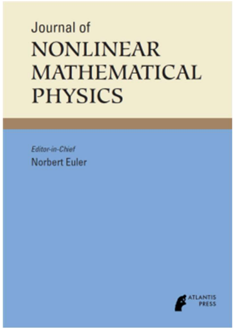

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On a canonical nine-body problem. Integrable Ermakov decomposition via reciprocal transformations

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Here, a recently introduced nine-body problem is shown to be decomposable via a novel class of reciprocal transformations into a set of integrable Ermakov systems. This Ermakov decomposition is exploited to construct more general integrable nine-body systems in which the canonical nine-body system is embedded.

Keywords: Many Body; Ermakov; Reciprocal.

2000 Mathematics Subject Classification: MSC 37K10

1. Introduction

The study of many-body problems with their established importance in both classical and quantum mechanics has an extensive literature motivated, in particular, by the pioneering models of Calogero [4,5], Moser [20] and Sutherland [48,49]. The surveys [6,19,22] and the literature cited therein may be consulted in this regard.

In recent work in [25], non-autonomous extensions of 3-body and 4-body systems incorporating those set down in [2, 7, 17] have been shown to be decomposible into integrable multi-component Ermakov systems. Two-component Ermakov systems adopt the form [23, 24, 26]

$$\begin{aligned} \ddot{\alpha} + \omega(t)\alpha &= \frac{1}{\alpha^2\beta}\Phi(\beta/\alpha), \\ \ddot{\beta} + \omega(t)\beta &= \frac{1}{\alpha\beta^2}\Psi(\alpha/\beta) \end{aligned} \tag{1.1}$$

and admit a distinctive integral of motion, namely, the invariant

$$\mathscr{E} = \frac{1}{2}(\alpha\dot{\beta} - \beta\dot{\alpha})^2 + \int^{\beta/\alpha}\Phi(z)dz + \int^{\alpha/\beta}\Psi(w)dw \tag{1.2}$$

together with concomitant nonlinear superposition principles. Here, a dot indicates a derivative with respect to the independent variable t. Ermakov systems of the type (1.1) have diverse physical applications, notably in nonlinear optics [8, 13–16, 27, 28, 51]. There, in particular, they arise in the description of the evolution of size and shape of the light spot and wave front in elliptical Gaussian beams. In 2+1-dimensional rotating shallow water hydrodynamics, Hamiltonian two-component Ermakov systems of the type (1.1) have been derived in [29] which describe the time evolution of the semi-axes of the elliptic moving shoreline on an underlying circular paraboloidal basin. Ermakov-Ray-Reid systems have also been obtained in magnetogasdynamics in [30,50] and novel pulsrodon-type phenomena thereby isolated analogous to that observed in an elliptic warm-core oceanographic eddy context [31,47]. Nonlinear coupled systems of Ermakov-type have also been shown in [32] to arise in the description of gas cloud evolution as originally investigated by Dyson [11]. In [28], it was shown that the occurrence of integrable Hamiltonian Ermakov-Ray-Reid systems in nonlinear physics and continuum mechanics, remarkably, extends to the spiralling elliptic soliton system of [9] and to its generalisation in the Bose-Einstein context of [1].

In connection with many-body problems, a nine-body system has recently been investigated in detail in [3]. The mode of treatment for this canonical system was thereby extended to a class of 3^k many-body problems. Here, an alternative approach to nine-body problems of the type in [3] is adopted wherein they are shown via a novel class of reciprocal transformations to be decomposible into an equivalent set of integrable Ermakov systems. This Ermakov connection is exploited here to embed the original nine-body system in a wide solvable class involving a triad of arbitrary functions $J_1(y/x),\ J_2(y/x)$ and $J_3(y/x)$ where x,y are Jacobi variables. These $J_i(y/x),\ i=1,2,3$ are associated with a general parametrisation of two-component Hamiltonian Ermakov systems as originally introduced in [29].

2. A Class of Nine-Body Problems. Ermakov Reduction

Here, we consider a class of nine-body problems

$$\ddot{x}_i = \frac{\partial Z}{\partial x_i}, \quad i = 1,2,...,9 \tag{2.1}$$

of the type introduced in [3] with

$$\begin{aligned} Z = &\sum_{1\leqslant i\leqslant j<3} \frac{\lambda_{1,1}}{(x_i - x_j)^2} + \sum_{4\leqslant i<j\leqslant 6} \frac{\lambda_{2,1}}{(x_i - x_j)^2} + \sum_{7\leqslant i<j\leqslant 9} \frac{\lambda_{3,1}}{(x_i - x_j)^2} \\ &+ \sum_{1\leqslant i<j\leqslant 3} \frac{3\lambda_{1,2}}{(x_{3i-2} + x_{3i-1} + x_{3i} - x_{3j-2} - x_{3j-1} - x_{3j})^2} - \frac{\delta}{2\left(\sum\limits_{i=1}^{9} x_i\right)^2} + \frac{\mu}{\sum\limits_{i=1}^{9} x_i^2}. \end{aligned} \tag{2.2}$$

It is noted that $\delta = 0$ in the system investigated in [3].

Jacobi and centre of mass co-ordinates are now introduced according to

$$\begin{gathered} x = \frac{1}{\sqrt{2}}(x_1 - x_2)\,, \quad y = \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3)\,, \\ z = \frac{1}{\sqrt{2}}(x_4 - x_5)\,, \quad v = \frac{1}{\sqrt{6}}(x_4 + x_5 - 2x_6)\,, \\ r = \frac{1}{\sqrt{2}}(x_7 - x_8)\,, \quad s = \frac{1}{\sqrt{6}}(x_7 + x_8 - 2x_9) \\ w = \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3),\ p = \frac{1}{\sqrt{3}}(x_4 + x_5 + x_6),\ q = \frac{1}{\sqrt{3}}(x_7 + x_8 + x_9)\,. \end{gathered} \tag{2.3}$$

Under this linear transformation, the invariance property

$$\Sigma := x_1^2 + x_2^2 + \cdots + x_9^2 = x^2 + y^2 + z^2 + v^2 + r^2 + s^2 + w^2 + p^2 + q^2 \tag{2.4}$$

is seen to hold. Moreover,

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{6}}y+\frac{1}{\sqrt{3}}w, \quad x_2=-\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{6}}y+\frac{1}{\sqrt{3}}w, \quad x_3=-\sqrt{\frac{2}{3}}y+\frac{1}{\sqrt{3}}w, \\ x_4 &= \frac{1}{\sqrt{2}}z+\frac{1}{\sqrt{6}}v+\frac{1}{\sqrt{3}}p, \quad x_5=-\frac{1}{\sqrt{2}}z+\frac{1}{\sqrt{6}}v+\frac{1}{\sqrt{3}}p, \quad x_6=-\sqrt{\frac{2}{3}}v+\frac{1}{\sqrt{3}}p, \\ x_7 &= \frac{1}{\sqrt{2}}r+\frac{1}{\sqrt{6}}s+\frac{1}{\sqrt{3}}q, \quad x_8=-\frac{1}{\sqrt{2}}r+\frac{1}{\sqrt{6}}s+\frac{1}{\sqrt{3}}q, \quad x_9=-\sqrt{\frac{2}{3}}s+\frac{1}{\sqrt{3}}q. \end{aligned} \tag{2.5}$$

'In extenso' on re-labelling, (2.2) yields

$$\begin{aligned} Z = {} & \frac{\lambda_1}{(x_1-x_2)^2}+\frac{\lambda_2}{(x_2-x_3)^2}+\frac{\lambda_3}{(x_3-x_1)^2} \\ & +\frac{\mu_1}{(x_4-x_5)^2}+\frac{\mu_2}{(x_5-x_6)^2}+\frac{\mu_3}{(x_6-x_4)^2} \\ & +\frac{\nu_1}{(x_7-x_8)^2}+\frac{\nu_2}{(x_8-x_9)^2}+\frac{\nu_3}{(x_9-x_7)^2} \\ & +\frac{\sigma_1}{(x_1+x_2+x_3-x_4-x_5-x_6)^2}+\frac{\sigma_2}{(x_4+x_5+x_6-x_7-x_8-x_9)^2} \\ & +\frac{\sigma_3}{(x_7+x_8+x_9-x_1-x_2-x_3)^2} \\ & -\frac{\delta}{2(x_1+x_2+\ldots+x_9)^2}+\frac{\mu}{x_1^2+x_2^2+\ldots+x_9^2} \,. \end{aligned} \tag{2.6}$$

In terms of the Jacobi and centre of mass co-ordinates, the 9-body system determined by (2.1) with Z given by (2.6) becomes

$$\begin{aligned} \ddot{x}+\frac{2\mu x}{\Sigma^2} &= -\frac{\lambda_1}{x^3}+\frac{4\lambda_2}{(\sqrt{3}y-x)^3}-\frac{4\lambda_3}{(x+\sqrt{3}y)^3}\,, \\ \ddot{y}+\frac{2\mu y}{\Sigma^2} &= -\frac{4\sqrt{3}\lambda_2}{(\sqrt{3}y-x)^3}-\frac{4\sqrt{3}\lambda_3}{(x+\sqrt{3}y)^3}\,, \\ \ddot{z}+\frac{2\mu z}{\Sigma^2} &= -\frac{\mu_1}{z^3}+\frac{4\mu_2}{(\sqrt{3}v-z)^3}-\frac{4\mu_3}{(z+\sqrt{3}v)^3}\,, \\ \ddot{v}+\frac{2\mu v}{\Sigma^2} &= -\frac{4\sqrt{3}\mu_2}{(\sqrt{3}v-z)^3}-\frac{4\sqrt{3}\mu_3}{(z+\sqrt{3}v)^3}\,, \\ \ddot{r}+\frac{2\mu r}{\Sigma^2} &= -\frac{\nu_1}{r^3}+\frac{4\nu_2}{(\sqrt{3}s-r)^3}-\frac{4\nu_3}{(r+\sqrt{3}s)^3}\,, \\ \ddot{s}+\frac{2\mu s}{\Sigma^2} &= -\frac{4\sqrt{3}\nu_2}{(\sqrt{3}s-r)^3}-\frac{4\sqrt{3}\nu_3}{(r+\sqrt{3}s)^3}\,, \\ \ddot{w}+\frac{2\mu w}{\Sigma^2} &= \frac{2}{3}\left[-\frac{\sigma_1}{(w-p)^3}+\frac{\sigma_3}{(q-w)^3}\right]+\frac{\delta}{3(w+p+q)^3}\,, \\ \ddot{p}+\frac{2\mu p}{\Sigma^2} &= \frac{2}{3}\left[\frac{\sigma_1}{(w-p)^3}-\frac{\sigma_2}{(p-q)^3}\right]+\frac{\delta}{3(w+p+q)^3}\,, \\ \ddot{q}+\frac{2\mu q}{\Sigma^2} &= \frac{2}{3}\left[\frac{\sigma_2}{(p-q)^3}-\frac{\sigma_3}{(q-w)^3}\right]+\frac{\delta}{3(w+p+q)^3}\,. \end{aligned} \tag{2.7}$$

3. Application of a Reciprocal Transformation

Reciprocal-type transformations have diverse physical applications in such areas as gasdynamics and magnetogasdynamics [33,34], the solution of nonlinear moving boundary problems [12,35–38], the analysis of oil/water migration through a porous medium [39] and in Cattaneo-type hyperbolic nonlinear heat conduction [40]. Reciprocal transformations have also been applied in the theory of discontinuity-wave propagation [10]. In soliton theory, the conjugation of reciprocal and gauge transformations has been used to link integrable equations and the inverse scattering schemes in which they are embedded [18,21,41–45]. Here, a novel class of reciprocal transformations is introduced in the present context of many-body theory. This is used to reduce the nine-body system (2.7) in Jacobi and centre of mass co-ordinates to an equivalent set of integrable two-component Ermakov systems of the type (1.1) augmented by a classical single component Ermakov equation in a centre of mass component.

Thus, the class of reciprocal transformations

$$\left.\begin{array}{c} x^* = x/\rho, \quad y^* = y/\rho, \quad z^* = z/\rho, \\ v^* = v/\rho, \quad s^* = s/\rho, \quad w^* = w/\rho, \quad p^* = p/\rho, \quad q^* = q/\rho \\ \rho^* = \rho^{-1}, \quad dt^* = \rho^{-2} dt \end{array}\right\} \quad \mathbb{R}^* \tag{3.1}$$

such that $\mathbb{R}^{*2} = \mathrm{I}$ is introduced wherein ρ is governed by the base equation

$$\ddot{\rho} + 2\mu\rho\Sigma^{-2} = 0 \tag{3.2}$$

under $\mathbb{R}^*$, the nine-body system becomes

$$\begin{aligned} x^*_{t^*t^*} &= -\frac{\lambda_1}{x^{*3}} + \frac{4\lambda_2}{(\sqrt{3}y^* - x^*)^3} - \frac{4\lambda_3}{(x^* + \sqrt{3}y^*)^3}, \\ y^*_{t^*t^*} &= -\frac{4\sqrt{3}\lambda_2}{(\sqrt{3}y^* - x^*)^3} - \frac{4\sqrt{3}\lambda_3}{(x^* + \sqrt{3}y^*)^3}, \\ z^*_{t^*t^*} &= -\frac{\mu_1}{z^{*3}} + \frac{4\mu_2}{(\sqrt{3}v^* - z^*)^3} - \frac{4\mu_3}{(z^* + \sqrt{3}v^*)^3}, \\ v^*_{t^*t^*} &= -\frac{4\sqrt{3}\mu_2}{(\sqrt{3}v^* - z^*)^3} - \frac{4\sqrt{3}\mu_3}{(z^* + \sqrt{3}v^*)^3}, \\ r^*_{t^*t^*} &= -\frac{\nu_1}{r^{*3}} + \frac{4\nu_2}{(\sqrt{3}s^* - r^*)^3} - \frac{4\nu_3}{(r^* + \sqrt{3}s^*)^3}, \\ s^*_{t^*t^*} &= -\frac{4\sqrt{3}\nu_2}{(\sqrt{3}s^* - r^*)^3} - \frac{4\sqrt{3}\nu_3}{(r^* + \sqrt{3}s^*)^3}, \end{aligned} \tag{3.3}$$

together with

$$\begin{aligned} w^*_{t^*t^*} &= \frac{2}{3}\left[-\frac{\sigma_1}{(w^* - p^*)^3} + \frac{\sigma_3}{(q^* - w^*)^3}\right] + \frac{\delta}{3R^{*3}}, \\ p^*_{t^*t^*} &= \frac{2}{3}\left[\frac{\sigma_1}{(w^* - p^*)^3} - \frac{\sigma_2}{(p^* - q^*)^3}\right] + \frac{\delta}{R^{*3}}, \\ q^*_{t^*t^*} &= \frac{2}{3}\left[\frac{\sigma_2}{(p^* - q^*)^3} - \frac{\sigma_3}{(q^* - w^*)^3}\right] + \frac{\delta}{R^{*3}}, \end{aligned} \tag{3.4}$$

where

$$R^* = \frac{R}{\rho}\,, \quad R = \frac{1}{\sqrt{3}}(x_1 + x_2 + \cdots + x_9) = w + p + q\,. \tag{3.5}$$

In the above, (3.3) constitute three copies of integrable Hamiltonian Ermakov systems of the same kind as obtained in [25] for the original 3-body system of Calogero [4].

On introduction of the translational change of variables

$$W^* = w^* - p^*, \quad P^* = p^* - q^* \tag{3.6}$$

the system (3.4) produces the two-component Ermakov system

$$\begin{aligned} W^*_{t^*t^*} &= \frac{2}{3}\left[-\frac{2\sigma_1}{W^{*3}} + \frac{\sigma_2}{P^{*3}} - \frac{\sigma_3}{(W^* + P^*)^3}\right], \\ P^*_{t^*t^*} &= \frac{2}{3}\left[\frac{\sigma_1}{W^{*3}} - \frac{2\sigma_2}{P^{*3}} - \frac{\sigma_3}{(W^* + P^*)^3}\right] \end{aligned} \tag{3.7}$$

while addition of the constituent members of (3.3)–(3.4) shows that

$$R^*_{t^*t^*} = \frac{\delta}{3R^{*3}} \tag{3.8}$$

namely, a single component Ermakov equation in R^*.

The triad (3.3) constitutes three de-coupled Ermakov systems of the type (1.1) and with associated Hamiltonians

$$\begin{aligned} \mathscr{H}_{\text{I}} &= \frac{1}{2}\left[x^{*2}_{t^*} + y^{*2}_{t^*}\right] + \frac{1}{x^{*2}}\left[\frac{\lambda_1}{2} + \frac{2\lambda_2}{(\sqrt{3}y^*/x^* - 1)^3} + \frac{2\lambda_3}{(\sqrt{3}y^*/x^* + 1)^3}\right], \\ \mathscr{H}_{\text{II}} &= \frac{1}{2}\left[z^{*2}_{t^*} + v^{*2}_{t^*}\right] + \frac{1}{z^{*2}}\left[\frac{\mu_1}{2} + \frac{2\mu_2}{(\sqrt{3}v^*/z^* - 1)^3} + \frac{2\mu_3}{(\sqrt{3}v^*/z^* + 1)^3}\right], \\ \mathscr{H}_{\text{III}} &= \frac{1}{2}\left[r^{*2}_{t^*} + s^{*2}_{t^*}\right] + \frac{1}{r^{*2}}\left[\frac{\nu_1}{2} + \frac{2\nu_2}{(\sqrt{3}s^*/r^* - 1)^3} + \frac{2\nu_3}{(\sqrt{3}s^*/r^* + 1)^3}\right]. \end{aligned} \tag{3.9}$$

The sub-system (3.4), on the other hand admits the Hamiltonian

$$\begin{aligned} \mathscr{H}_{\text{IV}} = \frac{1}{2}\left[w^{*2}_{t^*} + p^{*2}_{t^*} + q^{*2}_{t^*}\right] - \frac{1}{3}\left[\frac{\sigma_1}{(w^* - p^*)^2} + \frac{\sigma_2}{(p^* - q^*)^2} + \frac{\sigma_3}{(w^* - q^*)^2}\right] \\ + \frac{\delta}{6(w^* + p^* + q^*)^2} \end{aligned} \tag{3.10}$$

that is,

$$\mathscr{H}_{\text{IV}} = \frac{1}{6}\left[2W^{*2}_{t^*} + (P^*_{t^*} + R^*_{t^*})^2\right] - \frac{1}{3}\left[\frac{\sigma_1}{W^{*2}} + \frac{\sigma_2}{P^{*2}} + \frac{\sigma_3}{(W^* + P^*)^2}\right] + \frac{\delta}{6R^{*2}} \tag{3.11}$$

wherein R^* is determined by the Ermakov equation (3.8).

Thus, the nine-body system encapsulated in (3.3)–(3.4) is seen to be reducible to a quartet of two-component Ermakov systems of the classical type (1.1), augmented by the single component Ermakov equation (3.8) in R^*. Moreover, the Ermakov-Ray-Reid systems in (3.3) and (3.7), in addition to their admittance of characteristic Ermakov invariants, also admit second integrals of motion, namely, the Hamiltonians $\mathscr{H}_{\text{I}} \cdots \mathscr{H}_{\text{IV}}$ and hence are integrable.

The preceding determines $x^*(t^*),\ y^*(t^*),\ z^*(t^*),\ v^*(t^*),\ r^*(t^*),\ s^*(t^*)$ together with $w^* - p^*,\ p^* - q^*$ and $R^* = w^* + p^* + q^*$ and hence $w^*(t^*),\ p^*(t^*)$ and $q^*(t^*)$. In addition, the base equation (3.2) in ρ, in the reciprocal variables becomes

$$\rho^*_{t^*t^*} - \frac{2\mu\rho^*}{[x^* + y^{*2} + \cdots + q^{*2}]^2} = 0\,. \tag{3.12}$$

The latter becomes determinate on insertion of $x^*,\ y^*, \ldots, q^*$ as obtained by means of the integrable Hamiltonian Ermakov systems. With $\rho^*(t^*)$ to hand, $t = t(t^*)$ is determined by the pair of reciprocal relations

$$\rho^* = \rho^{-1}, \quad dt^* = \rho^{-2} dt \tag{3.13}$$

so that

$$dt = \rho^{*-2} dt^* \tag{3.14}$$

while the original Jacobi and centre of mass variables $x(t),\ y(t), \ldots,\ q(t)$ are given parametrically via t^* by the residual reciprocal relations

$$x = x^*(t^*)/\rho^*(t^*), \quad y = y^*(t^*)/\rho^*(t^*), \quad \ldots, \quad q = q^*(t^*)/\rho^*(t^*)\,. \tag{3.15}$$

4. An Extended Solvable Nine-Body System. Application of the Ermakov Connection

The isolation of integrable nonlinear systems of Ermakov-type is a subject of current interest (see e.g. [46]). Here, the Ermakov reduction of the nine-body system (2.1)–(2.2) may be exploited to embed it in a more general novel class of solvable nonlinear systems. Thus, it was shown in [29] that the class of Ermakov-Ray-Reid systems.

$$\begin{aligned} \ddot{\alpha} &= \frac{2}{\alpha^3} J(\beta/\alpha) + \frac{\beta}{\alpha^4} dJ(\beta/\alpha)/d(\beta/\alpha)\,, \\ \ddot{\beta} &= -\frac{1}{\alpha^3} dJ(\beta/\alpha)/d(\beta/\alpha) \end{aligned} \tag{4.1}$$

parametrised in terms of arbitrary $J(\beta/\alpha)$, admits the Hamiltonian

$$\mathscr{H} = \frac{1}{2}[\,\dot{\alpha}^2 + \dot{\beta}^2\,] + \frac{1}{\alpha^2} J(\beta/\alpha) \tag{4.2}$$

and Ermakov invariant

$$\mathscr{E} = \frac{1}{2}(\alpha\dot{\beta} - \dot{\alpha}\beta)^2 + \left(\frac{\alpha^2 + \beta^2}{\alpha^2}\right) J(\beta/\alpha) \tag{4.3}$$

which together allow the solution of the system (4.1) in an algorithmic manner.

In the present nine-body context, the Ermakov connection shows that the system (3.3) may be embedded in the integrable triad of Hamiltonian Ermakov-Ray-Reid systems

$$\begin{aligned}
x^*_{t^*t^*} &= -\frac{2}{x^{*3}}J_1(y^*/x^*) + \frac{y^*}{x^{*4}}dJ_1(y^*/x^*)/d(y^*/x^*)\ , \\
y^*_{t^*t^*} &= -\frac{1}{x^{*3}}J_1(y^*/x^*)\ , \\
z^*_{t^*t^*} &= -\frac{2}{z^{*3}}J_2(v^*/z^*) + \frac{v^*}{z^{*4}}dJ_2(v^*/z^*)/d(v^*/z^*)\ , \\
v^*_{t^*t^*} &= \frac{1}{z^{*3}}J_2(v^*/z^*)\ , \\
r^*_{t^*t^*} &= -\frac{2}{r^{*3}}J_3(r^*/s^*) + \frac{s^*}{r^{*4}}dJ_3(r^*/s^*)/d(r^*/s^*)\ , \\
s^*_{t^*t^*} &= \frac{1}{r^{*3}}J_3(r^*/s^*)\ ,
\end{aligned} \tag{4.4}$$

parametrised in terms of arbitrary $J_1(y^*/x^*)$, $J_2(v^*/z^*)$ and $J_3(r^*/s^*)$. These systems augmented by the integrable triad (3.4) determine a solvable nine-component system which is reciprocally associated to a novel class of nine-body problems in which the original system (2.1)–(2.2) is embedded.

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