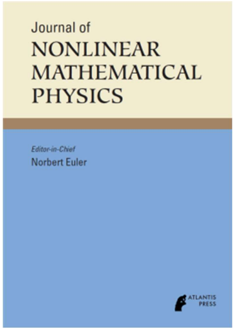

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Analytical Cartesian solutions of the multi-component Camassa-Holm equations

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Here, we give the existence of analytical Cartesian solutions of the multi-component Camassa-Holm (MCCH) equations. Such solutions can be explicitly expressed, in which the velocity function is given by $\boldsymbol{u} = \boldsymbol{b}(t) + A(t)\boldsymbol{x}$ and no extra constraint on the dimension N is required. The advantage of our method is that we turn the process of analytically solving MCCH equations into algebraically constructing the suitable matrix $A(t)$. As the applications, we obtain some interesting results: 1) If $\boldsymbol{u}$ is a linear transformation on $\boldsymbol{x} \in \mathbb{R}^N$, then p takes a quadratic form of $\boldsymbol{x}$. 2) If $A = f(t)I + D$ with $D^T = -D$, we obtain the spiral solutions. When $N = 2$, the solution can be used to describe "breather-type" oscillating motions of upper free surfaces. 3) If $A = (\frac{\dot{\alpha}_i}{\alpha_i})_{N\times N}$, we obtain the generalized elliptically symmetric solutions. When $N = 2$, the solution can be used to describe the drifting phenomena of the shallow water flow.

Keywords: Solution, Analytical Cartesian solution, Camassa-Holm equation, Curve integration theory, Multi-component Camassa-Holm equations.

Mathematics Subject Classification 2010: 35C05, 76B03, 76M60.

1. Introduction

Exact solutions of mathematical physics are usually very important. Not only because they can help us to understand physical phenomena they describe in nature, but also because they can serve as benchmarks for checking and improving numerical codes developed for studying more complex problems. Therefore, a lot of powerful methods has been developed, such as inverse scattering method, Hirota direct method, Darboux transformation and Bäcklund transformation et al [1-10]. However, none of these methods is universal due to the diversity and complexity of PDEs. Therefore, it will be of interest to find other effective methods that can lead to exact solutions.

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In this paper, we shall adopt a new method to construct solutions of the multi-component Camassa-Holm-type (CH) equations, which takes the following form [11]:

$$\begin{cases} \rho_t = -\nabla\rho\cdot\boldsymbol{u} - \rho(\nabla\cdot\boldsymbol{u}), \\ \boldsymbol{m}_t = -\boldsymbol{u}\cdot\nabla\boldsymbol{m} - (\nabla\boldsymbol{u})^T\cdot\boldsymbol{m} - \boldsymbol{m}(\nabla\cdot\boldsymbol{u}) - (\nabla\rho)^T\rho\,, \end{cases} \tag{1.1}$$

where ρ is the density, and $\boldsymbol{u}$, $\boldsymbol{m}$ denote the velocity and momentums of fluid on the n-torus $\mathbb{S}^n \simeq \mathbb{R}^n/\mathbb{Z}^n$. In general, it is assumed that there exists a linear operator A such that $\boldsymbol{m} = A\boldsymbol{u}$, and that A is in a form of $\alpha\mu + \beta - \Delta$ with $\{\alpha,\beta\} = \{0,1\}$ and $\alpha+\beta \neq 2$. While $\mu(\boldsymbol{u}) = \int_{\mathbb{S}^n}\boldsymbol{u}(x)dx$ denotes the mean value operator. The above system was introduced as a framework for studying and modeling fluid dynamics, especially for shallow water waves, turbulence modeling and geophysical fluids [12, 13].

To motivate our study, we shall review some related progresses on the multi-component CH system. The original interest in it may go back to the Camassa-Holm equation

$$m_t = -m_x u - 2u_x m, \quad \text{with} \quad m = u - u_{xx} \tag{1.2}$$

which was derived by Camassa, Holm, Johnson and Constantin *et al.* as a shallow water approximation (*see Refs.* [14–17]). As an important integrable model for describing the dynamics of shallow water waves, the CH equation has been studied extensively and intensively in a number of papers [14-26]. For instance, the complete integrability (as an infinite-dimensional Hamiltonian system) was established via inverse scattering method for suitable classes of initial data in [18, 19]. The bi-Hamiltonian structure and an infinite number of conservation laws of the CH equation were constructed in [14]. The links to the first negative flow of the KdV hierarchy were investigated in [20]. Since the traveling waves of greatest height of the governing equations for water waves are usually peaked [21, 22], peakons solutions are important. Such solutions were derived and their dynamics were analyzed in [23, 24]. Other solutions such as multi-soliton solutions, algebro-geometric solutions were studied in [25, 26]. Inspired by the nice properties of CH equation, the two-component CH equation:

$$\begin{cases} \rho_t = -\rho u_x - \rho_x u\,, \\ m_t = -m_x u - 2u_x m - \rho\rho_x\,, \end{cases} \quad \text{with} \quad m = u - u_{xx} \tag{1.3}$$

was introduced by Chen and Falqui in [27, 28] and also derived by Constantin and Ivanov as a model for shallow water waves in [29]. Experts find that it possed similar properties to the classical CH equation (*see* [30–32]). One of the important models closely related to the CH equation is the μ-Hunter-Saxton (μ-HS) equation

$$2\mu(u)u_x = u_{xxt} + 2u_x u_{xx} + uu_{xxx}\,, \tag{1.4}$$

which was proposed by Khesin et al [33] and simultaneously by Lenells et al with the name of μ-CH equation [34]. When $\mu(u) = 0$, it becomes the HS equation

$$-u_{xxt} = 2u_x u_{xx} + uu_{xxx}\,, \tag{1.5}$$

for modeling the propagation of nonlinear orientation waves in liquid crystals [35]. Both the μ-HS and HS equations have the two-component generalizations, which are considered as the system (1.3) with $m = \mu(u) - u_{xx}$ and $m = -u_{xx}$, respectively. Because these equations posses nice mathematical features and physical interpretations, they have gained much attention from integrable systems and PDE areas (*see* [36-39]).

We notice that most existing papers deal with the case $n = 1$ or 2. Investigations on the multi-component CH system mentioned above are rare, expect for the work done in [11, 34, 40, 41]. In particular, little work has been done on seeking exact solutions. From the view of mathematics and physics as explained in [40, 42, 43], the multi-component CH system is also very interesting. This deeply motivates us to undertake the present investigations.

Here, we would like to seek exact solutions with the velocity field of the form

$$\boldsymbol{u}(\boldsymbol{x},t) = A(t)\boldsymbol{x} + b(t), \qquad \boldsymbol{x} \in \mathbb{R}^N$$

for the multi-component CH system. This kind solution is part of a long history of finding exact solutions for fluid flows, especially for the Euler and Navier-Stokes (NS) equations [44-51]. A principle result in this direction is the work of Craik and Criminale [49] which gave a comprehensive analysis of solutions to the incompressible NS equations. We notice the continuity equation of multi-component CH equations (1.1) shares some similarities to the Euler and NS equations. Therefore, a natural question comes to us: Can we devise a plan to derive any solutions for the multi-component CH system? If the answer is positive, can we use such solutions to explain or predict any physical phenomenon? Bearing these questions in mind, we expand the investigation on the multi-component CH equations.

The structure of the paper is as follows: In section 2, we show that the multi-component CH equations admit the analytical Cartesian solutions if A satisfies certain matrix differential equations. In section 3, two solvable reductions are considered. Firstly, if A is an antisymmetric constant matrix, then the multi-component CH equations admit exact Cartesian solutions. Secondly, to construct more general solutions, the technique of matrix decomposition is used to make the matrix ODEs solvable. In particular, when $N = 2$, we obtain the rotational spiral solution and irrotational elliptical solutions obtained by Zhang, An and Yuen *et al.* The former can be used to describe the motion of "breather"-type oscillations of free surfaces in the upper ocean and the latter can be used to describe the drifting phenomena. In section 4, we discuss the property of such Cartesian solutions. Finally, a short conclusion is attached.

2. Existence of the exact Cartesian solutions

For convenience, we introduce a transformation via

$$p = \frac{1}{2}\rho^2 \tag{2.1}$$

then, the multi-component CH equations (1.1) are readily reduced into a form of

$$\begin{cases} \boldsymbol{m}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{m} + (\nabla\boldsymbol{u})^T\cdot\boldsymbol{m} + \boldsymbol{m}\,\mathrm{div}\,\boldsymbol{u} + \nabla p = 0, \\ p_t + 2p\,\mathrm{div}\,(\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)p = 0, \end{cases} \tag{2.2}$$

wherein $\boldsymbol{m} = (\alpha\mu + \beta - \Delta)\boldsymbol{u}$.

Here, our main goal is to seek suitable function p that enables us to obtain the analytical solutions wherein the velocity function $\boldsymbol{u}$ takes a linear form

$$\boldsymbol{u} = \boldsymbol{b}(t) + A\boldsymbol{x}$$

for the multi-component CH equations. In the above, $\boldsymbol{b}(t)$ is an N-dimensional vector function and A is an $N \times N$ matrix function, which are defined via

$$\boldsymbol{b}(t) = (b_1(t), b_2(t), \dots, b_N(t))^T, \qquad A = (a_{ij}(t))_{N\times N}.$$

It is noticed that ρ is redefined by p via (2.1), therefore, we only need to deal with p for solving the multi-component CH equations (2.2).

Theorem 2.1. *Defining B as part of a matrix Riccati equation*

$$B = \frac{1}{2}\big[A_t + (A + A^T)A + \mathrm{tr}(A)A\big]. \tag{2.3}$$

If A and B satisfy the following matrix differential equations

$$B^T = B, \tag{2.4}$$

$$B_t + 2\,\mathrm{tr}(A)B + BA + A^T B = 0, \tag{2.5}$$

then the multi-component CH equations (2.2) *admit the following explicit analytical solutions*

$$\boldsymbol{u} = \boldsymbol{b}(t) + A\boldsymbol{x}, \tag{2.6}$$

$$p = -(\alpha\mu + \beta)[\, \boldsymbol{x}^T\boldsymbol{b}_t + \boldsymbol{x}^T\boldsymbol{b}\mathrm{tr}(A) + \boldsymbol{x}^T(A + A^T)\boldsymbol{b} + \boldsymbol{x}^T B\boldsymbol{x} - c(t)\,]. \tag{2.7}$$

In the above, $\boldsymbol{b}(t)$ *is a vector function and* $c(t)$ *is a scalar function* $c(t)$*, which satisfy the following matrix ODE equations:*

$$\boldsymbol{d}_t + 2\boldsymbol{d}\,\mathrm{tr}(A) + A^T\boldsymbol{d} + 2B^T\boldsymbol{b} = 0, \tag{2.8}$$

$$c_t + 2c\,\mathrm{tr}(A) - \boldsymbol{b}^T\boldsymbol{d} = 0, \tag{2.9}$$

with

$$\boldsymbol{d} = \boldsymbol{b}_t + \big[A + A^T + \boldsymbol{I}\,\mathrm{tr}(A)\big]\boldsymbol{b}. \tag{2.10}$$

Proof. We now first prove the proposed analytical solution (2.6) will lead to (2.7) by solving the equation $(2.2)_1$. Substitution (2.6) into the first equation of (2.2) yields

$$\boldsymbol{m}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{m} + (\nabla\boldsymbol{u})^T\cdot\boldsymbol{m} + \boldsymbol{m}\,\mathrm{div}\,\boldsymbol{u} + \nabla p \tag{2.11}$$

$$= (\lambda - \Delta)(\boldsymbol{b}_t + A_t\boldsymbol{x}) + [(\boldsymbol{b} + A\boldsymbol{x})\cdot\nabla][(\lambda - \Delta)(\boldsymbol{b} + A\boldsymbol{x})] \tag{2.12}$$

$$+ [\nabla(\boldsymbol{b} + A\boldsymbol{x})]^T\cdot[(\lambda - \Delta)(\boldsymbol{b} + A\boldsymbol{x})] + [(\lambda - \Delta)(\boldsymbol{b} + A\boldsymbol{x})]\mathrm{div}(\boldsymbol{b} + A\boldsymbol{x}) + \nabla p \tag{2.13}$$

$$= \lambda[\boldsymbol{b}_t + A_t\boldsymbol{x} + (\boldsymbol{b}\cdot\nabla)A\boldsymbol{x} + (A\boldsymbol{x}\cdot\nabla)A\boldsymbol{x} + (\nabla A\boldsymbol{x})^T\cdot(\boldsymbol{b} + A\boldsymbol{x}) + (\boldsymbol{b} + A\boldsymbol{x})\mathrm{tr}(A) + \frac{1}{\lambda}\nabla p] \tag{2.14}$$

$$= \lambda\{\boldsymbol{b}_t + \boldsymbol{b}\mathrm{tr}(A) + (A + A^T)\boldsymbol{b} + [A_t + (A + A^T)A + \mathrm{tr}(A)A]\boldsymbol{x} + \frac{1}{\lambda}\nabla p\} = 0, \tag{2.15}$$

with $\lambda = \alpha\mu + \beta$.

For the convenience of subsequent computations, an auxiliary matrix is introduced via

$$B=(b_{ij})_{N\times N}=\frac{1}{2}\left[A_t+(A+A^T)A+\mathrm{tr}(A)A\right], \tag{2.16}$$

with

$$b_{ij}=\frac{1}{2}\left(a_{ij,t}+\sum_{k=1}^{N}(a_{ik}a_{kj}+a_{ki}a_{kj})+(a_{11}+\cdots+a_{nn})a_{ij}\right),$$

therefore, the equation (2.11) can be readily rewritten into the component form

$$Q_i(x_1,\ldots,x_N,t)\equiv -b_{it}-\mathrm{tr}(A)b_i-\sum_{k=1}^{N}[(a_{ik}+a_{ki})b_k-2b_{ik}x_k]=\frac{1}{\lambda}\frac{\partial p}{\partial x_i},\ \ i=1,2,\ldots,N. \tag{2.17}$$

It is noticed that for solving $p(\boldsymbol{x},t)$ from the above equation, all the N equations must be compatible with each other. In other words, the vector functions $(Q_1,Q_2,\ldots,Q_N)$ should be a potential field of p wherein the sufficient and necessary conditions are

$$\frac{\partial Q_i(x_1,\ldots,x_N,t)}{\partial x_j}=\frac{\partial Q_j(x_1,\ldots,x_N,t)}{\partial x_i},\ \ i,\,j=1,2,\ldots,N, \tag{2.18}$$

which holds if and only if

$$b_{ij}=b_{ji},\ \ i,\,j=1,2,\ldots,N.$$

The above condition implies that $B=(b_{ij})_{N\times N}=\frac{1}{2}[A_t+(A+A^T)A+\mathrm{tr}(A)A]$ is a symmetric matrix, which is just the condition (2.4).

The condition (2.18) shows that the function $p(\boldsymbol{x},t)$ can be written into a complete differential form

$$dp(\boldsymbol{x},t)=\sum_{i=1}^{N}\frac{\partial p(\boldsymbol{x},t)}{\partial x_i}dx_i=\sum_{i=1}^{N}\lambda Q_i(x_1,\ldots,x_N,t)dx_i.$$

Therefore, we obtain that the second kind of curvilinear integral of $p(\boldsymbol{x},t)$ is independent of path. So that it allows us to take a special integration route from $(0,0,\ldots,0)$ to $(x_1,x_2,\ldots,x_N)$

$$\begin{aligned}p(\boldsymbol{x},t)&=\sum_{i=1}^{N}\int_{(0,0,\ldots,0)}^{(x_1,x_2,\ldots,x_N)}\lambda Q_i(x_1,x_2,\ldots,x_N,t)dx_i\\&=\lambda\left[\int_0^{x_1}Q_1(x_1,0,\ldots,0,t)dx_1+\int_0^{x_2}Q_2(x_1,x_2,0,\ldots,0,t)dx_2+\cdots+\int_0^{x_N}Q_N(x_1,x_2,\ldots,x_N,t)dx_N\right]\end{aligned}$$

Directly complicated calculation shows that

$$\begin{aligned}p(\boldsymbol{x},t)&=\sum_{i=1}^{N}\int_{(0,0,\ldots,0)}^{(x_1,x_2,\ldots,x_N)}\lambda Q_i(x_1,x_2,\ldots,x_N,t)dx_i\\&=\lambda\left[\int_0^{x_1}Q_1(x_1,0,\ldots,0,t)dx_1+\cdots+\int_0^{x_N}Q_N(x_1,x_2,\ldots,x_N,t)dx_N\right]\\&=-\lambda\left[\sum_{i=1}^{N}\left[b_{it}+\sum_{k=1}^{N}a_{kk}b_i+\sum_{k=1}^{N}(a_{ik}+a_{ki})b_k\right]x_i+\sum_{i=1}^{N}b_{ii}x_i^2+2\sum_{i,k=1,\,i<k}^{N}b_{ik}x_ix_k-c(t)]\right]\\&=-\lambda\left[\boldsymbol{x}^T\boldsymbol{b}_t+\boldsymbol{x}^T\boldsymbol{b}\,\mathrm{tr}(A)+\boldsymbol{x}^T(A+A^T)\boldsymbol{b}+\boldsymbol{x}^TB\boldsymbol{x}-c(t)\right].\end{aligned}$$

At this point, we have finished proving that the functions given by (2.6) and (2.7) satisfy the first equation of (2.2). In the sequel, we shall prove that such functions also satisfy the second equation of (2.2). On use of the relations (2.5), (2.8) and (2.9), we have

$$\begin{aligned}
&p_t + 2p\,\mathrm{div}(\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)p \\
&= -\lambda\{\boldsymbol{x}^T[\boldsymbol{b}_t + \mathrm{tr}(A)\boldsymbol{b} + (A+A^T)\boldsymbol{b}]_t + \boldsymbol{x}^T B_t \boldsymbol{x} - c_t + 2\,\mathrm{tr}(A)\left[\boldsymbol{x}^T(\boldsymbol{b}_t + \mathrm{tr}(A)\boldsymbol{b})\right.\\
&\qquad \left. + \boldsymbol{x}^T(A+A^T)\boldsymbol{b} + \boldsymbol{x}^T B\boldsymbol{x} - c\right] + (A\boldsymbol{x}+\boldsymbol{b})^T[\boldsymbol{b}_t + \mathrm{tr}(A)\boldsymbol{b} + (A+A^T)\boldsymbol{b} + 2B\boldsymbol{x}]\}\\
&= -\lambda\left\{\boldsymbol{x}^T[B_t + 2\,\mathrm{tr}(A)B + 2A^T B]\boldsymbol{x} + \boldsymbol{x}^T[\boldsymbol{d}_t + 2\boldsymbol{d}\,\mathrm{tr}(A) + A^T\boldsymbol{d} + 2B^T\boldsymbol{b}] - [c_t + 2\,\mathrm{tr}(A)c - \boldsymbol{b}^T\boldsymbol{d}]\right\}\\
&= 0,
\end{aligned} \tag{2.19}$$

with $\boldsymbol{d} = \boldsymbol{b}_t + [A + A^T + \boldsymbol{I}\,\mathrm{tr}(A)]\boldsymbol{b}$. While the condition

$$x^T[B_t + 2\,\mathrm{tr}(A)B + 2A^T B]x = 0,$$

implies that $B_t + 2\,\mathrm{tr}(A)B + 2A^T B$ is antisymmetry, namely

$$[B_t + 2\,\mathrm{tr}(A)B + 2A^T B]^T = -[B_t + 2\,\mathrm{tr}(A)B + 2A^T B].$$

Hence we obtain the following relation

$$B_t + 2\,\mathrm{tr}(A)B + BA + A^T B = 0$$

as described in (2.5) in Theorem 2.1. The proof is completed. □

To conclude, here we have theoretically obtained the existence of explicit Cartesian vector solutions of (2.2) in Theorem 2.1. The Cartesian solutions are mainly governed by A via the relation (2.5), which is a complex matrix ODE system involving N^2 scalar equations. As we know, compared with a scalar equation, it is usually very difficult to construct general solutions of a given matrix ODE system. Therefore, to make some solutions available, special techniques are devised in the subsequent section.

3. Special reductions and corresponding solutions

Now, we return back to the condition (2.5) given in Theorem 2.1. Due to the inherent complexity to solve matrix ODE, our attentions are restricted to the cases that lead to some special solutions.

I. The first reduction: A is an antisymmetric constant matrix

It is noted that when A is an antisymmetric constant matrix, the solution can be readily constructed, which is established in the following theorem:

Theorem 3.1. *If A is an anti-symmetric constant matrix, then the multi-component CH equation* (2.2) *admits a general solution via*

$$\boldsymbol{u} = \boldsymbol{b}(t) + A\boldsymbol{x}, \tag{3.1}$$

$$p = -(\alpha\mu + \beta)[\boldsymbol{x}^T\boldsymbol{b}_t - c(t)], \tag{3.2}$$

where $\boldsymbol{b}(t)$ and $c(t)$ satisfies

$$\boldsymbol{b}(t) = \exp(At)\boldsymbol{b}_1 + \boldsymbol{b}_2, \tag{3.3}$$

$$c(t) = \boldsymbol{b}_1^T A \boldsymbol{b}_1 t + \boldsymbol{b}_2^T \exp(At)\boldsymbol{b}_1 + b_3, \tag{3.4}$$

where $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are constant vectors of integration, and b_3 is a constant of integration.

Proof. Since the solution derived here is just a special case of Theorem 2.1, we just need to validate that the conditions (2.4), (2.5), (2.8) and (2.9) can be satisfied in Theorem 3.1.

When A is an antisymmetric constant matrix, the conditions (2.4) and (2.5) can be easily checked. Insertion of A into (2.8) and (2.9) delivers

$$\boldsymbol{b}_{tt} - A\boldsymbol{b}_t = 0, \qquad c_t - \boldsymbol{b}^T \boldsymbol{b}_t = 0. \tag{3.5}$$

Computation shows the solutions of them are just (3.3) and (3.4). □

As the application of Theorem 3.1, here we give an illustrative example:

Example 3.1. Taking $N = 2$, for the 2-dimensional multi-component CH equations, setting the anti-symmetric matrix A as

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix},$$

where a is an arbitrary constant. According to (3.3) and (3.4), we have

$$\boldsymbol{b}(t) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \cos at + \begin{pmatrix} c_2 \\ -c_1 \end{pmatrix} \sin at + \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}, \qquad c(t) = c_5$$

where c_i $(i = 1, \ldots, 5)$ are constants of integration. Therefore, the solution of the 2D multi-component CH equations is

$$\begin{aligned} \boldsymbol{u} &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \cos at + \begin{pmatrix} c_2 \\ -c_1 \end{pmatrix} \sin at + \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}, \\ p &= -(\alpha\mu + \beta)[(c_2 \cos at - c_1 \sin at) a x_1 - (c_2 \sin at + c_1 \cos at) a x_2 - c_5]. \end{aligned} \tag{3.6}$$

To shed lights on the behaviors that the solution derived may exhibit, we perform the numerical simulations in Fig. 1. There we choose $c_1 = c_2 = 1$, $c_3 = c_4 = 0$. From the figure, we can see that the steady flow is rotational and the velocities of all fluid particles point directly inwards to the origin, which are quite different from those in the radially symmetric solution case.

Example 3.2. Taking $N = 3$, for the 3-dimensional multi-component CH equations, setting the anti-symmetric matrix A as

$$A = \frac{a}{\sqrt{3}} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad \boldsymbol{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ b_3(t) \end{pmatrix} = \begin{pmatrix} (\sqrt{3}c_1 - c_2)\cos at - (c_1 + \sqrt{3}c_2)\sin at \\ -(\sqrt{3}c_1 + c_2)\cos at - (c_1 - \sqrt{3}c_2)\sin at \\ 2c_1 \sin at + 2c_2 \cos at \end{pmatrix},$$

So according to (3.1) and (3.2) we can get the following rotational solution:

$$\begin{aligned} &u_1 = b_1(t) + \frac{a}{\sqrt{3}}(x_2 - x_1), \quad u_2 = b_2(t) + \frac{a}{\sqrt{3}}(x_3 - x_1), \quad u_3 = b_3(t) + \frac{a}{\sqrt{3}}(x_1 - x_2), \\ &p = -(\alpha\mu + \beta)(b_{1t}x_1 + b_{2t}x_2 + b_{3t}x_3 - c(t)). \end{aligned} \tag{3.7}$$

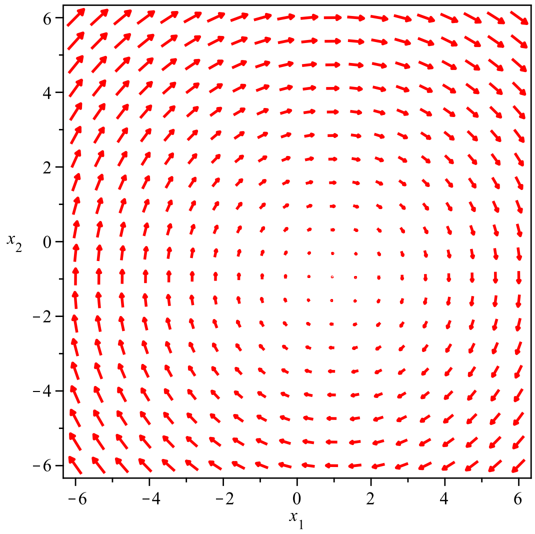

Fig. 1. The structure of the 2-dimensional rotational solution with velocity $\boldsymbol{u} = (u_1, u_2)^T$ given in (3.6). The length of the arrow stands for the strength of the velocity field.

II. The second type reduction: A is a decomposable t-dependent matrix

The special case considered here is A being a certain decomposable time-dependent matrix. In particular, we discuss the two cases: one is $A = f(t)I + D$ with D antisymmetric and the other is $A = \left(\frac{\dot{\alpha}_i}{\alpha_i}\right)_{N\times N}$. Under these two cases, the corresponding analytical solutions can be obtained.

Theorem 3.2. *Suppose that the matrix A can be decomposed into*

$$A = D + E, \quad D^T = -D, \tag{3.8}$$

where $D = A^{\text{off}}$ *and* $E = A^{\text{diag}}$ *change the off-diagonal part and diagonal part of A, respectively. If D and E satisfy the following matrix differential equations:*

$$D_t = -2ED - D\text{tr}(E), \tag{3.9}$$

$$C_1 \equiv E_{tt} + 6EE_t + 3\,\text{tr}(E)(E_t + 2E^2) + 4E^3 + E\,\text{tr}(E_t) + 2E[\text{tr}(E)]^2 = 0, \tag{3.10}$$

$$C_2 \equiv E_t D - DE_t + \text{tr}(E)(ED - DE) + 2(E^2 D - DE^2) = 0, \tag{3.11}$$

then

$$B = \frac{1}{2}[E_t + 2E^2 + E\,\text{tr}(E)], \tag{3.12}$$

is a symmetric matrix satisfying the conditions (2.4) *and* (2.5) *given in Theorem* 2.1.

Proof. With the aid of (3.8) and (3.9), one can have

$$\begin{aligned} B &= \frac{1}{2}[A_t + (A + A^T)A + \text{tr}(A)A] = \frac{1}{2}[D_t + E_t + (D + E + D^T + E^T)(D + E) + \text{tr}(E)(D + E)] \\ &= \frac{1}{2}[D_t + E_t + 2ED + 2E^2 + D\,\text{tr}(E) + E\,\text{tr}(E)] \\ &= \frac{1}{2}[E_t + 2E^2 + E\,\text{tr}(E)\,], \end{aligned}$$

which shows the condition (2.4) is satisfied and B is a symmetric matrix.

In the sequel, we shall show the condition (2.5) can be also satisfied under conditions given in this theorem. On using (3.12) and (3.8), we can obtain

$$\begin{aligned}
&2[B_t + 2\,\mathrm{tr}(A)B + BA + A^T B] \\
&= E_{tt} + 4EE_t + \mathrm{tr}(E)E_t + E\,\mathrm{tr}(E_t) + 2\,\mathrm{tr}(E)[E_t + 2E^2 + E\,\mathrm{tr}(E)] \\
&\qquad + [E_t + 2E^2 + E\,\mathrm{tr}(E)](D+E) + (D^T + E)[E_t + 2E^2 + E\,\mathrm{tr}(E)] \\
&= E_{tt} + 6EE_t + 3\,\mathrm{tr}(E)(E_t + 2E^2) + 4E^3 + E\,\mathrm{tr}(E_t) + 2E[\mathrm{tr}(E)]^2 \\
&\qquad\qquad + E_t D - DE_t + \mathrm{tr}(E)(ED - DE) + 2(E^2 D - DE^2) \\
&\equiv C_1 + C_2 = 0.
\end{aligned}$$

The proof is completed. □

In the following, we shall mainly focus on two special cases. One case is when $E = f(t)I, D \neq 0$ to be discussed in Corollary 3.1. The other case is when $E = (\frac{\dot{\alpha}_i}{\alpha_i})_{N\times N}$ and $D = 0$ to be discussed in Corollary 3.2. There exact solutions can be obtained.

Corollary 3.1. *In Theorem* 3.2*, setting*

$$E = f(t)I, \tag{3.13}$$

where I is a unitary matrix, then the matrix differential equation (3.9) *admits the solution of*

$$D = e^{-(N+2)\int f(t)dt}\boldsymbol{C}, \tag{3.14}$$

where $\boldsymbol{C}$ is an antisymmetric constant matrix, that is

$$\boldsymbol{C} = (c_{ij})_{N\times N}, \quad c_{ii} = 0, \quad c_{ij} = -c_{ji}, \quad i \neq j. \tag{3.15}$$

While Eq. (3.10) *is reducible to*

$$f_{tt} + 2(2N+3)ff_t + 2(N+1)(N+2)f^3 = 0. \tag{3.16}$$

Proof. On inserting (3.13) into (3.9), one can have

$$D_t = -(N+2)fD. \tag{3.17}$$

Hence, the solution of D is derived in (3.14). While substituting (3.13) into (3.7), one can directly obtain (3.16) and the condition (3.11) is satisfied automatically. □

It is noticed that the equation (3.16) can be rewritten into a form of

$$[f_t + (N+1)f^2]_t + 2(N+2)f[f_t + (N+1)f^2] = 0 \tag{3.18}$$

Thus, in what follows, we shall show its solvability in two cases:

Case 1. If $f_t + (N+1)f^2 = 0$, then the system (3.18) has a solution

$$f = \frac{1}{(N+1)t + k_1}, \tag{3.19}$$

where k_1 is a constant of integration. Then insertion it into (3.14) gives

$$D = \frac{\boldsymbol{C}}{[(N+1)t + k_1]^{\frac{N+2}{N+1}}} = \frac{1}{[(N+1)t + k_1]^{\frac{N+2}{N+1}}}(c_{ij})_{N\times N}, \quad 1 \le i < j \le N. \tag{3.20}$$

with $\boldsymbol{C}$ is an antisymmetric matrix of integration whose elements satisfy (3.15).

Case 2. If $f_t + (N+1)f^2 \neq 0$, on introduction a function g via

$$f = \partial_t \ln g \tag{3.21}$$

where $\partial_t = \frac{\partial}{\partial_t}$, therefore, the equation (3.18) is now reformulated into

$$\partial_t \ln \frac{g g_{tt} + N g_t^2}{g^2} = \partial_t \ln g^{-2(N+2)}. \tag{3.22}$$

Integration (3.22) with respect to t shows that

$$g g_{tt} + N g_t^2 = k_2 g^{-2(N+1)},$$

which can be readily written into

$$(g_t g^N) = k_2 g^{-(N+3)}.$$

Therefore, we can obtain the following relation

$$\left(g^{N+1}\right)_{tt} = k_2(N+1) g^{-(N+3)}. \tag{3.23}$$

In the above, k_2 is an integration constant. Here we go with $k_2 \neq 0$. By multiplying g_t^{N+1} on both sides of (3.23), integration shows

$$g_t^{N+1} = \pm \frac{N+1}{g} \sqrt{k_3 g^2 - k_2}, \tag{3.24}$$

whence

$$\int \frac{g^{N+1} dg}{\sqrt{k_3 g^2 - k_2}} = \pm t + k_4. \tag{3.25}$$

Its solutions can be readily obtained by trigonometric substitutions, whose forms are dependent on the parameters k_2 and k_3. Here we just take $k_2 > 0$ and $k_3 > 0$ as the example to illustrate. Under this condition, we introduce the trigonometric substitution of this form

$$\sec\phi = \sqrt{k_2/k_3}g := \frac{1}{k}g, \tag{3.26}$$

then substitution it into (3.25), we obtain

$$\begin{aligned}\frac{k^{N+2}}{\sqrt{k_2}}\int \sec^{N+2}\phi d\phi &= \frac{k^{N+2}}{\sqrt{k_2}(N+1)}[-\sec^{N}\phi\tan\phi + N\int \sec^{N-1}\phi d\phi] \\ &= \pm t + k_4. \end{aligned}\tag{3.27}$$

Once N is given, the value of (3.27) can be known. So that g and f can be generated accordingly via (3.21). Once f is generated, the function of D will be obtained via (3.14). Therefore the analytical solutions under Case 2 is derived.

Remark 3.1. For other choices of k_2 and k_3, one needs to introduce the corresponding trigonometric substitution to find the iterative relation similar to (3.27). Here the calculations procedures are omitted.

Remark 3.2. We needs to point out that when $k_2 = 0$, from (3.23), we can obtain that g is governed by

$$g^{N+1} = h_1 t + h_2. \tag{3.28}$$

Insertion it into (3.21) shows that

$$f = \frac{g_t}{g} = \frac{{g^{N+1}}_t}{(N+1)g^{N+1}} = \frac{h_1}{(N+1)(h_1 t + h_2)} \tag{3.29}$$

If we choose $h_1 = 1$ and $h_2 = k_1$, it is nothing but the solution given in (3.19) in Case 1. Thus, we conclude that the solutions (3.27) obtained in Case 2 are more general.

As the application of the above corollary, we give some examples:

Example 3.3. Taking $N = 2$, according to Case 1, we choose $k_1 = 0$, $\boldsymbol{b(t)} = 0$, $c(t) = 0$ and A as

$$A = D + f(t)I = \frac{1}{3t}\begin{pmatrix} 1 & \frac{1}{\sqrt[3]{3t}} \\ -\frac{1}{\sqrt[3]{3t}} & 1 \end{pmatrix},$$

so that

$$B = \frac{1}{2}[E_t + 2E^2 + E\mathrm{tr}(E)\,] = \frac{1}{9t^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, we can obtain an exact solution

$$\begin{aligned} &u_1 = \frac{1}{3t}(x_1 + \frac{x_2}{\sqrt[3]{3t}}), \quad u_2 = \frac{1}{3t}(x_2 - \frac{x_1}{\sqrt[3]{3t}}) \\ &p = -\frac{\alpha\mu + \beta}{9t^2}(x_1^2 + x_2^2). \end{aligned}\tag{3.30}$$

It is a kind of spiral solution that presented by Zhang and Zheng for Euler equations [52]. Interestingly, here we obtain for the multi-component CH equation. The behaviors of the spiral solution

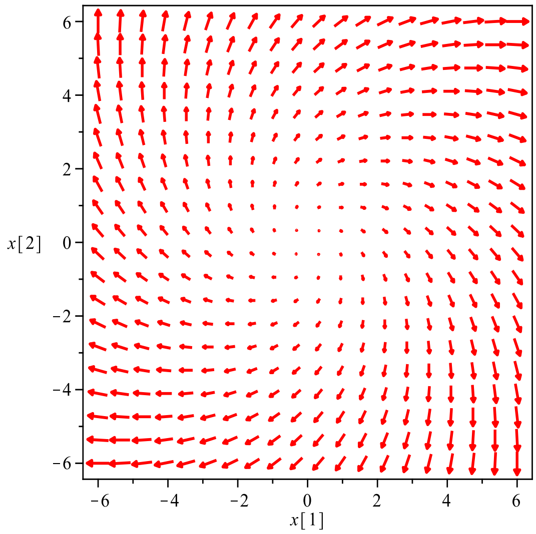

Fig. 2. The structure of the 2-dimensional spiral solution with $\boldsymbol{u} = (u_1, u_2)^T$ given in (3.30). The length of the arrow stands for the strength of the velocity field.

is exhibited in Fig. 2 and Fig. 3. We find from Fig. 2 that the velocities of all fluid particles point directly from the origin to the outside, which are quite different from the rotational solution obtained in (3.6). From Fig. 3, we find the surfaces of p jump up and down with time changing. It is remarkable that such jumping up and down phenomena just coincide with the motion of the "breather-type" oscillations of upper free surfaces in the ocean.

Corollary 3.2. *In Theorem* 3.2*, if setting*

$$E = \left(\frac{\dot{\alpha}_i}{\alpha_i}\right)_{N\times N}, \quad D = 0 \tag{3.31}$$

Then it is obvious that the relations (2.9)-(2.11) *in Theorem* 3.2 *are satisfied. Meanwhile,* $B = \frac{1}{2}[A_t + (A + A^T)A + \mathrm{tr}(A)A]$ *satisfies the conditions* (2.4) *and* (2.5).

Example 3.4. Taking $N = 1$, for 1-dimensional CH equation, we choose $c(t) = 0$ and $A = \frac{\dot{\alpha}_1}{\alpha_1}$, so that

$$B = \frac{1}{2}[A_t + (A + A^T)A + \mathrm{tr}(A)A] = \frac{1}{2}\left(\frac{\ddot{\alpha}_1}{\alpha_1} + \frac{2\dot{\alpha}_1^2}{\alpha_1^2}\right) \tag{3.32}$$

Therefore, we obtain an analytical solution

$$u = \frac{\dot{\alpha}_1}{\alpha_1}x + b(t), \quad p = -\frac{\alpha\mu + \beta}{2}\left[(\boldsymbol{b}_t + 3\frac{\boldsymbol{b}\alpha_1}{\alpha_1})x + 2\left(\frac{\ddot{\alpha}_1}{\alpha_1} + \frac{2\dot{\alpha}_1^2}{\alpha_1^2}\right)x^2 - c(t)\right] \tag{3.33}$$

where α_i is governed by an Emden equation

$$\ddot{\beta}_1 = \frac{C_1}{\beta_1^{\frac{1}{3}}}, \qquad \beta_1 = \alpha_1^3. \tag{3.34}$$

In particular, if we choose $\boldsymbol{b}(t) = 0$ and $c(t) = 0$, this solution is just what has obtained by Yuen [53].

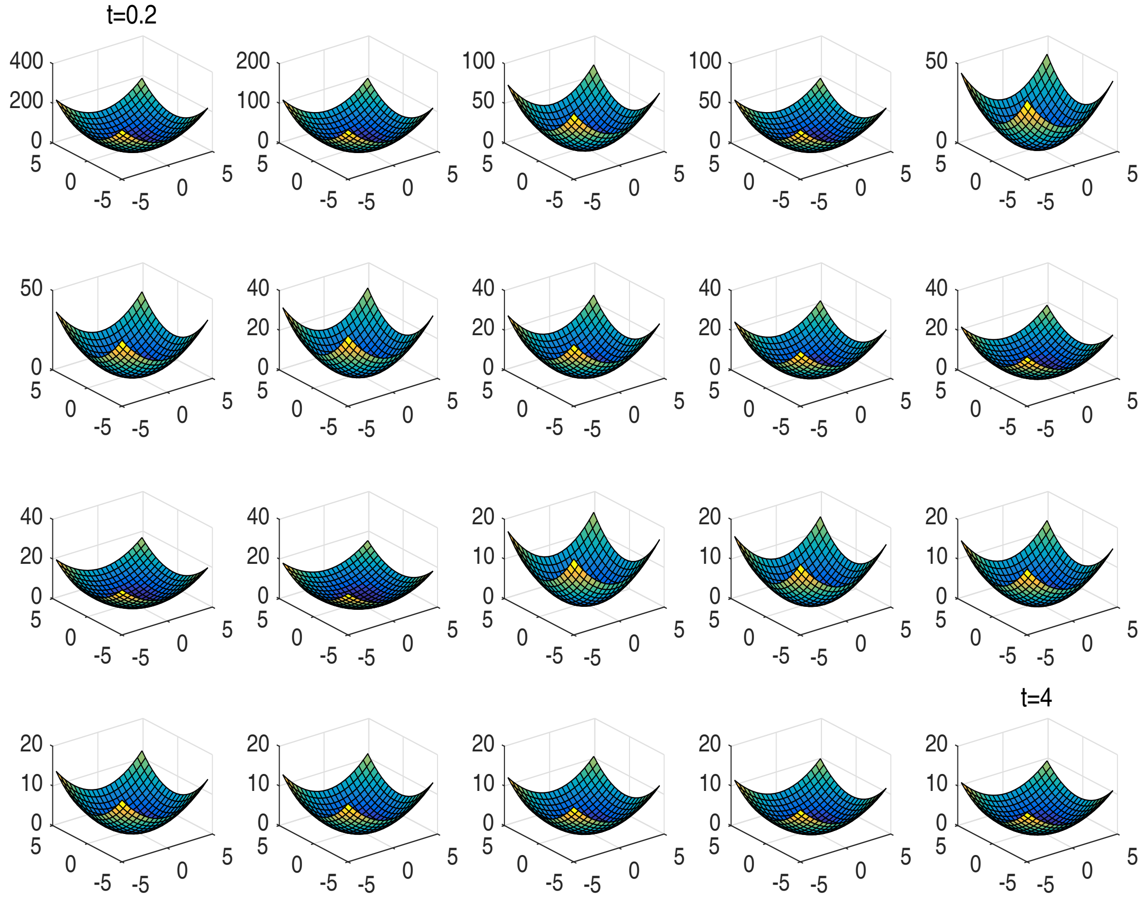

Fig. 3. Time evolutions of the solution $p = -\frac{\alpha\mu+\beta}{9t^2}(x_1^2+x_2^2)$ given in (3.30). We choose $\alpha = 1$, $\mu = -1$ and $\beta = 0$. The time interval is $\Delta t = 0.2$ and $t \in [0.2, 2]$.

Example 3.5. If $N > 1$, we choose $A = \left(\dfrac{\dot{\alpha}_i}{\alpha_i}\right)_{N\times N}$, so that

$$B = (b_{ij})_{N\times N} = \frac{1}{2}\left[A_t + (A + A^T)A + \mathrm{tr}(A)A\right] = \frac{1}{2}\left(\frac{\ddot{\alpha}_i}{\alpha_i} + \frac{2\dot{\alpha}_i^2}{\alpha_i^2} + \frac{\dot{\alpha}_i}{\alpha_i}\sum_{k\neq i}^{n}\frac{\dot{\alpha}_k}{\alpha_k}\right) \tag{3.35}$$

So that we obtain the general analytical solution

$$\begin{aligned} u_i &= \frac{\dot{\alpha}_i}{\alpha_i}x_i + b_i(t), \\ p &= -(\alpha\mu+\beta)[\boldsymbol{x}^T\boldsymbol{b}_t + \boldsymbol{x}^T\boldsymbol{b}(\frac{2\dot{\alpha}_i^2}{\alpha_i^2} + \sum_{k=1}^{n}\frac{\dot{\alpha}_k}{\alpha_k}) + \frac{1}{2}\boldsymbol{x}^T(\frac{\ddot{\alpha}_i}{\alpha_i} + \frac{2\dot{\alpha}_i^2}{\alpha_i^2} + \frac{\dot{\alpha}_i}{\alpha_i}\sum_{k\neq i}^{n}\frac{\dot{\alpha}_k}{\alpha_k})_{N\times N}\boldsymbol{x} - c(t)] \end{aligned} \tag{3.36}$$

where α_i is governed by the generalized Emden equation

$$\ddot{\beta}_i + \frac{1}{3}\dot{\beta}_i\sum_{k\neq i}^{n}\frac{\dot{\beta}_k}{\beta_k} = \frac{C_i}{\beta_i^{-\frac{1}{3}}\prod_{k=1}^{n}\beta_k^{\frac{2}{3}}}, \qquad \beta_i = \alpha_i^3, \tag{3.37}$$

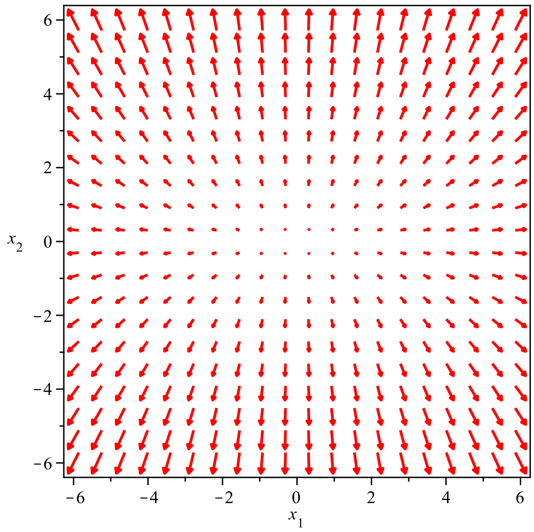

Fig. 4. The structure of the 2-dimensional irrotational solution with $\boldsymbol{u} = (u_1, u_2)^T$ given in (3.36). The length of the arrow stands for the strength of the velocity field.

with C_i is the integration constant. We find that it is just a kind of elliptically symmetric solutions generalized what are obtained by An and Yuen [41]. In order to show the behaviors the solution may posses, we present the numerical simulations when $N = 2$. It is seen from Fig. 4 that the velocity field $\boldsymbol{u}$ is *irrotational*. From Fig. 5, we find the center of the flow moves forward with time changing. Such moving behaviors that the solution exhibits is just the drifting phenomena of the shallow water flow.

4. Quasi-linear superposition principle of solutions

In order to shed some light on the properties and structures of the solutions, we now return back to the solutions (2.6) and (2.7). We take $\boldsymbol{b}(t) = \boldsymbol{0}$ and $c(t) = 0$ for simplicity so that the solutions is reducible to

$$\boldsymbol{u}(\boldsymbol{x}) = A\boldsymbol{x}, \quad p = -(\alpha\mu + \beta)\boldsymbol{x}^T B\boldsymbol{x}. \tag{4.1}$$

This renders one to consider that the solution $\boldsymbol{u}(\boldsymbol{x})$ is generated by a linear transformation A on $\boldsymbol{x} \in \mathbb{R}^N$:

$$A : \boldsymbol{x} \to A\boldsymbol{x} = \boldsymbol{u},$$

while $p = p(\boldsymbol{x}, t)$ is a quadratic form associated with the matrix $B = (A + A^T)A + \mathrm{tr}(A)A$

$$B : \boldsymbol{x} \to -\boldsymbol{x}^T B\boldsymbol{x} = p(\boldsymbol{x}, t).$$

It is noticed that there exists a quasi-linear superposition principle for Cartesian solution (4.1) that is analogous to the linear equations:

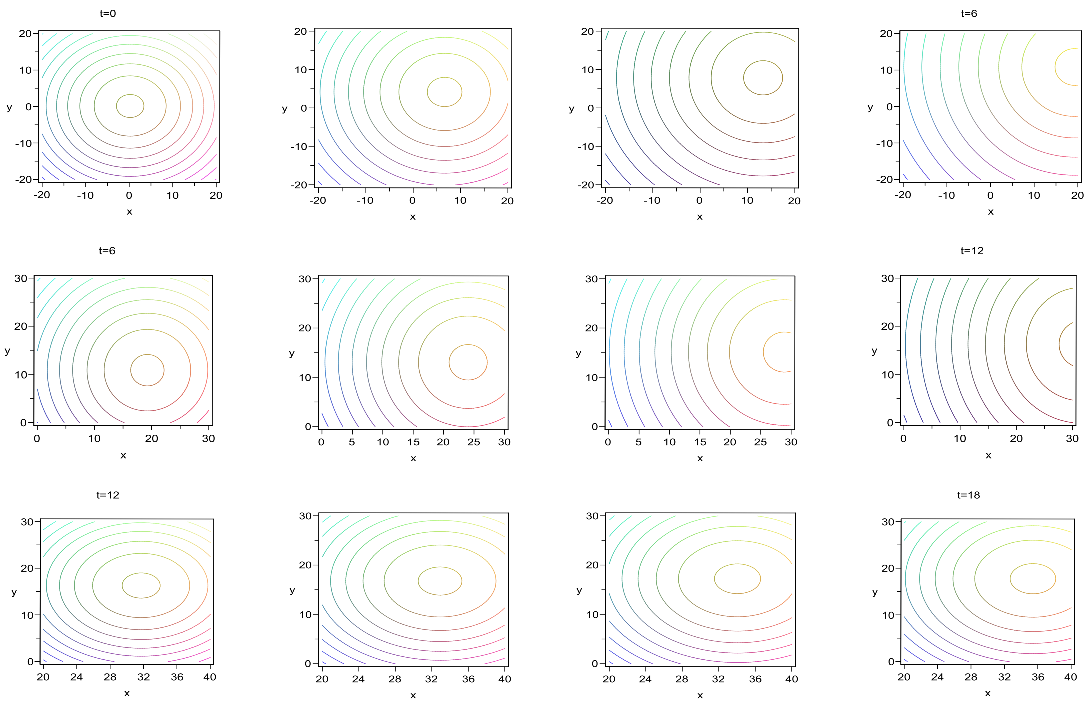

Fig. 5. Time evolutions of the solution p given in (3.36). Here $C_1 = 5, C_2 = 3, \alpha = 1, \mu = -1, \beta = 0$ and the time interval $\Delta = 3$.

Theorem 4.1. *Suppose that A and B satisfy the conditions in Theorem* 2.1*, and* $\boldsymbol{u}(\boldsymbol{x})$*,* $\boldsymbol{u}(\boldsymbol{y})$ *are two solutions whose take the form of* (4.1)*. Then*

$$\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}) = \boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{u}(\boldsymbol{y}),$$
$$p(\boldsymbol{x}+\boldsymbol{y}) = p(\boldsymbol{x}) + p(\boldsymbol{y}) - 2\boldsymbol{x}^T B\boldsymbol{y}$$

is a solution of the multi-component CH equations (2.2).

5. Conclusions and discussions

The multi-component CH system is an important mathematical physics model, which has been wide used in fluid mechanics, geophysics, oceanic dynamics and nonlinear optics *et al*. In the paper, based on the matrix theory and curve integration theory, we have shown that the multi-component CH equations admit analytical Cartesian solutions. Such solutions globally exist and admit a quasi-linear superposition principle under the condition $\boldsymbol{b}(t) = \boldsymbol{0}$ and $c(t) = 0$. As applications, we give some interesting solutions and their numerical simulations. Among these solutions, some are more general than other researchers obtained before. While, to our knowledge, some are completely new. However, there are still some interesting problems that need further consideration:

1) It is seen that seeking the suitable matrix solution A in (2.5) proves key to derivation of analytical solutions of the multi-component CH equations. However, due to the inherent complexity of solving matrix different equations, it constitutes an obstacle to give the general solutions of A. So that, in this paper, we only consider three special cases: A being an antisymmetric constant matrix, $A = f(t)I + D$

with $D^T = -D$, and $A = \left(\frac{\dot{\alpha}_i}{\alpha_i}\right)_{N\times N}$. Therefore, it would be of interest to consider how to find the more general reductions for (2.5) leads to the more general solutions of multi-component CH equation available.

2) It is noted that the solutions $\boldsymbol{u}$ we obtained take the linear form with respect to the spatial variables $\boldsymbol{x}$. Thus, it is natural to inquire whether the nonlinear form type solutions $\boldsymbol{u}$ exist, for example, can we find $\boldsymbol{u}$ be of quadratic form? Since we know that when $\boldsymbol{u}$ is the linear form, p takes the relevant quadratic form. Hence, it is worthy of investigating when $\boldsymbol{u}$ is a quadratic form, whether the suitable function p exists. If the answer is positive, how can we use the solutions to explain any related physical phenomena?

3) We find that the continuity function of the multi-component CH equations takes the same form as that of Euler equations, Navier-Stokes equations and Euler-Poisson equations, which are important fluid models. Therefore, we believe that the method proposed in this paper provides a possible way to solve these fluid equations.

4) It is known the multi-component CH system is the higher dimensional variation of classical CH equation. The important features of the latter equation is its complete integrablity and admittance the peakon solutions. Therefore, it would be worthy of considering whether the form equations have the same features.

Based on the importance and applications of the multi-component CH system, all these interesting problems mentioned above will be deeply investigated in our future work.

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