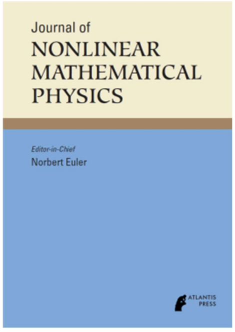

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Moving Boundary Problems for Heterogeneous Media. Integrability via Conjugation of Reciprocal and Integral Transformations

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The combined action of reciprocal and integral-type transformations is here used to sequentially reduce to analytically tractable form a class of nonlinear moving boundary problems involving heterogeneity. Particular such Stefan problems arise in the description of the percolation of liquids through porous media in soil mechanics.

Keywords: Stefan Problems; Heterogeneous Media; Reciprocal Transformations; Integral Transformations.

Mathematics Subject Classification: MSC 80A22

1. Introduction

Reciprocal transformations have been previously applied in [15,16] to Stefan problems for nonlinear heat equations of the type derived by Storm [35] to describe heat conduction in a range of simple metals. In [17], a reciprocal transformation was employed to determine conditions for the onset of melting in such metals subjected to applied boundary flux. Melting conditions as derived by Tarzia [36] and Solomon *et al.* [34] for analogous moving boundary problems for the classical heat equation were thereby extended.

Physical systems incorporating modulation, either spatial or temporal, arise in a wide range of physical settings. Thus, in classical continuum mechanics, they occur, *inter alia*, in elastodynamics, visco-elastodynamics and in crack and boundary loading problems in the elastostatics of inhomogeneous media [2, 6, 10].

Moving boundary problems incorporating inhomogeneity occur naturally in soil mechanics. Thus, they arise notably in the analysis of the transport of liquid through soils as modelled in the homogeneous case by the classical work of Richards [14]. In [4], a Lie-Bäcklund analysis was adopted to isolate integrable reductions of a nonlinear model based on a generalised Darcy's law descriptive of liquid transport through an unsaturated inhomogeneous medium under certain geometric constraints. In [18], a class of moving boundary problems for a nonlinear transport equation which arises in such a heterogeneous soil mechanics context was shown via a reciprocal transformation to admit exact parametric representation. The latter reduction was recently set in a more general context in [19] and alignment obtained with the homogeneous capillarity model of Richards.

In [20], a novel reciprocal transformation has been recently introduced which allows the reduction of certain multi-component, non-autonomous systems of generalised Ermakov-type to integrable canonical form. Here, such a reciprocal transformation is combined with a standard reciprocal transformation and an integral transformation with origin in work of [5] on boundary value problems for Burgers' equation to reduce a broad class of moving boundary problems involving heterogeneity to a canonical Stefan-type problem amenable to exact solution.

2. A Class of Heterogeneous Moving Boundary Problems

The motivation for the present work originates in an autonomisation procedure as set down in [1] for the Ermakov-Ray-Reid system [12, 13, 21, 22]

$$u_{xx} + \omega(x)u = \frac{1}{u^2 v}\Phi(v/u) \ , \quad v_{xx} + \omega(x)v = \frac{1}{v^2 u}\Psi(u/v) \ . \tag{2.1}$$

Thus, if the latter nonlinear coupled system is augmented by the linear base equation

$$\rho_{xx} + \omega(x)\rho = 0 \ , \tag{2.2}$$

then with the new dependent variables

$$u^* = u/\rho \ , \quad v^* = v/\rho \tag{2.3}$$

and independent variable

$$x^* = \sigma/\rho \tag{2.4}$$

where σ, ρ are linearly independent solutions of (2.2) with unit Wronskian $\rho\sigma_x - \sigma\rho_x$ then the Ermakov-Ray-Reid system (2.1) is reduced to the associated autonomous form

$$u^*_{x^*x^*} = \frac{1}{u^{*2}v^*}\Phi(v^*/u^*) \ , \quad v^*_{x^*x^*} = \frac{1}{v^{*2}u^*}\Psi(u^*/v^*) \ . \tag{2.5}$$

It is seen that with the unit Wronskian constraint as in [1], the relation (2.4) yields

$$dx^* = \rho^{-2}dx \ . \tag{2.6}$$

It is noted parenthetically that the analogous autonomisation of the classical Ermakov equation

$$\rho_{xx} + \omega(x)\rho = \frac{\varepsilon}{\rho^3} \tag{2.7}$$

allows the ready derivation of its associated nonlinear superposition principle.

Here, use will be made of reduction to autonomous form of the class of 1+1-dimensional nonlinear evolution equations

$$\frac{\partial T}{\partial t} = \lambda \frac{\partial^2}{\partial x^2}\left[\frac{1}{\rho^2(a+bT)}\right] + \frac{\mu}{\rho^2}\frac{\partial}{\partial x}\left[\frac{1}{\rho^2(a+bT)}\right] + \frac{\phi(\rho, \rho_x, ...; x)}{\rho^2(a+bT)} \tag{2.8}$$

with heterogeneity term $\rho = \rho(x)$ determined by a generalised Ermakov equation

$$\lambda\rho_{xx} + \mu\rho_x/\rho^2 + \phi(\rho, \rho_x, ...; x)\rho = \frac{\varepsilon}{\rho^3} \tag{2.9}$$

wherein, λ, μ and ε are real constants. Thus, it is seen, that under the transformation

$$\left.\begin{aligned} dx^* = \rho^{-2}dx \ , \quad t^* = t, \\ T^* = \rho^3(a+bT) \end{aligned}\right\} \quad \mathbb{R}^* \tag{2.10}$$

(2.8) is reduced to the unmodulated canonical form

$$\frac{\partial T^*}{\partial t^*} = \lambda b \frac{\partial^2}{\partial x^{*2}}\left[\frac{1}{T^*}\right] + \mu b \frac{\partial}{\partial x^*}\left[\frac{1}{T^*}\right] + \frac{\varepsilon b}{T^*} \, . \tag{2.11}$$

If one sets $\rho^* := \rho^{-1}$, then the underlying reciprocal property $\mathbb{R}^{*2}|_{a=0,b=1} = \mathrm{I}$ is retrieved. It is remarked that a nonlinear evolution equation of the type (2.11) with $\varepsilon = 0$ has been previously derived in [23] in connection with boundary value problems descriptive of two-phase flow under gravity in a porous medium.

In the sequel, by way of illustration, we proceed with $\varepsilon = 0$ together with

$$\lambda = 1 \, , \quad \mu = -\delta \, , \quad \delta \neq 0 \, , \quad \zeta = 0 \, , \tag{2.12}$$

whence, combination of (2.8) and (2.9) produces the conservation law

$$\frac{\partial}{\partial t}(\rho T) = \frac{\partial}{\partial x}\left[\rho^2 \frac{\partial}{\partial x}(\Delta/\rho)\right] - \delta \frac{\partial}{\partial x}(\Delta/\rho) \, , \tag{2.13}$$

with $\Delta := \dfrac{1}{\rho^2(a+bT)}$.

A class of moving boundary problems for the heterogeneous evolution equation (2.13) is now considered, namely

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho T) = \frac{\partial}{\partial x}\left[\rho^2 \frac{\partial}{\partial x}(\Delta/\rho)\right] - \delta \frac{\partial}{\partial x}(\Delta/\rho) \, , \quad 0 < x < X(t) \, , \quad t > 0 \\
&-\rho^2 \frac{\partial}{\partial x}(\Delta/\rho) + \delta(\Delta/\rho) = U(t) \, , \quad \text{on} \quad x = 0 \, , \quad t > 0 \\
&\left.\begin{aligned} -\rho^2 \frac{\partial}{\partial x}(\Delta/\rho) + \delta(\Delta/\rho) &= \alpha(\rho)\dot{X}(t) \\ \Delta/\rho &= \zeta(t) \, , \end{aligned}\right\} \quad \text{on} \quad x = X(t) \, , \quad t > 0 \\
&X(0) = 0 \, .
\end{aligned} \tag{2.14}$$

The preceding constitutes a Stefan-type problem with variable latent heat. Such moving boundary problems are of current research interest (see e.g. [3,33] and literature cited therein).

In view of (2.9) and (2.12) it is seen that the modulation is determined by

$$\rho_x = -\frac{\delta}{\rho} + \xi \, , \tag{2.15}$$

where ξ is an arbitrary constant of integration. Here, we proceed with $\xi = 0$ so that

$$\rho = \sqrt{2(-\delta x + \eta)} \, , \tag{2.16}$$

where the constants therein are such that $\eta > 0, \ \delta < 0$. The reciprocal variable x^* is then given by

$$x^* = \frac{1}{2(-\delta)} \ln\left[-\frac{\delta x + \eta}{\eta}\right] \tag{2.17}$$

where it has been required that $x = 0 \sim x^* = 0$ and $x > 0$.

Under $\mathbb{R}^*$, on setting $b=-1$, the class of moving boundary problems (2.14) becomes

$$\begin{aligned}
&\frac{\partial T^*}{\partial t^*} = \frac{\partial}{\partial x^*}\left[\frac{1}{T^{*2}}\frac{\partial T^*}{\partial x^*}\right] - \frac{\delta}{T^{*2}}\frac{\partial T^*}{\partial x^*}\,, \quad 0<x^*<X^*(t^*)\,, \quad t^*>0 \\
&\frac{1}{T^{*2}}\frac{\partial T^*}{\partial x^*} + \frac{\delta}{T^*} = U(t^*)\,, \quad \text{on} \quad x^*=0\,, \quad t^*>0 \\
&\left.\begin{aligned}\frac{1}{T^{*2}}\frac{\partial T^*}{\partial x^*} + \frac{\delta}{T^*} = \alpha^*\dot{X}^*\,, \\ T^* = 1/\zeta(t^*)\,,\end{aligned}\right\} \quad \text{on} \quad x^*=X^*(t^*)\,, \quad t^*>0 \\
&X^*(0)=0
\end{aligned} \tag{2.18}$$

where

$$X^*(t^*) = \frac{1}{2(-\delta)}\ln\left[-\frac{\delta X(t)+\eta}{\eta}\right]\,, \quad \dot{X}^* = \rho^{-2}\dot{X}(t) \tag{2.19}$$

so that

$$\alpha^* = \rho^2\alpha(\rho)\,. \tag{2.20}$$

On introduction of the additional reciprocal transformation

$$\left.\begin{aligned} dx^\dagger = T^*dx^* + \left[\frac{1}{T^{*2}}\frac{\partial T^*}{\partial x^*} + \frac{\delta}{T^*}\right]dt^*\,, \quad t^\dagger = t^*\,, \\ T^\dagger = \frac{1}{T^*} \end{aligned}\right\} \quad \mathbb{R}^\dagger \tag{2.21}$$

the nonlinear equation in T^* in (2.18) becomes

$$\frac{\partial T^\dagger}{\partial t^\dagger} = \frac{\partial^2 T^\dagger}{\partial x^{\dagger 2}} - 2\delta T^\dagger \frac{\partial T^\dagger}{\partial x^\dagger}\,, \tag{2.22}$$

namely Burgers' equation.

Boundary Conditions

Below in I–III are derived explicitly the boundary conditions reciprocal to those of the moving boundary problem (2.18) and which are to be applied to the Burgers' equation (2.22).

I

$$\frac{1}{T^{*2}}\frac{\partial T^*}{\partial x^*} + \frac{\delta}{T^*} = U(t^*) \quad \text{on} \quad x^*=0\,, \quad t^*>0\,. \tag{2.23}$$

Here, $\mathbb{R}^\dagger$ shows that

$$-\frac{\partial T^\dagger}{\partial x^\dagger} + \delta T^{\dagger 2} = UT^\dagger \quad \text{on} \quad x^\dagger|_{x^*=0}\,, \quad t^\dagger>0$$

where to determine $x^\dagger|_{x^*=0}$, we use the reciprocal relations

$$\frac{\partial x^\dagger}{\partial x^*} = T^*\,, \quad \frac{\partial x^\dagger}{\partial t^*} = \frac{1}{T^{*2}}\frac{\partial T^*}{\partial x^*} + \frac{\delta}{T^*} = \int_0^{x^*}\frac{\partial}{\partial x^*}\left[\frac{1}{T^{*2}}T^*_{x^*}\right]dx^* + \left.\frac{1}{T^{*2}}\frac{\partial T^*}{\partial x^*}\right|_{x^*=0} + \frac{\delta}{T^*}\,. \tag{2.24}$$

Thus,

$$
\begin{aligned}
\frac{\partial x^\dagger}{\partial t^*} &= \int_0^{x^*} \left[\frac{\partial T^*}{\partial t^*} - \delta \frac{\partial}{\partial x^*}\left(\frac{1}{T^*}\right)\right] dx^* + \frac{1}{T^{*2}} \frac{\partial T^*}{\partial x^*}\bigg|_{x^*=0} + \frac{\delta}{T^*} \\
&= \frac{\partial}{\partial t^*} \int_0^{x^*} T^*(\sigma,t^*) d\sigma - \delta\left(\frac{1}{T^*}\right) + \delta\left(\frac{1}{T^*}\right)\bigg|_{x^*=0} + \frac{1}{T^{*2}}\frac{\partial T^*}{\partial x^*}\bigg|_{x^*=0} + \frac{\delta}{T^*} \\
&= \frac{\partial}{\partial t^*} \int_0^{x^*} T^*(\sigma,t^*) d\sigma + U(t^*)
\end{aligned}
\tag{2.25}
$$

so that

$$
x^\dagger = \int_0^{x^*} T^*(\sigma,t^*) d\sigma + V(t^*) \tag{2.26}
$$

where $\dot{V} = U(t^*)$. Hence, under $\mathbb{R}^\dagger$, the boundary condition (2.23) on $x^* = 0$ becomes

$$
-\frac{\partial T^\dagger}{\partial x^\dagger} + \delta T^{\dagger^2} = U(t^\dagger) T^\dagger \quad \text{on} \quad x^\dagger = V(t^\dagger) \, . \tag{2.27}
$$

II

$$
\frac{1}{T^*}\frac{\partial T^*}{\partial x^*} + \frac{\delta}{T^*} = \alpha^* \dot{X}^* \quad \text{on} \quad x^* = X^*(t^*) \, . \tag{2.28}
$$

Here, the reciprocal transformation $\mathbb{R}^\dagger$ shows that

$$
\frac{dX^\dagger}{dt^\dagger} = \frac{1}{T^\dagger}\frac{dX^*}{dt^*} - \frac{1}{T^\dagger}\frac{\partial T^\dagger}{\partial x^\dagger} + \delta T^\dagger \, , \tag{2.29}
$$

whence, on elimination of $\dot{X}^*$ between (2.28) and (2.29), it is seen that

$$
\alpha^* \left[T^\dagger \frac{dX^\dagger}{dt^\dagger} + \frac{\partial T^\dagger}{\partial x^\dagger} - \delta T^{\dagger 2} \right] = -\frac{1}{T^\dagger}\frac{\partial T^\dagger}{\partial x^\dagger} + \delta T^\dagger \quad \text{on} \quad x^\dagger = X^\dagger(t^\dagger) \, . \tag{2.30}
$$

Thus,

$$
-\frac{\partial T^\dagger}{\partial x^\dagger} + \delta T^{\dagger^2} = \frac{\alpha^* T^{\dagger^2}}{1 + \alpha^* T^\dagger} \frac{dX^\dagger}{dt^\dagger} \quad \text{on} \quad x^\dagger = X^\dagger(t^\dagger) \tag{2.31}
$$

where

$$
X^\dagger(t^\dagger) = x^\dagger|_{x^*=X^*(t^*)} \, . \tag{2.32}
$$

III

$$
T^* = 1/\zeta(t^*) \quad \text{on} \quad x^* = X^*(t^*) \tag{2.33}
$$

This yields

$$
T^\dagger = \zeta(t^\dagger) \quad \text{on} \quad x^\dagger = X^\dagger(t^\dagger) \, . \tag{2.34}
$$

To determine $X^\dagger(t^\dagger)$, the reciprocal transformation $\mathbb{R}^\dagger$ shows that

$$\frac{\partial x^\dagger}{\partial x^*} = T^* \; , \quad \frac{\partial x^\dagger}{\partial t^*} = \frac{1}{T^{*2}}\frac{\partial T^*}{\partial x^*} + \frac{\delta}{T^*} = \int_{X^*(t^*)}^{x^*} \frac{\partial}{\partial x^*}\left[\frac{1}{T^{*2}}T^*_{x^*}\right]dx^* + \frac{1}{T^{*2}}\frac{\partial T^*}{\partial x^*}\bigg|_{x^*=X^*} + \frac{\delta}{T^*}$$

$$= \int_{X^*(t^*)}^{x^*}\left[\frac{\partial T^*}{\partial t^*} - \delta\frac{\partial}{\partial x^*}\left(\frac{1}{T^*}\right)\right]dx^* + \frac{1}{T^{*2}}\frac{\partial T^*}{\partial x^*}\bigg|_{x^*=X^*} + \frac{\delta}{T^*} \; . \tag{2.35}$$

Thus,

$$\frac{\partial x^\dagger}{\partial t^*} = \frac{\partial}{\partial t^*}\left[\int_{X^*(t^*)}^{x^*} T^*(\sigma,t^*)d\sigma\right] + T^*\dot{X}^*|_{x=X^*} - \frac{\delta}{T^*} + \frac{\delta}{T^*}\bigg|_{x^*=X^*} + \frac{1}{T^{*2}}\frac{\partial T^*}{\partial x^*}\bigg|_{x^*=X^*} + \frac{\delta}{T^*}$$

$$= \frac{\partial}{\partial t^*}\left[\int_{X^*(t^*)}^{x^*} T^*(\sigma,t^*)d\sigma\right] + (\zeta^{-1}+\alpha^*)\dot{X}^* \; . \tag{2.36}$$

Here, we proceed with

$$\zeta^{-1} + \alpha^* = constant = \xi^* \tag{2.37}$$

so that (2.36) yields

$$x^\dagger = \int_{X^*(t^*)}^{x^*} T^*(\sigma,t^*)d\sigma + \xi^* X^* \tag{2.38}$$

whence

$$X^\dagger = x^\dagger|_{x^*=X^*} = \xi^* X^* \tag{2.39}$$

with the initial condition

$$X^\dagger|_{t^\dagger=0} = \xi^* X^*|_{t^*=0} = 0 \tag{2.40}$$

on use of $(2.19)_1$.

Summary I

The conjugation of the reciprocal transformations $\mathbb{R}^*$ and $\mathbb{R}^\dagger$ applied to the class of moving boundary problems (2.14) incorporating heterogeneity produces a reciprocally associated class of moving boundary problems for Burgers' equation, namely

$$\begin{aligned}
&\frac{\partial T^\dagger}{\partial t^\dagger} = \frac{\partial^2 T^\dagger}{\partial x^{\dagger 2}} - 2\delta T^\dagger \frac{\partial T^\dagger}{\partial x^\dagger} \; , \quad 0 < x^\dagger < X^\dagger(t^\dagger) \; , \quad t^\dagger > 0 \\
&-\frac{\partial T^\dagger}{\partial x^\dagger} + \delta T^{\dagger 2} = U(t^\dagger)T^\dagger \; , \quad \text{on} \quad x^\dagger = V(t^\dagger) \; , \quad t^\dagger > 0 \\
&\left.\begin{aligned} -\frac{\partial T^\dagger}{\partial x^\dagger} + \delta T^{\dagger 2} &= \frac{\alpha^*\zeta^2(t^\dagger)}{1+\alpha^*\zeta(t^\dagger)}\frac{dX^\dagger}{dt^\dagger} \; , \\ T^\dagger &= \zeta(t^\dagger) \; , \end{aligned}\right\} \quad \text{on} \quad x^\dagger = X^\dagger(t^\dagger) \; , \quad t^\dagger > 0 \\
&\qquad X^\dagger|_{t^\dagger=0} = 0 \; .
\end{aligned} \tag{2.41}$$

where $dV/dt^\dagger = U(t^\dagger)$.

3. Canonical Reduction via an Integral Transformation

The moving boundary problems (2.41) will here be seen to be amenable to an elegant integral transformation of a generalised Hopf-Cole type as introduced by Calogero and DeLillo in [5]. This adopts the form

$$T^\dagger = -(1/\delta)\left[\ln\left|C(t^\dagger) - \int_{X^\dagger(t^\dagger)}^{x^\dagger} \Psi(\sigma,t^\dagger)d\sigma\right|\right]_{x^\dagger}, \tag{3.1}$$

with

$$\Psi = \delta C(t^\dagger)T^\dagger(x^\dagger,t^\dagger)\exp\left[-\delta\int_{X^\dagger(t^\dagger)}^{x^\dagger} T^\dagger(\sigma,t^\dagger)d\sigma\right]. \tag{3.2}$$

Thus,

$$\Psi_{\bar t} = \left[\delta\dot{C}T^\dagger + \delta C T^\dagger_{t^\dagger} + \delta C T^\dagger\left[-\delta\frac{\partial}{\partial t^\dagger}\int_{X^\dagger(t^\dagger)}^{x^\dagger} T^\dagger(\sigma,t^\dagger)d\sigma\right]\right]\exp\left[-\delta\int_{X^\dagger(t^\dagger)}^{x^\dagger} T^\dagger(\sigma,t^\dagger)d\sigma\right] \tag{3.3}$$

where the Leibniz rule shows that

$$\begin{aligned}\frac{\partial}{\partial t^\dagger}\left[\int_{X^\dagger(t^\dagger)}^{x^\dagger} T^\dagger(\sigma,t^\dagger)d\sigma\right] &= \int_{X^\dagger(t^\dagger)}^{x^\dagger}\frac{\partial T^\dagger}{\partial t^\dagger} - \dot{X}^\dagger T^\dagger|_{x^\dagger=X^\dagger(t^\dagger)} \\ &= \int_{X^\dagger(t^\dagger)}^{x^\dagger}[\,T^\dagger_{x^\dagger} - \delta T^{\dagger 2}\,]_{x^\dagger}dx^\dagger - \dot{X}^\dagger T^\dagger|_{x^\dagger=X^\dagger(t^\dagger)}\end{aligned} \tag{3.4}$$

whence

$$\begin{aligned}\Psi_{t^\dagger} = &\left[\delta\dot{C}T^\dagger + \delta C T^\dagger_{t^\dagger} + \delta C T^\dagger\left(-\delta(T^\dagger_{x^\dagger} - \delta T^{\dagger 2}) + \delta(T^\dagger_{x^\dagger} - \delta T^{\dagger 2})\Big|_{x^\dagger=X^\dagger(t^\dagger)}\right)\right. \\ &\left.+\ \delta^2 C T^\dagger\dot{X}^\dagger T^\dagger|_{x^\dagger=X^\dagger(t^\dagger)}\right]\exp\left[-\delta\int_{X^\dagger(t^\dagger)}^{x^\dagger} T^\dagger(\sigma,t^\dagger)d\sigma\right].\end{aligned} \tag{3.5}$$

Moreover,

$$\Psi_{x^\dagger x^\dagger} = \left[\delta C T^\dagger_{x^\dagger x^\dagger} - 2\delta^2 C T^\dagger T^\dagger_{x^\dagger} - \delta T^\dagger(\delta C T^\dagger_{x^\dagger} - \delta^2 C T^{\dagger 2})\right]\exp\left[-\delta\int_{X^\dagger(t^\dagger)}^{x^\dagger} T^\dagger(\sigma,t^\dagger)d\sigma\right] \tag{3.6}$$

so that, *in extenso*

$$\begin{aligned}\Psi_{t^\dagger} - \Psi_{x^\dagger x^\dagger} = &\left[\delta\dot{C}T^\dagger + \delta C T^\dagger_{t^\dagger} + \delta C T^\dagger\left[-\delta(T^\dagger_{x^\dagger} - \delta T^{\dagger 2}) + \delta(T^\dagger_{x^\dagger} - \delta T^{\dagger 2})\Big|_{x^\dagger=X^\dagger(t^\dagger)}\right]\right. \\ &\left.+\ \delta^2 C T^\dagger\dot{X}^\dagger T^\dagger|_{x^\dagger=X^\dagger(t^\dagger)} - \delta C T^\dagger_{x^\dagger x^\dagger} + 2\delta^2 C T^\dagger T^\dagger_{x^\dagger} + \delta^2 C T^\dagger T^\dagger_{x^\dagger} - \delta^3 T^{\dagger 2}\right]. \\ &\exp\left[-\delta\int_{X^\dagger(t^\dagger)}^{x^\dagger} T^\dagger(\sigma,t^\dagger)d\sigma\right] \\ = &\left[\delta T^\dagger\left[\dot{C} + \delta C(T^\dagger_{x^\dagger} - \delta T^{\dagger 2}|_{x^\dagger=X^\dagger} + \delta C T^\dagger\dot{X}^\dagger T^\dagger|_{x^\dagger=X^\dagger(t^\dagger)}\right]\right. \\ &\left.+\ \delta C\left[T^\dagger_{t^\dagger} - T^\dagger_{x^\dagger x^\dagger} + 2\delta T^\dagger T^\dagger_{x^\dagger}\right]\right]\exp\left[-\delta\int_{X^\dagger(t^\dagger)}^{x^\dagger} T^\dagger(\sigma,t^\dagger)d\sigma\right].\end{aligned} \tag{3.7}$$

Accordingly, the Burgers' equation (2.22) is mapped via the integral transformation (3.2) to the classical heat equation

$$\Psi_{t^\dagger} - \Psi_{x^\dagger x^\dagger} = 0 \tag{3.8}$$

on imposition of the requirement

$$\dot{C} + \delta C \left[(T^\dagger_{x^\dagger} - \delta T^{\dagger^2})|_{x^\dagger = X^\dagger} + T^\dagger|_{x^\dagger = X^\dagger(t^\dagger)} \dot{X}^\dagger \right] = 0\,. \tag{3.9}$$

In view of the conditions on the moving boundary $x^\dagger = X^\dagger(t^\dagger)$ as in (2.41), the condition (3.9) yields

$$\dot{C} + \frac{\delta C \zeta}{1 + \alpha^* \zeta} \dot{X}^\dagger = 0 \tag{3.10}$$

so that, in view of (2.37) and (2.39)

$$C(t^\dagger) = c_0 \exp\left[-(\delta/\zeta^*) X^\dagger\right] = c_0 \exp\left[-\delta X^*\right]\,, \quad c_0\ \varepsilon\ \mathbb{R}\,. \tag{3.11}$$

Now, (3.1) shows that

$$T^\dagger = \frac{\Psi/\delta}{C(t^\dagger) - \int_{X^\dagger(t^\dagger)}^{x^\dagger} \Psi(\sigma, t^\dagger) d\sigma} \tag{3.12}$$

whence

$$T^\dagger|_{x^\dagger = X^\dagger} = \frac{\Psi|_{x^\dagger = X^\dagger}}{\delta C(t^\dagger)} = \zeta(t^\dagger) \tag{3.13}$$

and

$$T^\dagger_{x^\dagger} = \frac{\Psi_{x^\dagger}}{\delta \left[C(t^\dagger) - \int_{X^\dagger(t^\dagger)}^{x^\dagger} \Psi(\sigma, t^\dagger) d\sigma \right]} + \frac{\Psi^2}{\delta \left[C(t^\dagger) - \int_{X^\dagger(t^\dagger)}^{x^\dagger} \Psi(\sigma, t^\dagger) d\sigma \right]^2}\,. \tag{3.14}$$

Accordingly,

$$-T^\dagger_{x^\dagger} + \delta T^{\dagger^2} = \frac{-\Psi_{x^\dagger}}{\delta \left[C(t^\dagger) - \int_{X^\dagger(t^\dagger)}^{x^\dagger} \Psi(\sigma, t^\dagger) d\sigma \right]} = -\frac{\Psi_{x^\dagger} T^\dagger}{\Psi}\,, \tag{3.15}$$

and the conditions on the moving boundary in (2.41) become

$$\left.\begin{aligned} \Psi_{x^\dagger} &= -(\alpha^*/\xi^*) \dot{X}^\dagger\,, \\ \Psi &= \delta C(t^\dagger) \zeta(t^\dagger) \end{aligned}\right\} \quad \text{on} \quad x^\dagger = X^\dagger(t^\dagger)\,. \tag{3.16}$$

The residual boundary condition in (2.41) yields

$$\Psi_{x^\dagger} = -U\Psi \quad \text{on} \quad x^\dagger = V(t^\dagger)\,, \quad t^\dagger > 0\,, \tag{3.17}$$

namely, a Robin-type requirement.

Summary II

Reduction of the original class of heterogeneous moving boundary problems (2.13) with modulation determined by (2.16) has been reduced via successive reciprocal-type and integral transformations to a canonical Stefan problem with a Robin boundary condition, namely

$$\begin{aligned} &\Psi_{t^\dagger} - \Psi_{x^\dagger x^\dagger} = 0 \,, \quad V(t^\dagger) < x^\dagger < X^\dagger(t^\dagger) \,, \quad t^\dagger > 0 \\ &\Psi_{x^\dagger} = -U(t^\dagger)\Psi \,, \quad \text{on} \quad x^\dagger = V(t^\dagger) \,, \quad t^\dagger > 0 \\ &\left.\begin{aligned} &\Psi_{x^\dagger} = -(\alpha^*/\xi^*)\dot{X}^\dagger \,, \\ &\Psi = \delta C(t^\dagger)\zeta(t^\dagger) \end{aligned}\right\} \quad \text{on} \quad x^\dagger = X^\dagger(t^\dagger) \,, \quad t^\dagger > 0 \\ &\quad X^\dagger|_{t^\dagger=0} = 0 \end{aligned} \tag{3.18}$$

where $\dot{V} = U$ and $V(t^\dagger) = x^\dagger|_{x^*=0}$ by virtue of (2.26).

In terms of the solution $\Psi(x^\dagger, t^\dagger)$ of the above moving boundary problem, the corresponding solution $T^\dagger(x^\dagger, t^\dagger)$ of the class of moving boundary problems (2.41) for Burgers' equation is given by the integral representation (3.1), that is, (3.12). The solution $T^*(x^*, t^*)$ of the reciprocally associated moving boundary problem (2.18) is then given parametrically via the relations

$$\left.\begin{aligned} T^* = \frac{1}{T^\dagger(x^\dagger, t^\dagger)} \,, \quad x^* = x^*(x^\dagger, t^\dagger) \,, \\ t^* = t^\dagger \end{aligned}\right\} \tag{3.19}$$

where $x^*(x^\dagger, t^\dagger)$ is obtained through the reciprocal relation

$$dx^* = T^\dagger dx^\dagger + \left(\frac{\partial T^\dagger}{\partial x^\dagger} - \delta T^{\dagger^2}\right) dt^\dagger \tag{3.20}$$

implicit in $\mathbb{R}^\dagger$ as given by (2.21). Thus,

$$x^\dagger = \int_{X^\dagger}^{x^\dagger} T^\dagger dx^\dagger + X^* = \int_{X^\dagger}^{x^\dagger} T^\dagger dx^\dagger + X^\dagger/\xi^* \tag{3.21}$$

in view of the relation (2.40). The associated solution of the original class of heterogeneous moving boundary problems (2.14) is then determined by the relations

$$\left.\begin{aligned} T = a - \frac{T^*(x^*, t^*)}{\rho^3} \\ x^* = \frac{1}{2(-\delta)} \ln\left[-\frac{\delta x}{\eta} + 1\right] \,, \quad t^* = t \end{aligned}\right\} \tag{3.22}$$

where ρ is given by (2.16).

4. A Solvable Stefan Problem with Variable Latent Heat

The system (3.18) constitutes a moving boundary problem of Stefan-type with variable latent heat. Here, it is considered with the specialisations

$$X^\dagger(t^\dagger) = 2\gamma\sqrt{t^\dagger} \,, \quad V = 2\alpha\sqrt{t^\dagger} \tag{4.1}$$

with the latter requirement on $V(t^\dagger)$ corresponding to

$$U(t^\dagger) = \frac{\alpha}{\sqrt{t^\dagger}} \ . \tag{4.2}$$

In addition, the conditions on the moving boundary $x^\dagger = X^\dagger(t^\dagger)$ are taken to be of the type adopted in [33], so that here

$$-(\alpha^*/\xi^*)|_{x^\dagger = X^\dagger} = -(\alpha(\rho)\rho^2/\xi^*)|_{x^\dagger = X^\dagger} = \lambda^\dagger X^\dagger(t^\dagger) \tag{4.3}$$

and

$$C(t^\dagger)\zeta(t^\dagger) = \sqrt{t^\dagger} \tag{4.4}$$

where in (4.3), $\lambda^\dagger$ is a real non-zero constant.

It was shown in [33] that the classical heat equation (3.8) admits the class of similarity solutions

$$\Psi(x^\dagger, t^\dagger) = 2\sqrt{t^\dagger}\ \eta(\xi^\dagger) \tag{4.5}$$

where $\xi^\dagger = x^\dagger/2\sqrt{t^\dagger}$ and

$$\frac{1}{2}\eta''(\xi^\dagger) + \xi^\dagger\eta'(\xi^\dagger) - \eta(\xi^\dagger) = 0 \tag{4.6}$$

with general solution

$$\eta(\xi) = A[\, e^{-\xi^{\dagger 2}} + \sqrt{\pi}\ \xi^\dagger \mathrm{erf}\ \xi^\dagger\,] + B\ \xi^\dagger \ , \tag{4.7}$$

where $A,\ B$ are arbitrary real constants.

Boundary Conditions

Below, the conditions imposed on the similarity solution (4.7) of the moving boundary problem (3.18) are summarised.

I

$$\Psi_{x^\dagger} = -\frac{\alpha}{\sqrt{t^\dagger}}\Psi \quad \text{on} \quad x^\dagger = 2\alpha\sqrt{t^\dagger}\ , \quad t^\dagger > 0\ . \tag{4.8}$$

This boundary condition yields

$$\eta'(\alpha) = -2\alpha\eta(\alpha) \tag{4.9}$$

so that

$$A\sqrt{\pi}\ \mathrm{erf}\ \alpha + B = -2\alpha[\, A(\, e^{-\alpha^2} + \sqrt{\pi}\ \alpha\ \mathrm{erf}\ \alpha\,) + B\alpha\,]\ , \tag{4.10}$$

II

$$\Psi_{x^\dagger} = \lambda^\dagger X^\dagger(t^\dagger)\dot{X}^\dagger(t^\dagger) \quad \text{on} \quad x^\dagger = X^\dagger(t^\dagger)\ , \quad t^\dagger > 0\ . \tag{4.11}$$

This requires that

$$\eta'(\gamma) = 2\gamma^2\lambda^\dagger \tag{4.12}$$

whence

$$A\sqrt{\pi}\,\mathrm{erf}\,\gamma + B = 2\gamma^2\lambda^\dagger\ . \tag{4.13}$$

III

$$\Psi = \delta\sqrt{t^\dagger} \quad \text{on} \quad x^\dagger = X^\dagger(t^\dagger)\ , \quad t^\dagger > 0\ . \tag{4.14}$$

This yields

$$\eta(\gamma) = \delta/2 \tag{4.15}$$

so that

$$A[\,e^{-\gamma^2} + \sqrt{\pi}\,\gamma\,\mathrm{erf}\,\gamma\,] + B\gamma = \delta/2\ . \tag{4.16}$$

The triad of equations (4.10), (4.13) and (4.16) serve to determine A, B and γ.

5. Conclusion

Reciprocal-type transformations have previously had diverse physical applications in such areas as gasdynamics, magnetogasdynamics, nonlinear heat conduction, the theory of discontinuity wave propagation and invariance properties of classical capillairity and nonlinear optics systems (see e.g. [7, 24–28] and literature cited therein). In terms of practical moving boundary problems, reciprocal transformations have, in particular, been applied in the analysis of methacrylate distribution in wood saturation processes [8]. In soliton theory, reciprocal transformations have been used to link inverse scattering schemes and the nonlinear integrable equations contained therein [9, 11, 29–32].

Here, two kinds of reciprocal transformation in conjunction with an integral transformation of a novel type introduced in [5] for Burgers' equation have been applied to reduce sequentially a class of nonlinear moving boundary problems incorporating heterogeneity to a canonical Stefan-type problem with variable latent heat. A class of such moving boundary problems is shown to admit exact similarity solutions.

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