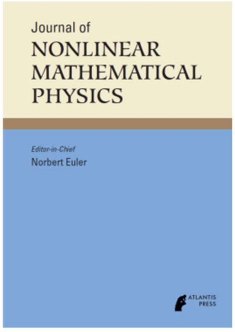


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# Solvable Systems Featuring 2 Dependent Variables Evolving in Discrete-Time via 2 Nonlinearly-Coupled First-Order Recursion Relations with Polynomial Right-Hand Sides

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The evolution equations mentioned in the title of this paper read as follows:

$$\tilde{x}_n = P^{(n)}(x_1, x_2), \quad n = 1, 2,$$

where $\ell$ is the "discrete-time" *independent* variable taking *integer* values ($\ell = 0, 1, 2, \ldots$), $x_n \equiv x_n(\ell)$ are the 2 dependent variables, $\tilde{x}_n \equiv x_n(\ell+1)$, and the 2 functions $P^{(n)}(x_1, x_2)$, $n = 1, 2$, are 2 *polynomials* in the 2 dependent variables $x_1(\ell)$ and $x_2(\ell)$. The results reported in this paper have been obtained by an appropriate modification of a recently introduced technique to obtain analogous results in *continuous-time t*—in which case $x_n \equiv x_n(t)$ and the above recursion relations are replaced by first-order ODEs. Their potential interest is due to the relevance of this kind of evolution equations in various applicative contexts.

## 1. Introduction

In this introductory Section 1, after providing some notational prescriptions, we tersely review previous relevant findings.

**Notation 1.1.** Hereafter $\ell = 0, 1, 2, \ldots$ denotes the *discrete-time independent* variable; the *dependent* variables are $x_n \equiv x_n(\ell)$ (generally with $n = 1, 2$), and the notation $\tilde{x}_n \equiv x_n(\ell+1)$ indicates the once-updated values of these variables. We shall also use other dependent variables, for instance $y_m \equiv y_m(\ell)$, and then of course likewise $\tilde{y}_m \equiv y_m(\ell+1)$. All variables such as $x$, $y$, $z$ (generally equipped with indices) are assumed to be *complex* numbers, unless otherwise indicated; it shall generally be clear from the context which of these and other quantities depend on time (as occasionally—but not always—*explicitly* indicated); parameters such as $a$, $\alpha$, $\beta$, $\gamma$, $A$, etc. (often equipped with indices) are generally time-independent *complex* numbers; and indices such as $n$, $m$, $j$ are generally *positive integers* (the values they may take shall be explicitly indicated or quite clear from the context). ■

**Remark 1.1.** In this paper the term *solvable* generally characterizes systems of evolution equations the initial-values problems of which are *explicitly solvable by algebraic operations*. ■

In the following Subsection 1.1 we tersely review—mainly via quotations (with minor adjustments) from a recent paper of ours [1]—a recent approach to identify *solvable* dynamical systems in *continuous-time t*, as introduction to the extension of (some of) these results to the case of *discrete-time* $\ell$, which is the topic of the present paper. Previous results on *solvable discrete-time* models

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are tersely reviewed in the subsequent Subsection 1.2. Our main findings are reported in Section 2 (also based on the results reported in Appendix A). A concluding Section 3 outlines tersely possible additional developments.

### 1.1. *Review of an analogous approach in the* continuous-time *context*

"Long time ago the idea has been introduced to identify dynamical systems (evolving in *continuous-time t*) which are *solvable* by using as a tool the relations between the time evolutions of the *coefficients* and the *zeros* of a generic time-dependent polynomial [2]. The basic idea of this approach is to relate the time-evolution of the $N$ zeros $x_n(t)$ of a generic time-dependent polynomial $p_N(z;t)$ of degree $N$ in its argument $z$,

$$p_N(z;t) = z^N + \sum_{m=1}^{N} \left[ y_m(t) z^{N-m} \right] = \prod_{n=1}^{N} \left[ z - x_n(t) \right] , \tag{1.1a}$$

to the time-evolution of its $N$ coefficients $y_m(t)$. Indeed, if the time evolution of the $N$ coefficients $y_m(t)$ is determined by a system of ODEs which is itself *solvable*, then the corresponding time-evolution of the $N$ zeros $x_n(t)$ is also *solvable*, via the following 3 steps: (i) given the initial values $x_n(0)$, the corresponding initial values $y_m(0)$ can be obtained from the *explicit* formulas—expressing the $N$ coefficients $y_m(t)$ of the polynomial (1.1a) in terms of its $N$ zeros $x_n(t)$—reading (for all time, hence in particular at $t = 0$)

$$y_m(t) = (-1)^m \sum_{1 \leq n_1 < n_2 < \cdots < n_m \leq N}^{N} \left\{ \prod_{\ell=1}^{M} \left[ x_{n_\ell}(t) \right] \right\} , \quad m = 1, 2, \ldots, N ; \tag{1.1b}$$

(ii) from the $N$ values $y_m(0)$ thereby obtained, the $N$ values $y_m(t)$ are then evaluated via the—assumedly *solvable*—system of ODEs satisfied by the $N$ coefficients $y_m(t)$; (iii) the $N$ values $x_n(t)$—i.e., the $N$ solutions of the dynamical system satisfied by the $N$ variables $x_n(t)$—are then determined as the $N$ zeros of the polynomial, see (1.1a), itself known at time $t$ in terms of its $N$ coefficients $y_m(t)$ (the computation of the zeros of a known polynomial being an *algebraic* operation; of course generally explicitly performable only for polynomials of degree $N \leq 4$)...

The viability of this technique to identify *solvable* dynamical systems depends of course on the availability of an *explicit* method to relate the time-evolution of the $N$ zeros of a *polynomial* to the corresponding time-evolution of its $N$ *coefficients*. Such a method was indeed provided in [2], opening the way to the identification of a vast class of *algebraically solvable* dynamical systems (see also, for instance, [3] and references therein); but that approach was essentially restricted to the consideration of *linear* time evolutions of the coefficients $y_m(t)$.

A development allowing to lift this quite strong restriction emerged relatively recently [4], by noticing the validity of the *identity*

$$\dot{x}_n = - \left[ \prod_{\ell=1, \ell \neq n}^{N} (x_n - x_\ell) \right]^{-1} \sum_{m=1}^{N} \left[ \dot{y}_m \left( x_n \right)^{N-m} \right] \tag{1.2}$$

which provides a convenient *explicit* relationship among the time evolutions of the $N$ *zeros* $x_n(t)$ and the $N$ *coefficients* $y_m(t)$ of the generic polynomial (1.1a). This allowed a major enlargement of the class of *algebraically solvable* dynamical systems identifiable via this approach: for many examples see [5] and references therein...

A new twist of this approach was then provided by its extension to *nongeneric* polynomials featuring—for *all* time—*multiple* zeros. The first step in this direction focussed on time-dependent polynomials featuring for *all* time a *single double zero* [6]; and subsequently significant progress has been made to treat the case of polynomials featuring a *single zero* of *arbitrary multiplicity* [7]. A convenient method was then provided which is suitable to treat the most general case of polynomials featuring an *arbitrary* number of *zeros* each of which features an *arbitrary multiplicity*. While all these developments might appear to mimic scholastic exercises analogous to the discussion among medieval scholars of how many angels might dance simultaneously on the tip of a needle, they do indeed provide *new tools* to identify *new* dynamical systems featuring interesting time evolutions (including systems displaying remarkable behaviors such as *isochrony* or *asymptotic isochrony*: see for instance [6] [7]); dynamical systems which—besides their intrinsic mathematical interest—are quite likely to play significant roles in applicative contexts...

We then focused on another twist of this approach to identify new *solvable* dynamical systems which was introduced quite recently [8]. It is again based on the relations among the time-evolution of the *coefficients* and the *zeros* of time-dependent polynomials [4] [5] with *multiple roots* (see [6], [7] and above); restricting moreover attention to such polynomials featuring *only* 2 *zeros*. Again, this might seem such a strong limitation to justify the doubt that the results thereby obtained be of much interest. But the effect of this restriction is to open the possibility to identify *algebraically solvable* dynamical models characterized by the following systems of 2 ODEs,

$$\dot{x}_n = P^{(n)}(x_1, x_2)\,, \quad n = 1, 2\,, \tag{1.3}$$

with $P^{(n)}(x_1,x_2)$ 2 *polynomials* in the 2 dependent variables $x_1(t)$ and $x_2(t)$; hence systems of considerable interest, both from a theoretical and an applicative point of view (see [8] and references quoted there)." [1]

This completes our review—via a long quotation from a previous paper—of recent developments concerning certain classes of standard dynamical systems in *continuous time*. In the present paper—after tersely reviewing, in the following Subsection 1.2, some past results in the *discrete-time* context—we focus on the derivation in such a context of analogous results to some of those reported in the *continuous-time* context in [1].

### 1.2. *Review of somewhat analogous past findings in the* **discrete-time** *context*

Somewhat analogous results to those reviewed in the *first* part of the previous Subsection 1.1 have been developed over time in the context of *discrete-time* evolutions, by focussing on the evolution of the *zeros* of generic monic polynomials the *coefficients* of which evolve in a *solvable* manner in *discrete time*.

The new results reported below consists essential of extensions to the *discrete-time* context of the results outlined in the *second* part of the preceding Subsection 1.1. Note however that here and below we actually dispense from a general discussion of the evolution of the *zeros* of a polynomial the *coefficients* of which evolve in *discrete time* in a *solvable* manner, both in the case of *generic* monic polynomials (as treated in Chapter 7 of [5]) and in the case of the special polynomials of higher degree than 2 which nevertheless feature for all time only 2 (of course *multiple*) *zeros* (as treated in [8], [1]); below we rather employ the simpler technique—described in the following Section 2—to identify *solvable* nonlinear evolution equations that emerged from that approach and which actually subtends most of the *explicit* findings reported in [1]. Hence from the previous findings for *discrete-time* evolutions—see [9] and Chapter 7 ("Discrete time") of [5]—we only use below the following *discrete-time* equivalent of the identity (1.2) (originating from the polynomial (1.1) with $t$ replaced by $\ell$),

$$\prod_{j=1}^{N} (\tilde{x}_n - x_j) + \sum_{m=1}^{N} \left[ (\tilde{y}_m - y_m) (\tilde{x}_n)^{N-m} \right] = 0 \,, \tag{1.4}$$

hence, for the $N = 2$ case,

$$(\tilde{x}_n - x_1)(\tilde{x}_n - x_2) + (\tilde{y}_1 - y_1)\tilde{x}_n + \tilde{y}_2 - y_2 = 0 \,, \quad n = 1, 2 \tag{1.5a}$$

of course with (see (1.1b))

$$y_1(\ell) = -\left[x_1(\ell) + x_2(\ell)\right] \,, \quad y_2(\ell) = x_1(\ell) x_2(\ell). \tag{1.5b}$$

## 2. A *solvable* system of 2 nonlinearly coupled evolution equations in *discrete-time* satisfied by 2 dependent variables

In this Section 2 we present our main results, consisting in the identification of a *solvable* systems of 2 nonlinearly-coupled *discrete-time* evolution equations belonging to the class

$$\tilde{x}_n = P^{(n)}(x_1, x_2) \,, \quad n = 1, 2 \,, \tag{2.6}$$

where the 2 functions $P^{(n)}(x_1, x_2)$ are 2 appropriately identified *polynomials* in the variables $x_1(\ell)$ and $x_2(\ell)$; a system which is clearly the natural generalization to *discrete time* of the *continuous-time* system (1.3). In particular we demonstrate the *solvable* character of the following dynamical system:

$$\tilde{z}_n = a_{n1}(z_1)^2 + a_{n2}(z_2)^2 + a_{n3} z_1 z_2 \,, \quad n = 1, 2 \,, \tag{2.7}$$

with the 6 parameters $a_{nj}$ ($n = 1, 2,\ j = 1, 2, 3$) explicitly given by 6 algebraic expressions in terms of 6 *arbitrary* parameters.

Let the 2 dependent variables $y_1(\ell)$ and $y_2(\ell)$ evolve in discrete time according to the following *discrete-time* evolution equations (the *solvability* of which is demonstrated in Appendix A):

$$\tilde{y}_1 = \alpha (y_1)^2 \,, \quad \tilde{y}_2 = \beta^2 (y_1)^2 y_2 + \gamma (y_1)^4 \,; \tag{2.8}$$

and assume again that the 2 variables $y_1(\ell)$ and $y_2(\ell)$ are related to the 2 variables $x_1(\ell)$ and $x_2(\ell)$ as follows (see (1.5b)):

$$y_1(\ell) = -\left[x_1(\ell) + x_2(\ell)\right] \,, \quad y_2(\ell) = x_1(\ell) x_2(\ell). \tag{2.9}$$

**Remark 2.1.** Let us re-emphasize that, if the *discrete-time* evolution of the 2 variables $y_1(\ell)$ and $y_2(\ell)$ is *solvable*, then the *discrete-time* evolution of the 2 variables $x_1(\ell)$ and $x_2(\ell)$ is as well *solvable*, because the *ansatz* (2.9) can be inverted via an *algebraic* operation, indeed quite *explicitly*, since it clearly implies that $x_1(\ell)$ and $x_2(\ell)$ are the 2 roots of the following monic polynomial of degree 2:

$$p_2(z) = z^2 + y_1 z + y_2 = (z - x_1)(z - x_2). \qquad \blacksquare$$

It is then a matter of trivial algebra—either via the formulas (1.5) or directly from (2.8) and (2.9)—to derive the following system of two *discrete-time* evolution equations satisfied by the 2 dependent variables $x_1(\ell)$ and $x_2(\ell)$:

$$(\tilde{x}_n)^2 + \alpha (x_1 + x_2)^2 \tilde{x}_n + (x_1 + x_2)^2 \left[ \beta^2 x_1 x_2 + \gamma (x_1 + x_2)^2 \right] = 0 \,, \tag{2.10a}$$

implying

$$\tilde{x}_n = -\frac{\alpha (x_1 + x_2)^2 + (-1)^n \Delta}{2} \,, \tag{2.10b}$$

with

$$\Delta^2 = (x_1 + x_2)^2 \left[ \left( \alpha^2 - 4\gamma \right) (x_1 + x_2)^2 - 4\beta^2 x_1 x_2 \right] . \tag{2.10c}$$

Assume now that

$$\gamma = \frac{\alpha^2 - \beta^2}{4} \,, \tag{2.11a}$$

so that the right-hand side of (2.10c) become an *exact* square implying

$$\Delta = \pm \beta \left[ (x_1)^2 - (x_2)^2 \right] . \tag{2.11b}$$

Then clearly

$$\tilde{x}_n = -\frac{\alpha (x_1 + x_2)^2 + (-1)^n \beta \left[ (x_1)^2 - (x_2)^2 \right]}{2} \,, \quad n = 1, 2 \,, \tag{2.12a}$$

namely

$$\tilde{x}_n = a_1^{(n)} (x_1)^2 + a_2^{(n)} (x_2)^2 + a_3^{(n)} x_1 x_2 \,, \quad n = 1, 2 \,, \tag{2.12b}$$

with

$$a_1^{(n)} = -\frac{\alpha + (-1)^n \beta}{2} \,, \quad a_2^{(n)} = -\frac{\alpha + (-1)^n \beta}{2} \,, \quad a_3^{(n)} = -\alpha \,. \tag{2.12c}$$

We have thereby identified a simple *solvable* system of type (2.6), featuring in its right-hand side 2 *homogeneous* polynomials of *second* degree in the 2 variables $x_1(\ell)$ and $x_2(\ell)$, the 6 coefficients $a_j^{(n)}$ ($n = 1, 2,\ j = 1, 2, 3$) of which depend on the 2 *a priori arbitrary* parameters $\alpha$ and $\beta$, see (2.12c).

From this system an analogous system featuring more arbitrary parameters can be identified via the simple trick of introducing 2 new dependent variables, $z_1(\ell)$ and $z_2(\ell)$, related linearly to the 2 variables $x_1(\ell)$ and $x_2(\ell)$:

$$z_1 = A_{11} x_1 + A_{12} x_2 \,, \quad z_2 = A_{21} x_1 + A_{22} x_2 \,, \tag{2.13a}$$

$$x_1 = (A_{22} z_1 - A_{12} z_2) / D \,, \quad x_2 = (-A_{21} z_1 + A_{11} z_2) / D \,, \tag{2.13b}$$

$$D = A_{11} A_{22} - A_{12} A_{21} \,. \tag{2.13c}$$

It is easily seen that the new system is then just the system (2.7), with the 6 parameters $a_{nj}$ ($n = 1, 2$, $j = 1, 2, 3$) *explicitly* expressed as follows in terms of the 4 *arbitrary* parameters $A_{nm}$ ($n = 1, 2$,

$m = 1, 2$) and the 2 *arbitrary* parameters $\alpha$ and $\beta$ (see (2.12c)):

$$a_{n1} = D^{-2} \left[ (A_{22})^2 \left( A_{n1} a_1^{(1)} + A_{n2} a_1^{(2)} \right) + (A_{21})^2 \left( A_{n1} a_2^{(1)} + A_{n2} a_2^{(2)} \right) - A_{22} A_{21} \left( A_{n1} a_3^{(1)} + A_{n2} a_3^{(2)} \right) \right] , \quad n = 1, 2 , \tag{2.14a}$$

$$a_{n2} = D^{-2} \left[ (A_{12})^2 \left( A_{n1} a_1^{(1)} + A_{n2} a_1^{(2)} \right) + (A_{11})^2 \left( A_{n1} a_2^{(1)} + A_{n2} a_2^{(2)} \right) - A_{11} A_{12} \left( A_{n1} a_3^{(1)} + A_{n2} a_3^{(2)} \right) \right] , \quad n = 1, 2, \tag{2.14b}$$

$$a_{n3} = D^{-2} \left[ -2 A_{12} A_{22} \left( A_{n1} a_1^{(1)} + A_{n2} a_1^{(2)} \right) - 2 A_{21} A_{11} \left( A_{n1} a_2^{(1)} + A_{n2} a_2^{(2)} \right) + (A_{11} A_{22} + A_{12} A_{21}) \left( A_{n1} a_3^{(1)} + A_{n2} a_3^{(2)} \right) \right] , \quad n = 1, 2 . \tag{2.14c}$$

For the inversion of these transformations—i.e., the issue of expressing the 4 parameters $A_{nm}$ ($n = 1, 2$; $m = 1, 2$) and the 2 parameters $\alpha$, $\beta$ (see (2.12c)) in terms of the 6 parameters $a_{n\ell}$ ($n = 1, 2$, $\ell = 1, 2, 3$)—we refer to the analogous discussion in [1].

## 3. Outlook

In this final Section 3 we outline tersely possible future developments of the findings reported above.

The findings reported in this paper extend to evolutions in *discrete-time* only some of the findings for evolutions in *continuous-time* reported in [8] and [1] and tersely reviewed above (in Section 1). It is therefore quite natural to envisage an extension from the *continuous-time* context to the *discrete-time* context of other results reported in [8] and [1].

A different research line might be directed towards *applications* of the *solvable* discrete-time evolution equation (2.7), including its generalization via an assigned shift of the dependent variables $z_n(\ell)$, say

$$z_n(\ell) = w_n(\ell) + f_n(\ell) , \quad n = 1, 2, \tag{3.1a}$$

with $f_n(\ell)$ two *arbitrarily assigned* functions of the discrete-time $\ell$, implying that the new dependent-variable $w_n(\ell)$ satisfies the (of course still *solvable*) discrete-time evolution equation

$$\begin{aligned} \tilde{w}_n &= a_{n1} (w_1)^2 + a_{n2} (w_2)^2 + a_{n3} w_1 w_2 + g_{n1} w_1 + g_{n2} w_2 + h_n , \\ g_{n1}(\ell) &\equiv 2 a_{n1} f_1(\ell) + a_{n3} f_2(\ell) , \quad g_{n2}(\ell) \equiv 2 a_{n2} f_2(\ell) + a_{n3} f_1(\ell) , \\ h_n(\ell) &\equiv a_{n1} \left[ f_1(\ell) \right]^2 + a_{n2} \left[ f_2(\ell) \right]^2 + a_{n3} f_1(\ell) f_2(\ell) - f_n(\ell + 1) , \\ n &= 1, 2. \end{aligned} \tag{3.1b}$$

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## Appendix A. A useful class of *solvable* systems of 2 nonlinear *discrete-time* evolution equations for the 2 variables $y_m(\ell)$

The system (2.8) of *discrete-time* evolution equations discussed in this Appendix A is too simple to justify considering its solution as a *new* finding; its solution is reported here because of its role in solving the novel, more interesting, *discrete-time* model discussed above. The solution of the initial-value problem of the first of the 2 *discrete-time* evolution equations (2.8),

$$\tilde{y}_1 = \alpha\,(y_1)^2\,, \tag{A.1a}$$

is an easy task:

$$y_1(\ell) = \alpha^{-1}\,[\alpha y_1(0)]^{2^\ell}\,, \quad \ell = 0,1,2,\ldots\,. \tag{A.1b}$$

To solve the initial-value problem for the second of the 2 *discrete-time* evolution equations (2.8),

$$\tilde{y}_2 = \beta^2\,(y_1)^2\,y_2 + \gamma(y_1)^4\,, \tag{A.2}$$

it is convenient to introduce the *ansatz*

$$y_2(\ell) = \left\{\beta^{2\ell}\prod_{s=0}^{\ell-1}[y_1(s)]^2\right\}Y(\ell), \tag{A.3a}$$

implying

$$Y(0) = y_2(0). \tag{A.3b}$$

**Remark A.1.** We always use the standard convention according to which, if $s_1 > s_2$,

$$\sum_{s=s_1}^{s_2} f(s) = 0\,, \quad \prod_{s=s_1}^{s_2} f(s) = 1 \tag{A.4}$$

for *any arbitrary* function $f(s)$ of the *discrete-time* variable $s$. ∎

It is then easily seen that (A.2) implies

$$Y(\ell+1) = Y(\ell) + F(\ell), \tag{A.5a}$$

with

$$F(\ell) = \gamma[y_1(\ell)]^4\,\beta^{-2(\ell+1)}\prod_{s=0}^{\ell}[y_1(s)]^{-2}\,, \tag{A.5b}$$

implying, via (A.1b),

$$F(\ell) = \gamma\alpha^{-4}(\alpha/\beta)^{2(\ell+1)}\,[\alpha y_1(0)]^2\,. \tag{A.5c}$$

Hence in conclusion, from (A.5) with (A.3b),

$$Y(\ell) = y_2(0) + \gamma(\alpha\beta)^{-2}\left[\frac{(\alpha/\beta)^{2\ell}-1}{(\alpha/\beta)^2-1}\right][\alpha y_1(0)]^2\,, \tag{A.6}$$

hence, via (A.3a),

$$y_2(\ell) = (\beta/\alpha)^{2\ell} [\alpha y_1(0)]^{2^{\ell+1}-2} \{y_2(0) \\ +\gamma(\alpha\beta)^{-2} \left[\frac{(\alpha/\beta)^{2\ell} - 1}{(\alpha/\beta)^2 - 1}\right] [\alpha y_1(0)]^2 \right\} . \quad \text{(A.7)}$$

The 2 formulas (A.1b) and (A.7) provide the *explicit* solution of the initial-values problem of the discrete-time evolution (2.8). Of course in (A.1b), to make this solution applicable to the final findings reported in Section 2, the assignment $\gamma = \left(\alpha^2 - \beta^2\right)/4$ must be made, see (2.11a).

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