A 3-Lie algebra and the dKP Hierarchy

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In terms of a 3-Lie algebra and the classical Poisson bracket \( \{ B_n, L \} \) of the dKP hierarchy, a special 3-bracket \( \{ B_m, B_n, L \} \) is proposed. When \( m = 0 \) or \( m = 1 \), the 3-lax equation \( \frac{dL}{dt} = \{ B_m, B_n, L \} \) is the dKP hierarchy and the corresponding proof is given. Meanwhile, for the generalized case \((m,n)\), the generalized dKP hierarchy is also investigated.

**Keywords:** 3-Lie algebra; dKP hierarchy; Integrable systems.

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1. Introduction

Nambu mechanics \([14, 15]\) is a generalization of classical Hamiltonian mechanics. In the context of integrable systems, the integrable hydrodynamical systems have been investigated via Nambu mechanics \([9, 10]\). Moreover various super-integrable systems, such as Calogero-Moser system, Kepler problem, three Hamiltonian structures of Landau problem, have been analyzed in the framework of Nambu mechanics \([1, 16]\), where a super-integrable system means that it is not only an integrable system in the Liouville-Arnold sense, but also possesses more constants of motion than degrees of freedom. With the development of infinite-dimensional 3-algebras \([2, 3, 6, 8]\), recently more attempts have been made to understand the connection between the infinite-dimensional 3-algebras and the integrable systems in the framework of Nambu mechanics. Chen et al. \([4]\) investigated the classical Heisenberg and \( w_3 \) 3-algebras, and established the relations between the dispersionless KdV hierarchy and these two infinite-dimensional 3-algebras. They found that the dispersionless KdV system is not only a bi-Hamiltonian system, but also a bi-Nambu-Hamiltonian system.
The \( W_{1+\infty} \) 3-algebra was constructed in [5] and its connection with the integrable system has also been investigated.

The dispersionless Kadomtsev-Petviashvili (dKP) hierarchy is a paradigm of the integrable systems, which arises as the quasi classical limit of the KP hierarchy. It consists of an infinite number of nonlinear differential equations. This kind of integrable system was introduced by Lebedev, Manin and Zakharov [13]. Many special solutions were obtained by Kodama and Gibbons [11]. Krichever [12] studied the dKP hierarchy and introduced the analogue of the tau function to integrate the consistency conditions for the free energy of the topological minimal models. It is well-known that the dKP hierarchy can be represented in terms of a Lax equation

\[
\frac{\partial L}{\partial t_n} = \{B_n, L\}.
\]

In this paper, we reinvestigate the property of the Lax equation \( B_n \) and \( L \) of the dKP hierarchy in the framework of 3-Lie algebra. In terms of a 3-bracket and Lax equation of the dKP hierarchy, we present a 3-Lax equation with respect to the Lax triple \( \{B_m, B_n, L\} \), and derive the corresponding (non)integrable nonlinear evolution equations for the cases of the different Lax triples \( \{B_m, B_n, L\} \).

This paper is organized as follows. In section 2, we briefly review the definition of the dKP hierarchy and define a 3-Lax equation

\[
\frac{\partial L}{\partial t_m} = \{B_m, B_n, L\}.
\]

Meanwhile, We prove that when \((m, n)\) takes the special values \((0, n+1)\) and \((1, n)\), the dKP hierarchy is derived. In addition, the generalized case of the 3-Lax equation \( \frac{\partial L}{\partial t_m} = \{B_m, B_n, L\} \) is discussed. We end this paper with the concluding remarks in section 3.

2. The dKP Hierarchy

The Lax equation with respect to a series of independent time variables \((t_1, t_2, \ldots)\) for the dKP hierarchy is

\[
\frac{\partial L}{\partial t_n} = \{B_n, L\}, \quad B_n = (L^n)_+,
\]

where \( L \) is a Laurent series of \( \lambda \) as the form

\[
L = \lambda + \sum_{i=2}^{+\infty} u_i \lambda^{-i+1},
\]

\((L^n)_+\) is the nonnegative powers of \( \lambda \) in the Laurent series \( L^n \), \( u_i = u_i(t_1, t_2, \ldots), i \geq 2 \), the bracket \( \{ , \} \) is the Poisson bracket in 2D phase space \((\lambda, \lambda, x = t_1)\)

\[
\{f, g\} = \frac{\partial f}{\partial \lambda} \frac{\partial g}{\partial x} - \frac{\partial g}{\partial \lambda} \frac{\partial f}{\partial x}.
\]

The dKP hierarchy (2.1) is a collection of nonlinear differential equations for \( u_n(x, t_2, \ldots) \) with respect to \((x, t_2, \ldots)\). This system is obtained by replacing micrdifferential operators \( \partial \) (in \( x \)) and their commutators of the KP hierarchy by Laurent series (in \( \lambda \)) and Poisson brackets. A number of characteristics of the KP hierarchy, indeed, persist in this hierarchy. For example, one can prove that the Lax equations (2.1) are equivalent to the zero-curvature equations

\[
\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + \{B_n, B_m\} = 0,
\]

with purely algebraic manipulation as done for the ordinary KP hierarchy [7].
Let us define a 3-bracket

$$\{f, g, h\} = f\{g, h\} + g\{h, f\} + h\{f, g\}, \quad (2.4)$$

where $f, g, h$ are smooth functions on 2D phase space ($\lambda, x$) and the bracket $\{ , \}$ is the Poisson bracket (2.3). The 3-bracket (2.4) satisfies the following properties:

1. skew-symmetry: $\{f_\sigma(1), f_\sigma(2), f_\sigma(3)\} = (-1)^{\varepsilon(\sigma)}\{f_1, f_2, f_3\}$,

2. Fundamental identity:

$$\{f_1, f_2, \{f_3, f_4, f_5\}\} = \{\{f_1, f_2, f_3\}, f_4, f_5\} + \{f_3, \{f_1, f_2, f_4\}, f_5\} + \{f_3, f_4, \{f_1, f_2, f_5\}\},$$

where $\varepsilon(\sigma)$ equals to 0 or 1 depending on the parity of the permutation $\sigma$. Therefore, with the 3-bracket (2.4), the set of all the smooth functions on the 2D phase space ($\lambda, x$) over a field of characteristic zero forms a 3-Lie algebra.

By means of the 3-bracket (2.4), we introduce the 3-Lax equation

$$\frac{\partial L}{\partial t_{m,n}} = \{B_m, B_n, L\}. \quad (2.5)$$

When $m = 0$, it is easy to see that (2.5) becomes (2.1).

Suppose $|\cdot|$ express the weight of $\cdot$. Let us assign the weights as follows,

$$|u_i| = i \quad (i \geq 2), \quad |x| = -1, \quad |1| = 0, \quad |f_{m,n}| = -(m+n),$$

then we have the following lemma.

**Lemma 2.1.** For the multivariate polynomial

$$\begin{align*}
(n-1)(1 + u_2 + \cdots + u_n)^{n+1} (u_2 + 2u_3 + \cdots + (n-1)u_n) \\
- (n+1)(1 + u_2 + \cdots + u_n)^{n} (2u_2 + \cdots + nu_n) + 3(1 + u_2 + \cdots + u_n)^{n+1},
\end{align*}$$

the terms with weight $n + 2 \quad (n \geq 2)$ disappear.

**Proof.** The terms with weight $n + 2$ in (2.6) can be written as the form $Cu_2^i u_3^j \cdots u_{n}^k$, where $0 \leq i_1, i_2, \ldots, i_n \leq n$ and $2i_2 + 3i_3 + \cdots + ni_n = n + 2$. The coefficient $C$ is

$$C = (n-1)^{\left[\begin{array}{c} C_{n+1,1}^{i_1} C_{n+1,2}^{i_2} \cdots C_{n+1,n-2}^{i_{n-2}} C_{n+1,n-1}^{i_{n-1}} C_{n+1,n-1}^{i_{n-1}} C_{n+1,n-1}^{i_{n-1}} \cdots C_{n+1,n}^{i_n} \\ + \cdots + C_{n+1,n}^{i_n} C_{n+1,n-1}^{i_{n-1}} C_{n+1,n-2}^{i_{n-2}} \cdots C_{n+1,1}^{i_1} \end{array}\right]} + (n+1)^{\left[\begin{array}{c} C_{n-2,1}^{i_1} C_{n-2,1}^{i_2} \cdots C_{n-2,n}^{i_n} C_{n-2,n}^{i_n} \cdots C_{n-2,n}^{i_n} \\ + \cdots + C_{n-2,n}^{i_n} C_{n-2,n-1}^{i_{n-1}} \cdots C_{n-2,n}^{i_n} \end{array}\right]} + 3C_{n+1,1}^{i_1} C_{n+1,1}^{i_2} \cdots C_{n+1,n}^{i_n}$$

$$= (n-1)^{\frac{(n+1)!}{i_2!i_3!\cdots i_n!(n+2-i_2-\cdots -i_n)!}} \frac{i_{n-1}(n-1) + i_{n-1}(n-2) + \cdots + i_2 \cdot 1}{n!}$$

$$- (n+1)^{\frac{(n+1)!}{i_2!i_3!\cdots i_n!(n+1-i_2-\cdots -i_n)!}} \frac{i_{n+1}(n+1) + i_{n-1}(n-1) + \cdots + i_2 \cdot 2}{n!}$$

$$+ 3 \frac{(n+1)!}{i_2!i_3!\cdots i_n!(n+1-i_2-\cdots -i_n)!}. \quad (2.7)$$
Using the condition $2i_2 + 3i_3 + \cdots + n_i_n = n + 2$, (2.7) equals to
\[
\frac{(n+1)!}{i_2!i_3!\cdots i_n!} \left[ \frac{(n-1)(n+2-i_2-\cdots -i_n)}{(n+2-i_2-\cdots -i_n)!} + \frac{-n+2+3}{(n+1-i_2-\cdots -i_n)!} \right] = 0.
\]

\[\square\]

**Theorem 2.1.** For the 3-brackets $\frac{n-1}{n+1} \{B_0, B_{n+1}, L\}$ and $\{B_1, B_n, L\}$, the corresponding coefficients of $\lambda^{-1}$ are the same, i.e. $\frac{n-1}{n+1} \frac{\partial u_2}{\partial u_{n+1}} = \frac{\partial u_2}{\partial u_1}$ by (2.5). Thus when $m = 1$ the 3-Lax equation (2.5) is in fact the dKP hierarchy as the case $m = 0$.

**Proof.** By direct calculations, we obtain
\[
B_n = \sum_{\substack{0 \leq i_1, \ldots, i_n \leq n \\text{i.i.d.} \\text{and} \\sum i_k = n \\text{mod} \ \text{div} \ \text{by} \ 3 \ \\text{and} \ \text{by} \ 2}} \frac{C^i}{C^{i_1}} C^{i_2} \cdots C^{i_n} u_2^{i_1} u_3^{i_2} \cdots u_n^{i_n} \lambda^{-(2i_2+3i_3+\cdots+i_n)}, \tag{2.8}
\]

and
\[
\{B_0, B_{n+1}, L\} = \sum_{\substack{0 \leq i_1, \ldots, i_n \leq n \\text{i.i.d.} \\text{and} \\sum i_k = n \\text{mod} \ \text{div} \ \text{by} \ 3 \ \\text{and} \ \text{by} \ 2}} \sum_{k=2}^{\infty} C^{i_1+1} C^{i_2} \cdots C^{i_n} u_2^{i_1} u_3^{i_2} \cdots u_n^{i_n} \lambda^{-(2i_2+3i_3+\cdots+i_n)}
\]
\[
- \sum_{\substack{0 \leq i_1, \ldots, i_n \leq n \\text{i.i.d.} \\text{and} \\sum i_k = n \\text{mod} \ \text{div} \ \text{by} \ 3 \ \\text{and} \ \text{by} \ 2}} \sum_{k=2}^{\infty} C^{i_1+1} C^{i_2} \cdots C^{i_n} u_2^{i_1} u_3^{i_2} \cdots u_n^{i_n} \frac{\partial (u_2^{i_1} u_3^{i_2} \cdots u_n^{i_n})}{\partial x} u_{k-1}.
\]

From the right side of (2.9), the coefficient of $\lambda^{-1}$ in $\frac{n-1}{n+1} \{B_0, B_{n+1}, L\}$ is
\[
(n-1)u_{n+2, x} + \sum_{k=2}^{n} \sum_{\substack{0 \leq i_1, \ldots, i_n \leq n \\text{i.i.d.} \\text{and} \\sum i_k = n \\text{mod} \ \text{div} \ \text{by} \ 3 \ \\text{and} \ \text{by} \ 2}} \frac{n-1}{i_1+1} C^{i_1} C^{i_2} \cdots C^{i_n} u_2^{i_1} u_3^{i_2} \cdots u_n^{i_n} \lambda^{-(2i_2+3i_3+\cdots+i_n)} (k-1) \frac{\partial (u_2^{i_1} u_3^{i_2} \cdots u_n^{i_n})}{\partial x}.
\]
Similarly, the coefficient of $\lambda^{-1}$ in $\{B_1, B_n, L\}$ is

$$(n-1)u_{n+2,3} + \sum_{k=2}^{n} \sum_{0 \leq i_1, \ldots, i_n \leq m \atop 2i_1+\cdots+3i_n = m+2-k} C_{n+1}^{i_1}C_{n-i_1}^{i_2} \cdots C_{n-i_1-i_2-\cdots-i_{n-1}}^{i_n} \frac{\partial}{\partial x} (u_2 u_3 \cdots u_n) = -3 \sum_{k=2}^{n} \sum_{0 \leq i_1, \ldots, i_n \leq m \atop 2i_1+\cdots+3i_n = m+2-k} C_{n}^{i_1}C_{n-i_1}^{i_2} \cdots C_{n-i_1-i_2-\cdots-i_{n-1}}^{i_n} \frac{\partial}{\partial x} (u_2 u_3 \cdots u_n) + 3 \sum_{k=2}^{n} \sum_{0 \leq i_1, \ldots, i_n \leq m \atop 2i_1+\cdots+3i_n = m+2-k} C_{n}^{i_1}C_{n-i_1}^{i_2} \cdots C_{n-i_1-i_2-\cdots-i_{n-1}}^{i_n} u_{k,x} u_2 u_3 \cdots u_n = 0. \tag{2.11}$$

To prove the theorem, we need prove the following equality

$$\sum_{k=2}^{n} \sum_{0 \leq i_1, \ldots, i_n \leq m \atop 2i_1+\cdots+3i_n = m+2-k} C_{n+1}^{i_1}C_{n-i_1}^{i_2} \cdots C_{n-i_1-i_2-\cdots-i_{n-1}}^{i_n} \frac{\partial}{\partial x} (u_2 u_3 \cdots u_n) = \sum_{k=2}^{n} \sum_{0 \leq i_1, \ldots, i_n \leq m \atop 2i_1+\cdots+3i_n = m+2-k} C_{n+1}^{i_1}C_{n-i_1}^{i_2} \cdots C_{n-i_1-i_2-\cdots-i_{n-1}}^{i_n} u_{k,x} u_2 u_3 \cdots u_n = 0. \tag{2.12}$$

On the left side of (2.12), the first term is the terms with weight $n+3$ among

$$\frac{\partial}{\partial x} \left[ \sum_{k=2}^{n} \sum_{0 \leq i_1, \ldots, i_n \leq m+1} \frac{n-1}{n+1} C_{n+1}^{i_1} (u_2 + \cdots + u_n)^{n+1-i_1} (k-1) u_k \right]
= (1 + u_2 + \cdots + u_n)^n (2u_2 + \cdots + nu_n)
= \frac{\partial}{\partial x} \left[ \frac{n-1}{n+1} (1 + u_2 + \cdots + u_n)^{n+1} (u_2 + 2u_3 + \cdots + (n-1)u_n) \right]
= (1 + u_2 + \cdots + u_n)^n (2u_2 + \cdots + nu_n) \right]. \tag{2.13}$$

Meanwhile the second term is the terms with weight $n+3$ among

$$3(1 + u_2 + \cdots + u_n)^n (u_2 x + \cdots + u_n x) = \frac{3}{n+1} \frac{\partial}{\partial x} [(1 + u_2 + \cdots + u_n)^{n+1}] \tag{2.14}$$

So the proof of (2.12) is equivalent to the proof of the terms with weight $n+2$ among

$$(n-1)(1 + u_2 + \cdots + u_n)^{n+1} (u_2 + 2u_3 + \cdots + (n-1)u_n)
= (n+1) (1 + u_2 + \cdots + u_n)^n (2u_2 + \cdots + nu_n) + 3(1 + u_2 + \cdots + u_n)^{n+1}
$$
disappear, which follows from Lemma 2.1. \hfill \Box

We further calculate the more general case of the 3-Lax equation (2.5). For convenience, let $I$ represent the condition $0 \leq i_1, \ldots, i_m \leq m; i_1 + i_2 + \cdots + i_m = m; 2i_2 + 3i_3 + \cdots + mi_m = m$, $J$ represent $0 \leq j_1, \ldots, j_n \leq n; j_1 + j_2 + \cdots + j_n = n; 2j_2 + 3j_3 + \cdots + nj_n = n$ and $K$ represent the condition $(2i_2 + \cdots + mi_m) + (2j_2 + \cdots + nj_n) = m + n - k + 1$. 

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Substituting (2.8) into (2.5) and comparing the coefficient of $\lambda^{-1}$ on both sides, we obtain

$$
\frac{\partial u_2}{\partial t_{m,n}} = (n-m)u_{m+n+1,x} + \sum_{k=2}^{m+n} \sum_{i,j,k} C_i^m C_j^{m-i} \cdots C_{i-1}^{m-n+i} C_{j-1}^{m-n+i-1} \left\{ [u_k(u_l^2 \cdots u_{m+n}^k)(u_l^2 \cdots u_{m+n}^k)]_{x} | -n-1 + (2j_1 + \cdots + n j_n) + 3(k-1) \right\} \left[u_k(u_l^2 \cdots u_{m+n}^k)(u_l^2 \cdots u_{m+n}^k)_{x} \right] + 3[n-(2j_1 + \cdots + n j_n)]u_{k,x}(u_l^2 \cdots u_{m+n}^k)(u_l^2 \cdots u_{m+n}^k). \tag{2.15}
$$

Similarly, we obtain

$$
\frac{\partial u_2}{\partial t_{0,m+n}} = (m+n)u_{m+n+1,x} + \sum_{k=2}^{m+n} \sum_{i,j,k} C_i^{m+n} C_j^{m+n-i} \cdots C_{i-1}^{m+n-i} C_{j-1}^{m+n-i-1} \left\{ [u_k(u_l^2 \cdots u_{m+n}^k)(u_l^2 \cdots u_{m+n}^k)]_{x} | 0 \leq i_1, \ldots, i_{m+n} \leq m+n \right\} \left[u_k(u_l^2 \cdots u_{m+n}^k)(u_l^2 \cdots u_{m+n}^k)_{x} \right] + 3(n-(2j_1 + \cdots + n j_n)]u_{k,x}(u_l^2 \cdots u_{m+n}^k)(u_l^2 \cdots u_{m+n}^k). \tag{2.16}
$$

When $m = 1$, as the result in the Theorem 2.1, (2.15) and (2.16) support $\frac{\partial u_2}{\partial t_{0,n-1}} = \frac{\partial u_2}{\partial t_{0,n}}$. But for the general case of $m, n \geq 2$ and $m \neq n$, there is not such kind of the equivalent relation between $\frac{\partial u_2}{\partial t_{0,m+n}}$ and $\frac{\partial u_2}{\partial t_{0,n}}$.

Substituting (2.8) into (2.5), when $m = 0, n = 2$, we obtain

$$
\sum_{i=1}^{m} \frac{\partial u_{i+1}}{\partial t_{0,2}} \lambda^{-i} = 2u_{3,x} \lambda^{-1} + \sum_{i=1}^{m} 2[u_{i+2,x} + (i-1)u_{i+1,2,x}] \lambda^{-i}. \tag{2.17}
$$

Comparing the coefficient of $\lambda^{-i}$, $i \geq 1$ on both sides of (2.17), we obtain

$$
\frac{\partial u_2}{\partial t_{0,2}} = 2u_{3,x}, \quad \frac{\partial u_{i+1}}{\partial t_{0,2}} = 2u_{i+2,x} + 2(i-1)u_{i+1,2,x}, \quad i \geq 2. \tag{2.18}
$$

Let $t_{02} = y$, (2.18) gives the recursion relations

$$
u_{1} = \frac{1}{2} \partial^{-1} u_{2,y}, \quad u_{i+2} = \frac{1}{2} \partial^{-1} u_{i+1,y} - (i-1) \partial^{-1} (u_{i,2,x}), \quad i \geq 2. \tag{2.19}
$$

Substituting (2.19) for the case $m = 0, n \geq 3$ and $m = 1, n \geq 2$ into (2.15), we get the dKP hierarchy. Substituting (2.19) for the case $m, n \geq 1$ and $m \neq n$ into (2.15), we get a general hierarchy. The properties of the general hierarchy deserve our further study.

3. Summary

By combining the Lax equation of the dKP hierarchy and the 3-bracket, we have defined a Lax triple $\{B_m, B_n, L\}$ and investigated the relations between different pairs of $(m, n)$. Here the 3-bracket is not a Nambu-Poisson bracket, but it satisfies the conditions of 3-Lie algebra. We have proved that when $m = 0$ and $m = 1$, the 3-Lax equation $\frac{\partial L}{\partial t_{m,n}} = \{B_m, B_n, L\}$ is the dKP hierarchy. For the generalized case of $(m, n)$, the 3-Lax equation means a generalized dKP hierarchy including the dKP hierarchy. Whether the hierarchy of the general case constitute a new integrable hierarchy remains to be studied.
Our analyses suggest that there exists much deeper connections between the infinite-dimensional 3-algebra and the integrable system. More properties with respect to their relations still deserve further study. We believe that 3-algebras may shed new light on the integrable systems.

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