phi)Z\partial^{-1}$	$(\Phi_1/\Phi)_x/(\Phi^{-1})_x$	$(\Phi^{-1})_x \cdot \int \Phi\Psi_1 dx$	$\Phi\tau_1$	$-\Phi_x\tau_1^2/\tau_0$
$T_I(\Psi)$	$T_I(\Psi)Z\partial$	$\int \Psi_x\Phi_1 dx/\Psi$	$(\Psi_1/\Psi_x)_x \cdot \Psi$	$\Psi_x\tau_0^2/\tau_1$	$\tau_0 \cdot \Psi$

The constrained mKP hierarchy [26] can be defined by imposing the following additional constraint on the Lax operator (2.2) of the mKP hierarchy,

$$L = \partial^l + \sum_{i=1}^{l-1} u_i\partial^i + \sum_{i=1}^{m} \Phi_i\partial^{-1}\Psi_i\partial. \tag{2.21}$$

When $l = 1$, $m = 1$, (2.21) will become into

$$L = \partial + \phi\partial^{-1}\psi\partial.$$

Compared with the mKP's Lax operator (see (2.2)), we can see that

$$u_0 = \phi\psi, \quad u_1 = -\phi\psi_x, \quad u_2 = \phi\psi_{xx}, \ldots.$$

Then by (2.1) we know that

$$\begin{aligned} u_{0t_2} &= \phi_{xx}\psi + 2\phi\psi\psi_x - \phi\psi_{xx} + 2\phi^2\psi\psi_x, \\ \phi_{t_2} &= \phi_{xx} + 2\phi\psi\phi_x, \\ \psi_{t_2} &= -\psi_{xx} + 2\phi\psi\psi_x; \end{aligned}$$

and

$$u_{0t_3} = \phi_{xxx}\psi + \phi\psi_{xxx} + 3\psi^2\left(\phi\phi_{xx} + (\phi_x)^2 + \phi^3\psi_x\right) + 3\phi^2\left(\psi^3\phi_x - (\psi_x)^2 + (\phi_x)^2\right) - 3\phi\phi_x\psi\psi_x,$$
$$\phi_{t_3} = \phi_{xxx} + 3\phi\psi\phi_{xx} + 3(\phi_x)^2\psi + 3(\phi\psi)^2\phi_x,$$
$$\psi_{t_3} = \psi_{xxx} - 3(\phi\psi)_x\psi_x + 3(\phi_x)^2 + 3(\phi\psi)^2\psi_x.$$

Remark. The relation between the KP hierarchy and the modified KP hierarchy can be extended to the constrained cases, which is shown in [26]. In fact, for the Lax operator of the constrained KP hierarchy

$$L_{KP} = \partial^l + \sum_{i=0}^{l-2} v_i\partial^i + \sum_{i=1}^{m} q_i\partial^{-1}r_i,$$

and f be the eigenfunction of the constrained KP hierarchy corresponding to L_{KP}, the transformed operator $\tilde{L} = f^{-1}\cdot L_{KP}\cdot f$ will be the Lax operator of the constrained mKP hierarchy, satisfying the following constraint

$$(\tilde{L})_{<1} = \sum_{i=1}^{m+1} \tilde{q}_i\partial^{-1}\tilde{r}_i\partial$$

with

$$\tilde{q}_i = f^{-1}q_i, \quad \tilde{r}_i = -\int f r_i dx, \quad i = 1,2,\ldots,m$$

and

$$\tilde{q}_{m+1} = f^{-1}L(f), \quad \tilde{r}_{m+1} = 1.$$

3. Gauge transformation of the cmKP hierarchy

In this section, we will investigate the gauge transformations of the cmKP hierarchy. Different from the mKP hierarchy, the gauge transformation of constrained mKP must preserve the form of Lax operator. In order to discuss this problem more conveniently, here we need to introduce some basic lemmas on pseudo-differential operators.

Lemma 3.1 ([25]). *For any pseudo-differential operator A and arbitrary functions f, g, one has the following operator identities:*

$$(Af\partial^{-1})_{<0} = A_{\geqslant 0}(f)\partial^{-1} + A_{<0}f\partial^{-1}, \tag{3.1}$$
$$(\partial^{-1}gA)_{<0} = \partial^{-1}A^*_{\geqslant 0}(g) + \partial^{-1}gA_{<0}. \tag{3.2}$$

Lemma 3.2. *For any pseudo-differential operator A, one has the following operator identities:*

$$\left(T_D(\Phi)A_{\geqslant 1}(T_D(\Phi))^{-1}\right)_{\leqslant 0} = -(T_D(\Phi)A_{\geqslant 1})(\Phi)\cdot\partial^{-1}\Phi^{-1}\partial, \tag{3.3}$$
$$\left(T_I(\Psi)A_{\geqslant 1}(T_I(\Psi))^{-1}\right)_{\leqslant 0} = -\Psi^{-1}\partial^{-1}\cdot\left(\partial^{-1}(T_I(\Psi)^{-1})^*(A_{\geqslant 1})^*(\Psi_x)\right)\cdot\partial. \tag{3.4}$$

Proof. For (3.3), we can use the first expression in Lemma 3.1,

$$\begin{aligned}\left(T_D(\Phi)A_{\geqslant 1}(T_D(\Phi))^{-1}\right)_{\leqslant 0} &= \left((\Phi^{-1})_x^{-1}\partial\Phi^{-1}A_{\geqslant 1}\partial^{-1}\right)_{<0}\partial \\ &\quad -\left((\Phi^{-1})_x^{-1}\partial\Phi^{-1}A_{\geqslant 1}\Phi\partial^{-1}\Phi^{-1}\right)_{<0}\partial \\ &= -\left((\Phi^{-1})_x^{-1}\partial\Phi^{-1}A_{\geqslant 1}\Phi\partial^{-1}\right)_{<0}\Phi^{-1}\partial \\ &= -(T_D(\Phi)A_{\geqslant 1})(\Phi)\partial^{-1}\Phi^{-1}\partial.\end{aligned}$$

For (3.4), by the second formula in Lemma 3.1,

$$\begin{aligned}\left(T_I(\Psi)A_{\geqslant 1}(T_I(\Psi))^{-1}\right)_{\leqslant 0} &= (\Psi^{-1}\partial^{-1}\Psi_x A_{\geqslant 1}\partial^{-1})_{<0}\partial + (\Psi^{-1}\partial^{-1}\Psi_x A_{\geqslant 1}\Psi_x^{-1}\Psi)_{<0}\partial \\ &= -(\Psi^{-1}\partial^{-1}\int \Psi_x A_{\geqslant 1}dx)_{<0}\partial + (\Psi^{-1}\partial^{-1}\Psi_x A_{\geqslant 1}\Psi_x^{-1}\Psi)_{<0}\partial \\ &= -\Psi^{-1}\partial^{-1}\left(\partial^{-1}(T_I(\Psi)^{-1})^*(A_{\geqslant 1})^*(\Psi_x)\right)\partial.\end{aligned}$$

□

Lemma 3.3. *For any functions q and r, one has the following operator identities:*

$$\begin{aligned}\left(T_D(\Phi)\cdot q\partial^{-1}r\partial\cdot(T_D(\Phi))^{-1}\right)_{\leqslant 0} &= T_D(\Phi)(q)\cdot\partial^{-1}\cdot\left(\partial^{-1}(T_D(\Phi)^{-1})^*\partial\right)(r)\cdot\partial \\ &\quad -\left(T_D(\Phi)q\partial^{-1}r\partial\right)(\Phi)\cdot\partial^{-1}\Phi^{-1}\partial, \qquad (3.5)\end{aligned}$$

$$\begin{aligned}\left(T_I(\Psi)\cdot q\partial^{-1}r\partial\cdot(T_I(\Psi))^{-1}\right)_{\leqslant 0} &= T_I(\Psi)(q)\cdot\partial^{-1}\cdot\left(\partial^{-1}(T_I(\Psi)^{-1})^*\partial\right)(r)\cdot\partial \\ &\quad -\Psi^{-1}\partial^{-1}\cdot\left(\partial^{-1}(T_I(\Psi)^{-1})^*\partial r\partial^{-1}q\partial\right)(\Psi)\cdot\partial. \qquad (3.6)\end{aligned}$$

Proof.

$$\begin{aligned}\left(T_D(\Phi)\cdot q\partial^{-1}r\partial\cdot(T_D(\Phi))^{-1}\right)_{\leqslant 0} &= \left((\Phi^{-1})_x^{-1}\partial\Phi^{-1}\cdot q\partial^{-1}r\partial\cdot\Phi\partial^{-1}(\Phi^{-1})_x\partial^{-1}\right)_{<0}\partial \\ &= \left((\Phi^{-1})_x^{-1}\partial\Phi^{-1}\cdot q\partial^{-1}r\partial\cdot\Phi(\Phi^{-1}\partial^{-1}-\partial^{-1}\Phi^{-1})\right)_{<0}\partial \\ &= T_D(\Phi)(q)\cdot\partial^{-1}r\partial - III, \qquad (3.7)\end{aligned}$$

where

$$\begin{aligned}III &= \left((\Phi^{-1})_x^{-1}\partial\Phi^{-1}q\partial^{-1}(\partial r-r_x)\Phi\partial^{-1}\Phi^{-1}\right)_{<0}\partial \\ &= \left(T_D(\Phi)q\partial^{-1}r\partial\right)(\Phi)\cdot\partial^{-1}\Phi^{-1}\partial + T_D(\Phi)(q)\cdot\partial^{-1}\cdot\Phi^{-1}\int r_x\Phi dx\cdot\partial.\end{aligned}$$

Substitute III into (3.7) and we can get (3.5).

As for (3.6),

$$\Big(T_I(\Psi)\cdot q\partial^{-1}r\partial\cdot(T_I(\Psi))^{-1}\Big)_{\leqslant 0} = (\Psi^{-1}\partial^{-1}\Psi_x\cdot q\partial^{-1}r\partial\cdot\Psi_x^{-1}\partial\Psi\partial^{-1})_{<0}\partial$$
$$= \Psi^{-1}\partial^{-1}\cdot qr\Psi\partial - IV + V, \tag{3.8}$$

where

$$\begin{aligned} IV &= (\Psi^{-1}\partial^{-1}\Psi_x q\partial^{-1}r_x\Psi_x^{-1}\Psi)_{<0}\partial \\ &= T_I(\Psi)(q)\cdot\partial^{-1}r_x\Psi_x^{-1}\Psi\cdot\partial - \Psi^{-1}\partial^{-1}\cdot\int\Psi_x q dx r_x\Psi_x^{-1}\Psi\cdot\partial, \\ V &= (\Psi^{-1}\partial^{-1}\Psi_x q\partial^{-1}r)_{<0}\partial \\ &= T_I(\Psi)(q)\cdot\partial^{-1}r\cdot\partial - \Psi^{-1}\partial^{-1}\cdot\int\Psi_x q dx r\cdot\partial. \end{aligned}$$

Due to

$$\Big(\partial^{-1}(T_I(\Psi)^{-1})^*\partial r\partial^{-1}q\partial\Big)(\Psi) = qr\Psi + \int\Psi_x q dx r_x\Psi_x^{-1}\Psi - \int\Psi_x q dx r,$$

(3.6) can be derived after the substitution of IV and V into (3.8). □

Proposition 3.1. *Under the gauge transformation $T_D(\Phi)$, the initial Lax operator of the constrained mKP hierarchy*

$$L^{(0)} = L^{(0)}_{\geqslant 1} + \sum_{i=1}^{m}\Phi_i^{(0)}\partial^{-1}\Psi_i^{(0)}\partial$$

will become into

$$L^{(1)} = T_D(\Phi)L^{(0)}(T_D(\Phi))^{-1} = L^{(1)}_{\geqslant 1} + L^{(1)}_{\leqslant 0},$$

where

$$\begin{aligned} L^{(1)}_{\leqslant 0} &= \Phi_0^{(1)}\partial^{-1}\Psi_0^{(1)}\partial + \sum_{i=1}^{m}\Phi_i^{(1)}\partial^{-1}\Psi_i^{(1)}\partial, \\ \Phi_0^{(1)} &= -(T_D(\Phi)L^{(0)})(\Phi), \quad \Psi_0^{(1)} = \Phi^{-1}, \\ \Phi_i^{(1)} &= T_D(\Phi)(\Phi_i^{(0)}), \quad \Psi_i^{(1)} = \Big(\partial^{-1}(T_D(\Phi)^{-1})^*\partial\Big)(\Psi_i^{(0)}). \end{aligned}$$

Proof. By using (3.3)–(3.5) and Lemma 2.2, the above results can be directly obtained. □

Notice that there is an extra term in the integral part of the $L^{(1)}_{\leqslant 0}$. To preserve the form of the integral part, we choose Φ that coincides with one of the original eigenfunctions that appear in the $L^{(0)}_{\leqslant 0}$, e.g , $\Phi = \Phi_1^{(0)}$. In this case one has $\Phi_1^{(1)} = 0$, and $\Phi_0^{(1)}$ and $\Psi_0^{(1)}$ take over the roles of $\Phi_1^{(1)}$ and $\Psi_1^{(1)}$. Here $\Phi_0^{(1)}$ and $\Psi_0^{(1)}$ are still the (adjoint) eigenfunctions, which are proved in the next proposition.

Proposition 3.2. *Under the gauge transformation operator $T_D(\Phi)$,*
(I) $\Phi_0^{(1)} = -\big(T_D(\Phi)L^{(0)}\big)(\Phi)$ *is still the eigenfunction,*
(II) $\Psi_0^{(1)} = \Phi^{-1}$ *is the adjoint eigenfunction.*

Proof. (I) Since

$$\begin{aligned}\partial_{t_n}(L^{(0)}(\Phi)) &= (\partial_{t_n}L^{(0)})(\Phi)+L^{(0)}\partial_{t_n}(\Phi)\\ &= [(L^{(0)})^{n/l}_{\geqslant 1},L^{(0)}](\Phi)+L^{(0)}(L^{(0)})^{n/l}_{\geqslant 1}(\Phi)\\ &= (L^{(0)})^{n/l}_{\geqslant 1}L^{(0)}(\Phi),\end{aligned}$$

From the above we can see $L^{(0)}(\Phi)$ is the eigenfunction of $L^{(0)}$. Thus $-(T_D(\Phi)L^{(0)})(\Phi)$ is an eigenfunction of $L^{(1)}$ according to Lemma 2.2.
(II) For $\Psi_0^{(1)} = \Phi^{-1}$ to be the adjoint eigenfunctions, we can prove by using (2.18).

$$\begin{aligned}\left(-\partial^{-1}\left((L^{(1)})^{n/l}\right)^{*}_{\geqslant 1}\partial\right)(\Phi^{-1}) &= -\partial^{-1}\left((\Phi^{-1})_x^{-1}\partial\Phi^{-1}(L^{(0)})^{n/l}_{\geqslant 1}\Phi\partial^{-1}(\Phi^{-1})_x\right)^{*}\partial(\Phi^{-1})\\ &\quad -\partial^{-1}\left((\Phi^{-1})_x^{-1}\partial\Phi^{-1})_{t_n}\Phi\partial^{-1}(\Phi^{-1})_x\right)^{*}(\Phi^{-1})_x\\ &= -\partial^{-1}\left((\Phi^{-1})_x\partial^{-1}\Phi(\Phi^{-1}\partial(\Phi^{-1})_x^{-1})_{t_n}\right)(\Phi^{-1})_x\\ &= (\Phi^{-1})_{t_n}.\end{aligned}$$

□

Similarly, one can get the following proposition .

Proposition 3.3. *Under the gauge transformation $T_I(\Psi)$, the initial Lax operator of the constrained mKP hierarchy*

$$L^{(0)} = (L^{(0)})_{\geqslant 1}+\sum_{i=1}^{m}\Phi_i^{(0)}\partial^{-1}\Psi_i^{(0)}\partial$$

will become into

$$L^{(1)} = T_I(\Psi)L^{(0)}(T_I(\Psi))^{-1} = L^{(1)}_{\geqslant 1}+L^{(1)}_{\leqslant 0},$$

where

$$\begin{aligned}&L^{(1)}_{\leqslant 0} = \Phi_0^{(1)}\partial^{-1}\Psi_0^{(1)}\partial+\sum_{i=1}^{m}\Phi_i^{(1)}\partial^{-1}\Psi_i^{(1)}\partial,\\ &\Phi_0^{(1)} = \Psi^{-1},\quad \Psi_0^{(1)} = -\left(\partial^{-1}(T_I(\Psi)^{-1})^{*}(L^{(0)})^{*}\partial\right)(\Psi),\\ &\Phi_i^{(1)} = T_I(\Psi)(\Phi_i^{(0)}),\quad \Psi_i^{(1)} = \left(\partial^{-1}(T_I(\Psi)^{*})^{-1}\partial\right)(\Phi_i^{(0)}).\end{aligned}$$

Proof. It can be proved by (3.4), (3.6) and Lemma 2.2. □

It is important to note here that the Lax operator has $m+1$ components after transformation. In order to maintain the original form, one of the transformed components of $L^{(1)}$ must be 0. Therefore, by letting $\Psi = \Psi_1^{(0)}$ in Proposition 8, one has $\Psi_1^{(1)} = 0$. And $\Phi_0^{(1)}$ and $\Psi_0^{(1)}$ take over the roles of $\Phi_1^{(1)}$ and $\Psi_1^{(1)}$. Similarly, $\Phi_1^{(0)}$ and $\Psi_1^{(0)}$ are still the (adjoint) eigenfunctions due to the proposition below.

Proposition 3.4. *Under the gauge transformation operator $T_I(\Psi)$,*
(I) $\Phi_0^{(1)} = \Psi^{-1}$ *is still the eigenfunction.*
(II) $\Psi_0^{(1)} = -\left(\partial^{-1}(T_I(\Psi)^{-1})^*(L^{(0)})^*\partial\right)(\Psi)$ *is adjoint eigenfunction.*

Proof. (I) In order to prove $\Phi_0^{(1)} = \Psi^{-1}$ is an eigenfunction, we just have to prove $\partial_{t_n}\Psi^{-1} = (L^{(1)})^{n/l}_{\geqslant 1}(\Psi^{-1})$. In fact according to (2.18),

$$\begin{aligned}(L^{(1)})^{n/l}_{\geqslant 1}(\Psi^{-1}) &= (\Psi^{-1}\partial^{-1}\Psi_x L^{(0)}\Psi_x^{-1}\partial\Psi)^{n/l}_{\geqslant 1}(\Psi^{-1})\\ &= (\Psi^{-1}\partial^{-1}\Psi_x(L^{(0)})^{n/l}_{\geqslant 1}\Psi_x^{-1}\partial\Psi)(\Psi^{-1}) + (\Psi^{-1}\partial^{-1}\Psi_x)_{t_n}\Psi_x^{-1}\partial\Psi(\Psi^{-1})\\ &= (\Psi^{-1})_{t_n}.\end{aligned}$$

(II) Due to

$$\begin{aligned}\partial_{t_n}\Big((\partial^{-1}(L^{(0)})^*\partial)(\Psi)\Big) &= (\partial^{-1}(L^{(0)})^*_{t_n})(\Psi_x) + (\partial^{-1}(L^{(0)})^*)(\Psi_{xt_n})\\ &= -\left(\partial^{-1}\left((L^{(0)})^{n/l}\right)^*_{\geqslant 1}L^{(0)*}\right)(\Psi_x)\\ &= -\left(\partial^{-1}\left((L^{(0)})^{n/l}\right)^*_{\geqslant 1}\partial\right)\Big((\partial^{-1}(L^{(0)})^*\partial)(\Psi)\Big),\end{aligned}$$

Thus $(\partial^{-1}(L^{(0)})^*\partial)(\Psi)$ is the adjoint eigenfunction of $L^{(0)}$. Thus $-\big(\partial^{-1}(T_I(\Psi)^{-1})^*(L^{(0)})^*\partial\big)(\Psi)$ is adjoint eigenfunction of $L^{(1)}$ according to Lemma 2.2. □

Summarize the results above in the next proposition.

Proposition 3.5. *Under the gauge transformation operator $T_D(\Phi_1^{(0)})$ and $T_I(\Psi_1^{(0)})$, the Lax operator of constrained mKP hierarchy*

$$L^{(0)} = L^{(0)}_{\geqslant 1} + \sum_{i=1}^{m}\Phi_i^{(0)}\partial^{-1}\Psi_i^{(0)}\partial$$

becomes into

$$L^{(1)} = L^{(1)}_{\geqslant 1} + \sum_{i=1}^{m}\Phi_i^{(1)}\partial^{-1}\Psi_i^{(1)}\partial.$$

- *Under the action of $T_D(\Phi_1^{(0)})$:*

$$\begin{aligned}&\Phi_1^{(1)} = -\Big(T_D(\Phi_1^{(0)})L^{(0)}\Big)(\Phi_1^{(0)}), \quad \Psi_1^{(1)} = (\Phi_1^{(0)})^{-1},\\ &\Phi_i^{(1)} = T_D(\Phi_1^{(k)})(\Phi_i^{(k)}), \quad \Psi_i^{(1)} = \partial^{-1}(T_D(\Phi_1^{(0)})^{-1})^*\partial(\Psi_i^{(0)}), \quad i = 2,\ldots,m.\end{aligned}$$

- *Under the action of $T_I(\Psi_1^{(0)})$:*

$$\begin{aligned}&\Phi_1^{(1)} = (\Psi_1^{(0)})^{-1}, \quad \Psi_1^{(1)} = -\partial^{-1}(T_I(\Psi_1^{(0)})^{-1})^*L^{(0)*}\partial(\Psi_1^{(0)})\\ &\Phi_i^{(1)} = T_I(\Psi_1^{(0)})(\Phi_i^{(0)}), \quad \Psi_i^{(1)} = \partial^{-1}(T_I(\Psi_1^{(0)})^*)^{-1}\partial(\Psi_i^{(0)}), \quad i = 2,\ldots,m.\end{aligned}$$

Remark. In fact, the special choices in above propostion of the eigenfunction and the adjoint eigenfunction, in order to preserve the forms of the constrained Lax operator, go back to the case of the constrained KP hierarchy in [1,2].

By using Proposition 3.5, we can find the result below.

Proposition 3.6. *For the gauge transformation operator of the cmKP hierarchy, we have*

$$T_I(\Psi_1^{(1)})T_D(\Phi_1^{(0)}) = T_D(\Phi_1^{(1)})T_I(\Psi_1^{(0)}) = 1.$$

Proof. We know $\Psi_1^{(1)} = (\Phi_1^{(0)})^{-1}$ from Proposition 3.5, thus

$$\begin{aligned} T_I(\Psi_1^{(1)})T_D(\Phi_1^{(0)}) &= (\Psi_1^{(1)})^{-1}\partial^{-1}(\Psi_1^{(1)})_x\left((\Phi_1^{(0)})^{-1}\right)_x^{-1}\partial(\Phi_1^{(0)})^{-1} \\ &= \Phi_1^{(0)}\partial^{-1}\left((\Phi_1^{(0)})^{-1}\right)_x\left((\Phi_1^{(0)})^{-1}\right)_x^{-1}\partial(\Phi_1^{(0)})^{-1} \\ &= 1. \end{aligned}$$

Similarly, $T_D(\Phi_1^{(1)})T_I(\Psi_1^{(0)}) = 1.$ □

4. The successive applications of T_D and T_I

From Proposition 3.6, we can know the pair of T_D and T_I will cancel in the products. Therefore, one can only consider the products of T_D or T_I. First, we recall the corresponding results of the mKP hierarchy [6]. Consider the following chain of the gauge transformation operators $T_D(q)$ and $T_I(r)$ of the mKP hierarchy.

$$\begin{aligned} &L \xrightarrow{T_D(q_1)} L^{(1)} \xrightarrow{T_D(q_2^{(1)})} L^{(2)} \to \cdots \to L^{(n-1)} \xrightarrow{T_D(q_n^{(n-1)})} L^{(n)} \\ &\xrightarrow{T_I(r_1^{(n)})} L^{(n+1)} \xrightarrow{T_I(r_1^{(n+1)})} \cdots \to L^{(n+k-1)} \xrightarrow{T_I(r_k^{(n+k-1)})} L^{(n+k)}. \end{aligned}$$

Denote

$$T^{(n,k)}(r_1,r_2,\ldots,r_k;q_1,q_2,\ldots,q_n) = T_I(r_k^{(n+k-1)})\cdots T_I(r_1^{(n)})T_D(q_n^{(n-1)})\cdots T_D(q_2^{(1)})T_D(q_1). \quad (4.1)$$

The generalized Wronskian determinant [13] is needed in the next propositions, which is defined in the following form

$$IW_{k,n} \equiv IW_{k,n}(r_k,r_{k-1},\ldots,r_1;q_1,\ldots,q_n) = \begin{vmatrix} \int q_1r_kdx & \int q_2r_kdx & \cdots & \int q_nr_kdx \\ \int q_1r_{k-1}dx & \int q_2r_{k-1}dx & \cdots & \int q_nr_{k-1}dx \\ \vdots & \vdots & \vdots & \vdots \\ \int q_1r_1dx & \int q_2r_1dx & \cdots & \int q_nr_1dx \\ q_1 & q_2 & \cdots & q_n \\ q_{1x} & q_{2x} & \cdots & q_{nx} \\ \vdots & \vdots & \vdots & \vdots \\ q_1^{(n-k-1)} & q_2^{(n-k-1)} & \cdots & q_n^{(n-k-1)} \end{vmatrix}.$$

In particular,

$$IW_{0,n} = W_n(q_1, q_2, \ldots, q_n), \tag{4.2}$$

which is Wronskian determinant of $q_1, q_2, \ldots, q_n$.

Lemma 4.1 ([6]). *$T^{(n,0)}$ and $(T^{(n,0)})^{-1}$ have the following forms*

$$T^{(n,0)}(q_1, q_2, \ldots, q_n) = \frac{1}{W_n(q_1, q_2, \ldots, q_n)} \begin{vmatrix} q_1 & q_2 & \cdots & q_n & 1 \\ q_{1x} & q_{2x} & \cdots & q_{nx} & \partial \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q^{(n)} & q_2^{(n)} & \cdots & q_n^{(n)} & \partial^n \end{vmatrix} \tag{4.3}$$

and

$$T^{(n,0)}(q_1, q_2, \ldots, q_n)^{-1} = \frac{(-1)^{n-1}}{W_n(q_1, q_2, \ldots, q_n)} \begin{vmatrix} q_1\partial^{-1} & q_1 & \cdots & q_1^{(n-2)} \\ q_2\partial^{-1} & q_2 & \cdots & q_2^{(n-2)} \\ \vdots & \vdots & \vdots & \vdots \\ q_n\partial^{-1} & q_n & \cdots & q_n^{(n-2)} \end{vmatrix}. \tag{4.4}$$

Lemma 4.2 ([6]). *$T^{(0,k)}$ and $(T^{(0,k)})^{-1}$ have the following forms*

$$T^{(0,k)}(r_1, \ldots, r_k) = \frac{(-1)^{k-1}}{W_k(r_1, \ldots, r_k)} \begin{vmatrix} r_{1x} & r_{1xx} & \cdots & r_1^{(k-1)} & \partial^{-1} r_{1x} \\ r_{2x} & r_{2xx} & \cdots & r_2^{(k-1)} & \partial^{-1} r_{2x} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{kx} & r_{kxx} & \cdots & r_k^{(k-1)} & \partial^{-1} r_{kx} \end{vmatrix}, \tag{4.5}$$

and

$$T^{(0,k)}(r_1, \ldots, r_k)^{-1} = \begin{vmatrix} 1 & r_{1x} & \cdots & r_{kx} \\ -\partial & r_{1xx} & \cdots & r_{kxx} \\ \vdots & \vdots & \vdots & \vdots \\ (-\partial)^k & r_1^{(k+1)} & \cdots & r_k^{(k+1)} \end{vmatrix} \frac{W_k(r_1, \ldots, r_k)}{W_{k+1}(1, r_1, r_2, \ldots, r_k)^2}. \tag{4.6}$$

Lemma 4.3 (Jacobi Expansion, [3,13]). *For Wronskian determinants,*

$$\left(\frac{W_n(f_1, f_2, \ldots, f_{n-1}, f_{n+1})}{W_n(f_1, f_2, \ldots, f_{n-1}, f_n)} \right)_x = \frac{W_{n+1}(f_1, f_2, \ldots, f_n, f_{n+1}) W_{n-1}(f_1, f_2, \ldots, f_{n-1})}{W_n(f_1, f_2, \ldots, f_{n-1}, f_n)^2}. \tag{4.7}$$

Using the gauge transformation operator T_D, one can construct the following n-step gauge transformation :

$$L^{(0)} \xrightarrow{T_D^{(0)}} L^{(1)} \xrightarrow{T_D^{(1)}} L^{(2)} \xrightarrow{T_D^{(2)}} \cdots \xrightarrow{T_D^{(n-1)}} L^{(n)},$$

where $T_D^{(i)} = T_D(\Phi_1^{(i)})$. According to the above formula, we can obtain the following proposition.

Proposition 4.1. *For the Lax operator of the constrained mKP hierarchy, under n steps of T_D, we have*

$$L^{(0)} = (L^{(0)})_{\geqslant 1} + \sum_{i=1}^{m} \Phi_i^{(0)} \partial^{-1} \Psi_i^{(0)} \partial, \tag{4.8}$$

$$\text{(i)} \quad \Phi_1^{(n)} = (-1)^n \frac{W_{n+1}(\Phi_1^{(0)}, \eta_1, \eta_2, \ldots, \eta_n)}{W_n(\Phi_1^{(0)}, \eta_1, \eta_2, \ldots, \eta_{n-1})}, \tag{4.9}$$

$$\text{(ii)} \quad \Phi_i^{(n)} = \frac{W_{n+1}(\Phi_1^{(0)}, \eta_1, \eta_2, \ldots, \eta_n, \Phi_i^{(0)})}{W_n(\Phi_1^{(0)}, \eta_1, \eta_2, \ldots, \eta_{n-1})}, \tag{4.10}$$

$$\text{(iii)} \quad \Psi_1^{(n)} = (-1)^{n-1} \frac{W_{n-1}(\Phi_1^{(0)}, \eta_1, \eta_2, \ldots, \eta_{n-2})}{W_n(\Phi_1^{(0)}, \eta_1, \eta_2, \ldots, \eta_{n-1})}, \tag{4.11}$$

$$\text{(iv)} \quad \Psi_i^{(n)} = (-1)^n \int \frac{IW_{1,n}(\Psi_{1x}^{(0)}; \Phi_1^{(0)}, \eta_1, \ldots, \eta_{n-1})}{W_n(\Phi_1^{(0)}, \eta_1, \eta_2, \ldots, \eta_{n-1})} dx, \tag{4.12}$$

where $\eta_k = (L^{(0)})^k(\Phi_1^{(0)})$, $i = 2, 3, \ldots, m$ and $k = 1, 2, 3, \ldots$.

Proof. (i) According to proposition 3.5,

$$\Phi_1^{(1)} = -T_D(\Phi_1^{(0)})(L^{(0)}(\Phi_1^{(0)})) = -T_D(\Phi_1^{(0)})(\eta_1) = -\eta_1^{(1)},$$
$$\Phi_1^{(2)} = (-1)^2 T_D(\Phi_1^{(1)}) T_D(\Phi_1^{(0)})(L^{(0)2}(\Phi_1^{(0)})) = (-1)^2 T_D(\Phi_1^{(1)}) T_D(\Phi_1^{(0)})(\eta_2) = (-1)^2 \eta_2^{(2)},$$
$$\ldots,$$

we can know $\Phi_1^{(k)} = (-1)^k \eta_k^{(k)}$ by induction, thus $T_D^{(i)} = T_D(\eta_1^{(i)})$.

$$\Phi_1^{(n)} = (-1)^n T_D^{(n-1)} T_D^{(n-2)} \ldots T_D^{(1)} T_D^{(0)} (L^{(0)n}(\Phi_1^{(0)})) = (-1)^n T^{(n,0)}(\Phi_1^{(0)}, \eta_1, \ldots, \eta_{n-1})(\eta_n).$$

Therefore with the help of (4.3), one can obtain (4.9).
(ii)

$$\Phi_i^{(n)} = T^{(n,0)}(\Phi_1^{(0)}, \eta_1, \ldots, \eta_{n-1})(\Phi_1^{(0)}).$$

In the same way as (i), we can get (ii).
(iii) By Proposition 3.5

$$\Psi_1^{(n)} = \left(\Phi_1^{(n-1)}\right)^{-1} = (-1)^{n-1} \frac{W_{n-1}(\Phi_1^{(0)}, \eta_1, \eta_2, \ldots, \eta_{n-2})}{W_n(\Phi_1^{(0)}, \eta_1, \eta_2, \ldots, \eta_{n-1})}.$$

(iv) According to Proposition 3.5 and (4.4),

$$\Psi_i^{(n)} = \partial^{-1}((T^{(n,0)})^{-1})^* \partial(\Psi_i^{(0)}),$$

we can get (iv). □

Similarly, we can construct another n-step gauge transformation using only T_I :

$$L^{(0)} \xrightarrow{T_I^{(0)}} L^{(1)} \xrightarrow{T_I^{(1)}} L^{(2)} \xrightarrow{T_I^{(2)}} \cdots \xrightarrow{T_I^{(n-1)}} L^{(n)}.$$

where $T_I^{(i)} = T_D(\Psi_1^{(i)})$. Under the successive application of T_I, we have the following conclusions.

Proposition 4.2. *For the Lax operator of the constrained mKP hierarchy, under n steps of T_I, we have*

$$L^{(0)} = (L^{(0)})_{\geqslant 1} + \sum_{i=1}^{m} \Phi_i^{(0)} \partial^{-1} \Psi_i^{(0)} \partial,$$

$$\text{(i)} \quad \Psi_1^{(n)} = \frac{W_{n+1}(\Psi_1^{(0)}, \widehat{\eta}_1, \widehat{\eta}_2, \ldots, \widehat{\eta}_n)}{W_{n+1}(1, \Psi_1^{(0)}, \widehat{\eta}_1, \widehat{\eta}_2, \ldots, \widehat{\eta}_{n-1})},$$

$$\text{(ii)} \quad \Psi_i^{(n)} = \frac{W_{n+1}(\Psi_1^{(0)}, \widehat{\eta}_1, \widehat{\eta}_2, \ldots, \widehat{\eta}_{n-1}, \Psi_i^{(0)})}{W_{n+1}(1, \Psi_1^{(0)}, \widehat{\eta}_1, \widehat{\eta}_2, \ldots, \widehat{\eta}_{n-1})},$$

$$\text{(iii)} \quad \Phi_1^{(n)} = \frac{W_n(1, \Psi_1^{(0)}, \widehat{\eta}_1, \widehat{\eta}_2, \ldots, \widehat{\eta}_{n-2})}{W_n(\Psi_1^{(0)}, \widehat{\eta}_1, \widehat{\eta}_2, \ldots, \widehat{\eta}_{n-1})},$$

$$\text{(iv)} \quad \Phi_i^{(n)} = \frac{IW_{1,n}(\Phi_{ix}^{(0)}; \Psi_{1x}^{(0)}, \eta_{1x}, \ldots, \eta_{n-1,x})}{W_n(\Psi_1^{(0)}, \widehat{\eta}_1 \cdots \widehat{\eta}_{n-1})},$$

where $i = 2, 3, \ldots, m$ and

$$\widehat{\eta}_k = \int (L^{(0)*})^k (\Psi_{1x}^{(0)}) dx. \quad k = 1, 2, 3, \ldots.$$

Proof. According to Proposition 3.5,

$$\Psi_1^{(1)} = -\partial^{-1}(T_I(\Psi_1^{(0)})^{-1})^* \partial(\partial^{-1}(L^{(0)*})\partial(\Psi_1^{(0)})) = -\partial^{-1}(T_I(\Psi_1^{(0)})^{-1})^* \partial(\widehat{\eta}_1) = -\widehat{\eta}_1^{(1)},$$

$$\begin{aligned}\Psi_1^{(2)} &= (-1)^2 \partial^{-1}(T_I(\Psi_1^{(1)} T_I(\Psi_1^{(0)})^{-1})^* \partial(\partial^{-1}(L^{(0)*})^2 \partial(\Psi_1^{(0)})) \\ &= (-1)^2 \partial^{-1}(T_I(\Psi_1^{(1)} T_I(\Psi_1^{(0)})^{-1})^* \partial(\widehat{\eta}_2) = (-1)^2 \widehat{\eta}_2^{(2)}, \\ &\cdots\end{aligned}$$

One can know $\Psi_1^{(k)} = (-1)^k \widehat{\eta}_k^{(k)}$ by induction. Thus $T_I^{(i)} = T_I(\widehat{\eta}_i^{(i)})$. Then all the results can be proved with the same method as the one in Proposition 4.1, by using Lemma 4.2 and Lemma 4.3. □

5. Conclusions and Discussions

The main results of this paper are as follows. First, we give two kinds of gauge transformations for the constrained modified KP hierarchy, which are summarized in Proposition 3.5. The corresponding gauge transformations should not only maintain the form of the Lax operator, but also keep the form of the Lax equation unchanged. For this, by selecting special generating eigenfunction and adjoint eigenfunction, the elementary gauge transformation operators of modified KP hierarchy $T_D(\Phi) = (\Phi^{-1})_x^{-1} \partial \Phi^{-1}$ and $T_I(\Psi) = \Psi^{-1} \partial^{-1} \Psi_x$, become the ones in the constrained case, which are verified in Proposition 3.1 and Proposition 3.3 respectively. Second, we find that the two gauge transformations T_D and T_I can cancel each other. Therefore, in the Section 4, we can only discuss the successive applications of T_D or T_I. The specific forms of the successive applications of T_D or T_I are given in Proposition 4.1 and Proposition 4.2 respectively.

The selections of the special generating eigenfunction and adjoint eigenfunction in $T_D(\Phi)$ and $T_I(\Psi)$ are very crucial, which can go back to Aratyn et al's work in [1, 2] for the constrained KP case. Another kind of important choices is given by Oevel [25]. The eigenfunction ϕ, satisfying $L_{KP}(\phi) = \lambda\phi$ for some constant λ, is chosen as the generating eigenfunction for the constrained KP case in [25]. Willox et al's work [30] provides one good rule (see (23) in [30]) of selecting eigenfunction in gauge transformation for the constrained KP case, containing the results of Aratyn and Oevel. Based upon these different choices, Willox et al [30] also considered an interesting binary transformation for the constrained KP hierarchy, which is the combination of the Darboux transformation of Aratyn's type with the one of Oevel's type.

By considering the relation between the constrained KP and mKP hierarchies [26], it will be very natural and interesting to ask what the relations between their gauge transformations are, and whether there are gauge transformations similar to those in [30] for the case of the constrained mKP hierarchy. For this, we will consider these questions in our future work.

Acknowledgments

This work is supported by China Postdoctoral Science Foundation (Grant No. 2016M591949) and Jiangsu Postdoctoral Science Foundation (Grant No. 1601213C). We thank anonymous referee for his/her useful comments and suggestions.

References

[1] H. Aratyn, E. Nissimov and S. Pacheva, Darboux-Backlund solutions of SL(p,q) KP-KdV hierarchies, constrained generalized Toda lattices, and two-matrix string model, *Phys. Lett. A* **201** (1995) 293–305.
[2] H. Aratyn, E. Nissimov and S. Pacheva, Virasoro symmetry of constrained KP hierarchies, *Phys. Lett. A* **228** (1997) 164–175.
[3] L.L. Chau, J.C. Shaw and M.H. Tu, Solving the constrained KP hierarchy by gauge transformations, *J. Math.Phys.* **38** (1997) 4128–4137.
[4] L.L. Chau, J.C. Shaw and H.C. Yen, Solving the KP hierarchy by gauge transformations, *Commun. Math. Phys.* **149** (1992) 263–278.
[5] J.P. Cheng, Miura and auto-Backlund transformations for the q-deformed modified KP hierarchies, *J. Nonlin. Math. Phys.* **24** (2017) 7–19.
[6] J.P. Cheng, The gauge transformation of modified KP hierarchy, *J. Nonlin. Math. Phys.* **25** (2018) 66–85.
[7] J.P. Cheng, J.Z. Wang and X.Y. Zhang, The gauge transformation of the q-deformed modified KP hierarchy, *J. Nonlin. Math. Phys.* **21** (2014) 533–542.
[8] Y. Cheng, Modifying the KP, the n^{th} constrained KP hierarchies and their Hamiltonian structures, *Commun. Math. Phys.* **171** (1995) 661–682.
[9] E. Date, M. Kashiwara and T. Miwa, Transformation groups for solition equations, in *Nonlinear integrable systems-classical theory and quantum theory*, ed. by M. Jimbo and T. Miwa, World Scientific, Singapore, 1983, pp. 39–119.
[10] L.A. Dickey, Modified KP and discrete KP, *Lett. Math. Phys.* **48** (1999) 277–289.
[11] L.A. Dickey, Soliton equations and Hamiltonian systems (2nd Edition), World scientific, Singapore, 2003.
[12] J.S. He, Y.H. Li and Y. Cheng, q-deformed KP hierarchy and q-deformed constrained KP hierarchy, *SIGMA* **2** (2006) 060.
[13] J.S. He, Y.S. Li and Y. Cheng, The determinant representation of the gauge transformation operators, *Chin. Ann. Math. B* **23** (2002) 475–486.
[14] J.S. He, Z.W. Wu and Y. Cheng, Gauge transformations for the constrained CKP and BKP hierarchies, *J. Math. Phys.* **48** (2007) 113519.
[15] M. Jimbo and T. Miwa, Solitons and infinite dimensional Lie algebras, *Publ. RIMS, Kyoto Univ.* **19** (1983) 943–1001.

[16] V. Kac and J. van de Leur, Equivalence of formulations of the MKP hierarchy and its polynomial tau-functions, arXiv: 1801. 02845.
[17] M. Kashiwara and T. Miwa, The τ function of the Kadomtsev-Petviashvili equation, Transfromation groups for soliton equations.I, *Proc. Japan Acad. A.* **57** (1981) 342–347.
[18] K. Kiso, A remark on the commuting flows defined by Lax equations, *Prog. Theor. Phys.* **83** (1990) 1108–1125.
[19] B. Konopelchenko and W. Oevel, An r-matrix approach to nonstandard classes of integrable equations, *Publ. RIMS, Kyoto Univ.* **29** (1993) 581–666.
[20] B.A. Kupershmidt, Mathematics of dispersive water waves, *Commun. Math. Phys.* **99** (1985) 51–73.
[21] N. Li and J.P. Cheng, The successive applications of two types of gauge transformations for the q-deformed modified Kadomtsev-Petvivashvili hierarchy, *Z. Naturforsch. A* **73** (2018) 345–356.
[22] M.H. Li, J.P. Cheng and J.S. He, The successive application of the gauge transformation for the modified semidiscrete KP hierarchy, *Z. Naturforsch. A* **71** (2016) 1093–1098.
[23] S.W. Liu, Y. Cheng and J.S. He, The determinant representation of the gauge transformation for the discrete KP hierarchy, *Sci. China Math.* **53** (2010) 1195–1206.
[24] W. Oevel and C. Rogers, Gauge transformations and reciprocal links in $2+1$ dimensions, *Rev. Math. Phys.* **5** (1993) 299–330.
[25] W. Oevel, Darboux theorems and Wronskian formulas for integrable system I: constrained KP flows, *Physica A* **195** (1993) 533–576.
[26] W. Oevel and S. Carillo, Squared eigenfunction symmetries for soliton equations: Part II, *J. Math. Anal. Appl.* **217** (1998) 179–199.
[27] T. Song, C. Z. Li and X. Y. Li, Gauge transformations of the multi-component BKP and CKP hierarchies, *Mod. Phys. Lett. B* **30** (2017) 1750280.
[28] T. Takebe, A note on the modified KP hierarchy and its (yet another) dispersionless limit, Lett. *Math. Phys.* **59** (2002) 157–172.
[29] T. Takebe and L. P. Teo, Coupled modified KP hierarchy and its dispersionless limit, *SIGMA* **2** (2006) 072.
[30] R. Willox, I. Loris and C.R. Gilson, Binary Darboux transformations for constrained KP hierarchies, *Inverse Problems* **13** (1997) 849–865.