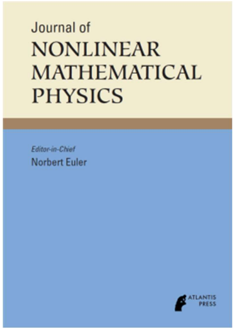

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Delta shock waves in conservation laws with impulsive moving source: some results obtained by multiplying distributions

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The present paper concerns the study of a Riemann problem for the conservation law $u_t + [\phi(u)]_x = k\delta(x - vt)$ where x, t, k, v and $u = u(x,t)$ are real numbers. We consider ϕ an entire function taking real values on the real axis and δ stands for the Dirac measure. Within a convenient space of distributions we will explicitly see the possible emergence of waves with the shape of shock waves, delta waves and delta shock waves. For this purpose, we define a rigorous concept of a solution which extends both the classical solution concept and a weak solution concept. All this framework is developed in the setting of a distributional product that is not constructed by approximation. We include the main ideas of this product for the reader's convenience. Recall that delta shock waves are relevant physical phenomena which may be interpreted as processes of concentration of mass or even as processes of formation of galaxies in the universe.

Keywords: Products of distributions; conservation laws with impulsive source; Riemann problems; traveling shock waves; Delta waves; Delta shock waves.

2010 Mathematics Subject Classification: 46F10, 35D99, 35L67

1. Introduction and contents

Let us consider the Riemann problem

$$u_t + [\phi(u)]_x = k\delta(x - vt), \tag{1.1}$$

$$u(x,0) = u_1 + (u_2 - u_1)H(x), \tag{1.2}$$

where $x \in \mathbb{R}$ is the space variable, $t \in \mathbb{R}$ is the time variable, $u(x,t) \in \mathbb{R}$ is the unknown state variable, k, v, u_1, $u_2 \in \mathbb{R}$ are given constants, H is the Heaviside function and δ stands for the Dirac measure supported at the origin. We suppose ϕ an entire function taking real values on the real axis. The goal is to evaluate the evolution of the profile (1.2) within the space W of distributions u defined by

$$u(x,t) = f(t) + g(t)H(x - vt) + h(t)\delta(x - vt),$$

where f, g, $h : \mathbb{R} \to \mathbb{R}$ are C^1-functions.

The main result (under certain conditions) is the solution,

$$u(x,t) = u_1 + (u_2 - u_1)H(x - vt) + At\delta(x - vt), \tag{1.3}$$

where

$$A = k + v(u_2 - u_1) - [\phi(u_2) - \phi(u_1)]. \tag{1.4}$$

This solution, when it exists, is unique in W. Thus, the emergence of traveling shock waves ($A = 0$ and $u_1 \neq u_2$), delta waves ($A \neq 0$ and $u_1 = u_2$) and delta shock waves ($A \neq 0$ and $u_1 \neq u_2$) becomes possible.

An interesting result can be obtained as a particular case, supposing that there exists a solution for (1.1), (1.2), in W, with $k = 0$: in this case (as we will see) we have necessarily $A = 0$. This means that, in W, the usual conservation law

$$u_t + [\phi(u)]_x = 0, \tag{1.5}$$

subjected to the initial condition (1.2) cannot develop neither delta waves nor delta shock waves; the impulse amplification (which concerns the term $At\delta(x - vt)$) during the time evolution is impossible in this setting. This is a characteristic of the equation (1.5) that does not happen for systems of the same type; remember that Dirac measures amplified along time appear in the solution of Riemann problems for systems of conservation laws with source zero (see [4–6, 10, 11, 14]). Also recall that this amplification is a relevant physical phenomenon which may be interpreted as a process of concentration of mass or even as a process of formation of galaxies in the universe [16]. Thus, and for the problem (1.1), (1.2), the concentration of matter in the front shock during the time evolution is possible only when $k \neq 0$. We will also prove that, the existence of an impulsive source does not cause necessarily the appearing of an impulse in the solution, that is, there exist cases, with $k \neq 0$, where the solution contains only functions (without any Dirac measure present). This will be easily shown in an example concerning Burgers equation with an impulsive moving source.

Often, in this kind of problems, the distributional solutions appear as weak limits. It may even happen that those weak limits cannot be substituted into equations or systems owing to the well known difficulties of multiplying distributions. On top of that, limit processes involving sequences of continuous functions may not yield to mathematically consistent solutions (see [13], Section II). Our method overcome those difficulties, as we will explain.

For the equation (1.1), we will adopt a concept of solution defined within the setting of a distributional product. This concept is a consistent extension of the classical solution concept and, in a sense explained at the end of Section 5, can also be seen as a new type of weak solution.

In our framework, the product of two distributions is a distribution that depends on the choice of a certain function α encoding the indeterminacy inherent to such products. This indeterminacy generally is not avoidable and in many cases it also has a physical meaning; concerning this point let us mention [1–3, 8]. Thus, the solutions of differential equations containing such products may depend (or not) of α. We call such solutions α-solutions. Thus, when the solutions depend on α, the future behavior of the system cannot be fully predicted. This fact might be due to physical features omitted in the formulation of the model with the goal of simplifying it. It is worthwhile to stress that, for the present problem (1.1), (1.2), the solutions, when they exist, are independent of α!

The concept of α-solution has shown to be a convenient tool in the study of singular solutions of nonlinear PDEs and systems (see for instance, [8–12, 14, 15]). Also recently, Chun Shen and Meina Sun [17] used our framework to study the zero-pressure gas dynamic system with the Coulomb like friction term; they advocate that "it is more easy to use the method of α-solutions to discover discontinuous solutions involving Dirac-delta measures in many branches of engineering, physics and mechanics".

Let us now summarize the present paper's contents. In Section 2, we present the main ideas of our method for multiplying distributions. In Section 3, we define powers of certain distributions. In Section 4, we define the composition of an entire function with a distribution. In Section 5, we define the concept of α-solution for the equation (1.1). In Section 6, we present the main result, that is, all possible solutions of the Riemann problem (1.1), (1.2) that belong to W. Easy examples are given in Section 7.

2. The multiplication of distributions

Let $\mathscr{C}^\infty$ be the space of indefinitely differentiable real or complex-valued functions defined on $\mathbb{R}^N$, $N \in \{1, 2, 3, \ldots\}$, and $\mathscr{D}$ the subspace of C^∞ consisting of those functions with compact support. Let $\mathscr{D}'$ be the space of Schwartz distributions and $L(\mathscr{D})$ the space of continuous linear maps $\phi : \mathscr{D} \to \mathscr{D}$, where we suppose $\mathscr{D}$ endowed with the usual topology. We will sketch the main ideas of our distributional product (the reader can look at (2.4), (2.7), and (2.9) as definitions, if he prefers to skip this presentation). For proofs and other details concerning this product see [7].

The construction of a general product in $\mathscr{D}'$

First, we define a product $T\phi \in \mathscr{D}'$ for $T \in \mathscr{D}'$ and $\phi \in L(\mathscr{D})$ by

$$\langle T\phi, \xi\rangle = \langle T, \phi(\xi)\rangle,$$

for all $\xi \in \mathscr{D}$; this makes $\mathscr{D}'$ a right $L(\mathscr{D})$-module. Next, we define an epimorphism $\tilde{\zeta} : L(\mathscr{D}) \to \mathscr{D}'$, where the image of ϕ is the distribution $\tilde{\zeta}(\phi)$ given by

$$\langle \tilde{\zeta}(\phi), \xi\rangle = \int \phi(\xi),$$

for all $\xi \in \mathscr{D}$ (in the present paper, all integrals are extended all over $\mathbb{R}^N$); given $S \in \mathscr{D}'$, we say that ϕ is a representative operator of S if $\tilde{\zeta}(\phi) = S$. For instance, if $\beta \in \mathscr{C}^\infty$ is seen as a distribution, the operator $\phi_\beta \in L(\mathscr{D})$ defined by $\phi_\beta(\xi) = \beta\xi$, for all $\xi \in \mathscr{D}$, is a representative operator of β because, for all $\xi \in \mathscr{D}$, we have

$$\langle \tilde{\zeta}(\phi_\beta), \xi\rangle = \int \phi_\beta(\xi) = \int \beta\xi = \langle \beta, \xi\rangle.$$

For this reason $\tilde{\zeta}(\phi_\beta) = \beta$. If $T \in \mathscr{D}'$, we also have

$$\langle T\phi_\beta, \xi\rangle = \langle T, \phi_\beta(\xi)\rangle = \langle T, \beta\xi\rangle = \langle T\beta, \xi\rangle,$$

for all $\xi \in \mathscr{D}$. Hence,

$$T\beta = T\phi_\beta.$$

Thus, given $T,\ S \in \mathscr{D}'$, we are tempted to define a natural product by setting $TS := T\phi$, where $\phi \in L(\mathscr{D})$ is a representative operator of S, i.e., ϕ is such that $\tilde{\zeta}(\phi) = S$. Unfortunately, this product is not well defined, because TS depends on the representative $\phi \in L(\mathscr{D})$ of $S \in \mathscr{D}'$.

This difficulty can be overcome, if we fix $\alpha \in \mathscr{D}$ with $\int \alpha = 1$ and define $s_\alpha : L(\mathscr{D}) \to L(\mathscr{D})$ by

$$[(s_\alpha\phi)(\xi)](y) = \int \phi[(\tau_y\check{\alpha})\xi], \tag{2.1}$$

for all $\xi \in \mathscr{D}$ and all $y \in \mathbb{R}^N$, where $\tau_y\check{\alpha}$ is given by $(\tau_y\check{\alpha})(x) = \check{\alpha}(x-y) = \alpha(y-x)$ for all $x \in \mathbb{R}^N$. It can be proved that for each $\alpha \in \mathscr{D}$ with $\int \alpha = 1$, $s_\alpha(\phi) \in L(\mathscr{D})$, s_α is linear, $s_\alpha \circ s_\alpha = s_\alpha$ (s_α is a projector of $L(\mathscr{D})$), $\ker s_\alpha = \ker \tilde{\zeta}$, and $\tilde{\zeta} \circ s_\alpha = \tilde{\zeta}$.

Now, for each $\alpha \in \mathscr{D}$, we can define a general α-product $\underset{\alpha}{\odot}$ of $T \in \mathscr{D}'$ with $S \in \mathscr{D}'$ by setting

$$T \underset{\alpha}{\odot} S := T(s_\alpha \phi), \tag{2.2}$$

where $\phi \in L(\mathscr{D})$ is a representative operator of $S \in \mathscr{D}'$. This α-product is independent of the representative ϕ of S, because if ϕ, ψ are such that $\tilde{\zeta}(\phi) = \tilde{\zeta}(\psi) = S$, then $\phi - \psi \in \ker \tilde{\zeta} = \ker s_\alpha$. Hence,

$$T(s_\alpha \phi) - T(s_\alpha \psi) = T[s_\alpha(\phi - \psi)] = 0.$$

Since ϕ in (2.2) satisfies $\tilde{\zeta}(\phi) = S$, we have $\int \phi(\xi) = \langle S, \xi \rangle$ for all $\xi \in \mathscr{D}$, and by (2.1)

$$[(s_\alpha \phi)(\xi)](y) = \langle S, (\tau_y \check{\alpha})\xi \rangle = \langle S\xi, \tau_y \check{\alpha} \rangle = (S\xi * \alpha)(y),$$

for all $y \in \mathbb{R}^N$, which means that $(s_\alpha \phi)(\xi) = S\xi * \alpha$. Therefore, for all $\xi \in \mathscr{D}$,

$$\begin{aligned} \langle T \underset{\alpha}{\odot} S, \xi \rangle &= \langle T(s_\alpha \phi), \xi \rangle = \langle T, (s_\alpha \phi)(\xi) \rangle = \langle T, S\xi * \alpha \rangle \\ &= [T * (S\xi * \alpha)\check{\ }](0) = [(S\xi)\check{\ } * (T * \check{\alpha})](0) = \langle (T * \check{\alpha})S, \xi \rangle, \end{aligned}$$

and we obtain an easier formula for the general product (2.2),

$$T \underset{\alpha}{\odot} S = (T * \check{\alpha})S. \tag{2.3}$$

In general, this α-product is neither commutative nor associative but it is bilinear and satisfies the Leibniz rule written in the form

$$D_k(T \underset{\alpha}{\odot} S) = (D_k T) \underset{\alpha}{\odot} S + T \underset{\alpha}{\odot} (D_k S),$$

where D_k is the usual k-partial derivative operator in distributional sense ($k = 1, 2, \ldots, N$).

Recall that the usual Schwartz products of distributions are not associative and the commutative property is a convention inherent to the definition of such products (see the classical monograph of Schwartz [18] pp. 117, 118, and 121, where these products are defined). Unfortunately, the α-product (2.3), in general, is not consistent with the classical Schwartz products of distributions with functions.

How to get a product consistent with the Schwartz product of a distribution with a C$^\infty$-function?

In order to obtain consistency with the usual product of a distribution with a C^∞-function, we are going to introduce some definitions and single out a certain subspace H_α of $L(\mathscr{D})$.

An operator $\phi \in L(\mathscr{D})$ is said to vanish on an open set $\Omega \subset \mathbb{R}^N$, if and only if $\phi(\xi) = 0$ for all $\xi \in \mathscr{D}$ with support contained in Ω. The support of an operator $\phi \in L(\mathscr{D})$ will be defined as the complement of the largest open set in which ϕ vanishes.

Let $\mathscr{N}$ be the set of operators $\phi \in L(\mathscr{D})$ whose support has Lebesgue measure zero, and $\rho(C^\infty)$ the set of operators $\phi \in L(\mathscr{D})$ defined by $\phi(\xi) = \beta\xi$ for all $\xi \in \mathscr{D}$, with $\beta \in C^\infty$. For each $\alpha \in \mathscr{D}$, with $\int \alpha = 1$, let us consider the space $H_\alpha = \rho(C^\infty) \oplus s_\alpha(\mathscr{N}) \subset L(\mathscr{D})$. It can be proved that $\zeta_\alpha := \tilde{\zeta}|_{H_\alpha} : H_\alpha \to C^\infty \oplus \mathscr{D}'_\mu$ is an isomorphism ($\mathscr{D}'_\mu$ stands for the space of distributions whose support has Lebesgue measure zero). Therefore, if $T \in \mathscr{D}'$ and $S = \beta + f \in C^\infty \oplus \mathscr{D}'_\mu$, a new α-product, $\dot{\alpha}$, can be defined by $T_{\dot{\alpha}}S := T\phi_\alpha$, where for each α, $\phi_\alpha = \zeta_\alpha^{-1}(S) \in H_\alpha$. Hence,

$$\begin{aligned} T_{\dot{\alpha}}S &= T\zeta_\alpha^{-1}(S) = T[\zeta_\alpha^{-1}(\beta + f)] \\ &= T[\zeta_\alpha^{-1}(\beta) + \zeta_\alpha^{-1}(f)] = T\beta + T \underset{\alpha}{\odot} f = T\beta + (T * \check{\alpha})f, \end{aligned}$$

and putting α instead of $\check{\alpha}$ (to simplify), we get

$$T_{\dot{\alpha}}S = T\beta + (T * \alpha)f. \tag{2.4}$$

Thus, the referred consistency is obtained when the C^∞-function is placed at the right-hand side: if $S \in C^\infty$, then $f = 0$, $S = \beta$, and $T_{\dot{\alpha}}S = T\beta$.

How to obtain the consistency to all Schwartz products of D'^p-distributions with C^p-functions?

The α-product (2.4) can be easily extended for $T \in \mathscr{D}'^p$ and $S = \beta + f \in C^p \oplus \mathscr{D}'_\mu$, where $p \in \{0, 1, 2, \ldots, \infty\}$, $\mathscr{D}'^p$ is the space of distributions of order $\leq p$ in the sense of Schwartz ($\mathscr{D}'^\infty$ means $\mathscr{D}'$), $T\beta$ is the Schwartz product of a $\mathscr{D}'^p$-distribution with a C^p-function, and $(T * \alpha)f$ is the usual product of a C^∞-function with a distribution. This extension is clearly consistent with all Schwartz products of $\mathscr{D}'^p$-distributions with C^p-functions, if the C^p-functions are placed at the right-hand side. It also keeps the bilinearity and satisfies the Leibniz rule written in the form

$$D_k(T_{\dot{\alpha}}S) = (D_kT)_{\dot{\alpha}}S + T_{\dot{\alpha}}(D_kS),$$

clearly under certain natural conditions; for $T \in \mathscr{D}'^p$, we must suppose $S \in C^{p+1} \oplus \mathscr{D}'_\mu$. Moreover, these products are invariant by translations, that is,

$$\tau_a(T_{\dot{\alpha}}S) = (\tau_aT)_{\dot{\alpha}}(\tau_aS),$$

where τ_a stands for the usual translation operator in distributional sense. These products are also invariant for the action of any group of linear transformations $h : \mathbb{R}^N \to \mathbb{R}^N$ with $|\det h| = 1$, that leave α invariant.

Thus, for each $\alpha \in \mathscr{D}$ with $\int \alpha = 1$, formula (2.4) allows us to evaluate the product of $T \in \mathscr{D}'^p$ with $S \in C^p \oplus \mathscr{D}'_\mu$; therefore, we have obtained a family of products, one for each α.

From now on, we always consider the dimension $N = 1$. For instance, if β is a continuous function we have for each α by applying (2.4),

$$\begin{aligned} &\delta_{\dot{\alpha}}\beta = \delta_{\dot{\alpha}}(\beta + 0) = \delta\beta + (\delta * \alpha)0 = \beta(0)\delta, \\ &\beta_{\dot{\alpha}}\delta = \beta_{\dot{\alpha}}(0 + \delta) = \beta 0 + (\beta * \alpha)\delta = [(\beta * \alpha)(0)]\delta, \\ &\delta_{\dot{\alpha}}\delta = \delta_{\dot{\alpha}}(0 + \delta) = \delta 0 + (\delta * \alpha)\delta = \alpha\delta = \alpha(0)\delta, \\ &\delta_{\dot{\alpha}}(D\delta) = (\delta * \alpha)D\delta = \alpha D\delta = \alpha(0)D\delta - \alpha'(0)\delta, \\ &(D\delta)_{\dot{\alpha}}\delta = (D\delta * \alpha)\delta = \alpha'\delta = \alpha'(0)\delta, \end{aligned} \tag{2.5}$$

$$H_{\dot{\alpha}}\delta = (H * \alpha)\delta = \left[\int_{-\infty}^{+\infty} \alpha(-\tau)H(\tau)\,d\tau\right]\delta = \left(\int_{-\infty}^{0} \alpha\right)\delta. \tag{2.6}$$

For each α, the support of the α-product (2.4) satisfies $\operatorname{supp}(T_{\dot{\alpha}}S) \subset \operatorname{supp} S$, as for usual functions, but it may happen that $\operatorname{supp}(T_{\dot{\alpha}}S) \not\subset \operatorname{supp} T$. For instance, if $a, b \in \mathbb{R}$, from (2.4) we have,

$$(\tau_a\delta)_{\dot{\alpha}}(\tau_b\delta) = [(\tau_a\delta)*\alpha](\tau_b\delta) = (\tau_a\alpha)(\tau_b\delta) = \alpha(b-a)(\tau_b\delta).$$

Other products we need in the present paper

It is also possible to multiply many other distributions preserving the consistency with all Schwartz products of distributions with functions. For instance, using the Leibniz formula to extend the α-products, it is possible to write

$$T_{\dot{\alpha}}S = Tw + (T*\alpha)f, \tag{2.7}$$

with $T \in \mathscr{D}'^{-1}$ and $S = w + f \in L^1_{\mathrm{loc}} \oplus \mathscr{D}'_\mu$, where $\mathscr{D}'^{-1}$ stands for the space of distributions $T \in \mathscr{D}'$ such that $DT \in \mathscr{D}'^0$ and Tw is the usual pointwise product of $T \in \mathscr{D}'^{-1}$ with $w \in L^1_{\mathrm{loc}}$. Recall that, locally, T can be read as a function of bounded variation (see [9], Sec. 2 for details). For instance, since $H \in \mathscr{D}'^{-1}$ and $H = H + 0 \in L^1_{\mathrm{loc}} \oplus \mathscr{D}'_\mu$, we have

$$H_{\dot{\alpha}}H = HH + (H*\alpha)0 = H. \tag{2.8}$$

More generally, if $T \in \mathscr{D}'^{-1}$ and $S \in L^1_{\mathrm{loc}}$, then $T_{\dot{\alpha}}S = TS$ because by (2.7) we can write

$$T_{\dot{\alpha}}S = T_{\dot{\alpha}}(S+0) = TS + (T*\alpha)0 = TS.$$

Thus, in distributional sense, the α-products of functions that, locally, are of bounded variation coincide with the usual pointwise product of these functions considered as a distribution. We stress that in (2.4) or (2.7) the convolution $T*\alpha$ is not to be understood as an approximation of T. Those formulas are exact.

Another useful extension is given by the formula

$$T_{\dot{\alpha}}S = D(Y_{\dot{\alpha}}S) - Y_{\dot{\alpha}}(DS), \tag{2.9}$$

for $T \in \mathscr{D}'^0 \cap \mathscr{D}'_\mu$ and $S, DS \in L^1_{\mathrm{loc}} \oplus \mathscr{D}'_c$, where $\mathscr{D}'_c \subset \mathscr{D}'_\mu$ is the space of distributions whose support is at most countable, and $Y \in \mathscr{D}'^{-1}$ is such that $DY = T$ (the products $Y_{\dot{\alpha}}S$ and $Y_{\dot{\alpha}}(DS)$ are supposed to be computed by (2.4) or (2.7)). The value of $T_{\dot{\alpha}}S$ given by (2.9) is independent of the choice of $Y \in \mathscr{D}'^{-1}$ such that $DY = T$ (see [9] p. 1004 for the proof). For instance, by (2.6) and (2.9), we have for any α,

$$\delta_{\dot{\alpha}}H = D(H_{\dot{\alpha}}H) - H_{\dot{\alpha}}(DH) = DH - H_{\dot{\alpha}}\delta = \delta - \left(\int_{-\infty}^{0}\alpha\right)\delta = \left(\int_{0}^{+\infty}\alpha\right)\delta, \tag{2.10}$$

so that $H_{\dot{\alpha}}\delta + \delta_{\dot{\alpha}}H = \delta$ for any α. The products (2.4), (2.7), and (2.9) are compatible, that is, if an α-product can be computed by two of them, the result is the same.

3. Powers of distributions

Let $M \subset \mathscr{D}'$ be a set of distributions such that, if $T_1, T_2 \in M$, then $T_1{}_{\dot{\alpha}}T_2$ is well defined and $T_1{}_{\dot{\alpha}}T_2 \in M$. For each $T \in M$ we define the α-power T^n_α by the recurrence relation

$$T^n_\alpha = (T^{n-1}_\alpha)_{\dot{\alpha}}T \text{ for } n \geq 1, \text{ with } T^0_\alpha = 1 \text{ for } T \neq 0; \tag{3.1}$$

naturally, if $0 \in M$, $0^n_\alpha = 0$ for all $n \geq 1$.

Since our distributional products are consistent with the Schwartz products of distributions with functions, when the functions are placed at the right-hand side, we have $\beta^n_\alpha = \beta^n$ for all $\beta \in C^0 \cap M$. Thus, this definition is consistent with the usual definition of powers of C^0-functions. Moreover, if M is such that $\tau_a T \in M$ for all $T \in M$ and all $a \in \mathbb{R}^N$, then we also have $(\tau_a T)^n_\alpha = \tau_a(T^n_\alpha)$.

Taking, for instance, $M = C^p \oplus (\mathscr{D}'^p \cap \mathscr{D}'_\mu)$ and supposing $T_1, T_2 \in M$, we have $T_1 = \beta_1 + f_1$, $T_2 = \beta_2 + f_2$ and by (2.4), we can write

$$\begin{aligned} T_1 \dot{\alpha} T_2 &= T_1\beta_2 + (T_1 * \alpha) f_2 = (\beta_1 + f_1)\beta_2 + [(\beta_1 + f_1) * \alpha] f_2 \\ &= \beta_1\beta_2 + f_1\beta_2 + [(\beta_1 + f_1) * \alpha] f_2 \in M. \end{aligned}$$

Therefore, we can define α-powers T^n_α of distributions $T \in C^p \oplus (\mathscr{D}'^p \cap \mathscr{D}'_\mu)$. For instance, if $m \in \mathbb{C}\backslash\{0\}$, we have $(m\delta)^0_\alpha = 1$, $(m\delta)^1_\alpha = m\delta$, and for $n \geq 2$, $(m\delta)^n_\alpha = m^n[\alpha(0)]^{n-1}\delta$, as can be easily seen by induction applying (2.5).

Setting $M = \mathscr{D}'^{-1}$ and supposing $T_1, T_2 \in \mathscr{D}'^{-1}$, we have $T_1 \dot{\alpha} T_2 \in \mathscr{D}'^{-1}$. Thus, we can also define α-powers T^n_α of distributions $T \in \mathscr{D}'^{-1}$ by the recurrence relation (3.1) and clearly we get,

$$T^n_\alpha = T^n,$$

that is, in distributional sense the α-powers of functions that, locally, are of bounded variation, coincide with the usual powers of these functions when considered as distributions.

In the sequel we will write, in all cases, T^n instead of T^n_α, supposing α fixed. For instance, if $m \in \mathbb{R}$ we will write $(m\delta)^1 = m\delta$ and for $n \geq 2$, $(m\delta)^n = m^n[\alpha(0)]^{n-1}\delta$.

Taking $M = \{a + (b-a)H + m\delta : a, b, m \in \mathbb{R}\}$ we have:

Theorem 3.1. *Given α, let us suppose $a, b, m \in \mathbb{R}$, $p = \int_{-\infty}^{0} \alpha$, $q = \int_{0}^{+\infty} \alpha$ and $\lambda = \alpha(0)m + (b-a)q$. Then,*

$$\left[a + (b-a)H + m\delta\right]^n = a^n + (b^n - a^n)H + m[P_{n-1}(a+\lambda)]\delta, \tag{3.2}$$

where P_{n-1} is the polynomial defined by the recurrence relation $P_0(s) = 1$ and for $n \geq 1$, $P_n(s) = sP_{n-1}(s) + pb^n + qa^n$.

For a proof, see [12], p. 335.

4. Composition of entire functions with distributions

Let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire function. Then we have,

$$\phi(s) = a_0 + a_1 s + a_2 s^2 + \cdots \tag{4.1}$$

for the sequence $a_n = \frac{\phi^{(n)}(0)}{n!}$ of complex numbers and all $s \in \mathbb{C}$. If $T \in M$ we define the composition $\phi \circ T$ by formula

$$\phi \circ T = a_0 + a_1 T + a_2 T^2 + \cdots \tag{4.2}$$

whenever this series converges in $\mathscr{D}'$. Clearly, this definition is consistent with the usual meaning of $\phi \circ T$, if $T \in M$ is a function. Moreover, if M is such that $\tau_a T \in M$ for all $T \in M$ and all $a \in \mathbb{R}$, we have $\tau_a(\phi\circ T)=\phi\circ(\tau_a T)$, if $\phi\circ T$ or $\phi\circ(\tau_a T)$ are well defined. Remember that, in general, $\phi\circ T$ depends on α. For instance, taking $M=\{m\delta : m\in\mathbb{C}\}$ it is easy to see that

$$\phi\circ(m\delta)=\begin{cases}\phi(0)+\phi'(0)m\delta & \text{if } \alpha(0)=0,\\ \phi(0)+\dfrac{\phi(m\alpha(0))-\phi(0)}{\alpha(0)}\delta & \text{if } \alpha(0)\neq 0.\end{cases}$$

We need the following statements:

Lemma 4.1. *Let $a,b\in\mathbb{C}$ and let p, q and the sequence P_n be defined as in theorem* 3.1. *Still suppose ϕ an entire function defined by* (4.1). *Then, the function $W_\phi:\mathbb{C}\to\mathbb{C}$ defined by $W_\phi(s)=\sum_{n=1}^{\infty}a_nP_{n-1}(s)$ is well defined and satisfies the following conditions:*

(a) *W_ϕ is an entire function;*

(b) $W_\phi(pa+qb)=\begin{cases}\dfrac{\phi(b)-\phi(a)}{b-a} & \text{if } b\neq a,\\ \phi'(a) & \text{if } b=a;\end{cases}$

(c) $W_\phi=0$ *if and only if* $\phi'=0$.

Theorem 4.1. *Given α, let $a,b,m\in\mathbb{C}$, $q=\int_0^{+\infty}\alpha$ and $\lambda=\alpha(0)m+q(b-a)$. Suppose also that $T=a+(b-a)H+m\delta$ and ϕ is an entire function defined by* (4.1). *Then,*

$$\phi\circ T=\phi(a)+[\phi(b)-\phi(a)]H+mW_\phi(a+\lambda)\delta,$$

where W_ϕ is defined in lemma 4.1.

For the proofs see [12], Section 4.

5. The α-solution concept

Let I be an interval of $\mathbb{R}$ with more that one point, and let $\mathscr{F}(I)$ be the space of continuously differentiable maps $\tilde{u}:I\to\mathscr{D}'$ in the sense of the usual topology of $\mathscr{D}'$. For $t\in I$, the notation $[\tilde{u}(t)](x)$ is sometimes used for emphasizing that the distribution $\tilde{u}(t)$ acts on functions $\xi\in\mathscr{D}$ depending on x.

Let $\Sigma(I)$ be the space of functions $u:\mathbb{R}\times I\to\mathbb{R}$ such that:

(a) for each $t\in I$, $u(x,t)\in L^1_{\text{loc}}(\mathbb{R})$;
(b) $\tilde{u}:I\to\mathscr{D}'$, defined by $[\tilde{u}(t)](x)=u(x,t)$ is in $\mathscr{F}(I)$.

The natural injection $u\mapsto\tilde{u}$ from $\Sigma(I)$ into $\mathscr{F}(I)$ identifies any function in $\Sigma(I)$ with a certain map in $\mathscr{F}(I)$. Since $C^1(\mathbb{R}\times I)\subset\Sigma(I)$, we can write the inclusions

$$C^1(\mathbb{R}\times I)\subset\Sigma(I)\subset\mathscr{F}(I).$$

Thus, identifying u with $\tilde{u}$ the equation (1.1) can be read as follows:

$$\frac{d\tilde{u}}{dt}(t)+D[\phi\circ\tilde{u}(t)]=k\tau_{vt}\delta, \tag{5.1}$$

Definition 5.1. Given α, the map $\tilde{u}\in\mathscr{F}(I)$ will be called an α-solution of the equation (5.1) on I, if $\phi\circ\tilde{u}(t)$ is well defined, and if this equation is satisfied for all $t\in I$.

This definition sees equation (1.1) as an evolution equation and we have the following results:

Theorem 5.1. *If $u(x,t)$ is a classical solution of* (1.1) *on $\mathbb{R}\times I$ then, for any α, the map $\tilde{u}\in\mathscr{F}(I)$ defined by $[\tilde{u}(t)](x)=u(x,t)$ is an α-solution of* (5.1) *on I.*

Note that, by a classical solution of (1.1) on $\mathbb{R}\times I$, we mean a C^1-function $u(x,t)$ that satisfies (1.1) on $\mathbb{R}\times I$.

Theorem 5.2. *If $u\colon \mathbb{R}\times I\to\mathbb{R}$ is a C^1-functions and, for a certain α, the map $\tilde{u}\in\mathscr{F}(I)$ defined by $[\tilde{u}(t)](x)=u(x,t)$ is an α-solution of* (5.1) *on I, then $u(x,t)$ is a classical solution of* (1.1) *on $\mathbb{R}\times I$.*

For the proof, it is enough to observe that any C^1-functions $u(x,t)$ can be read as continuously differentiable function $\tilde{u}\in\mathscr{F}(I)$ defined by $[\tilde{u}(t)](x)=u(x,t)$ and to use the consistency of the α-products with the classical Schwartz products of distributions with functions.

Definition 5.2. Given α, any α-solution $\tilde{u}$ of (5.1) on I, will be called an α-solution of the equation (1.1) on I.

As a consequence, an α-solution $\tilde{u}$ in this sense, read as a usual distributional u, affords a general consistent extension of the concept of a classical solution for the equation (1.1). Thus, and for short, we also call the distribution u an α-solution of (1.1).

As it is well known, a weak solution of a differential equation is a function for which the derivatives may not exist but satisfies the equation in some precise sense.

One of the most important definitions is based in the classical theory of distributions. In this theory, the study of differential equations is, of course, restricted to linear equations, owing to the well known difficulties of multiplying distributions. The classical setting usually considers linear partial differential equations with C^∞-coefficients, and a weak solution is generally defined as satisfying the equation in the sense of distributions.

Linear partial differential evolution equations in the unknown u can be re-interpreted as evolution equations with α-products, in the unknown $\tilde{u}(t)$, if the (re-interpreted) C^∞-coefficients are placed at the right-hand side of $\tilde{u}(t)$ and its derivatives. Actually, in this case, our α-products are consistent with the products of distributions with C^∞-functions. Thus, if $u\in\Sigma(I)$ is a weak solution, then, for any α, the corresponding map $\tilde{u}\in\mathscr{F}(I)$ is an α-solution. Conversely, if $u\in\Sigma(I)$ and for a certain α the corresponding map $\tilde{u}\in\mathscr{F}(I)$ is an α-solution, then u is a weak solution. In this sense, the α-solution concept can be identified with the weak solution concept. Meanwhile, an advantage arises: the coefficients of such equations can now be considered as distributions, if the α-products involved are well defined and the solutions are considered as elements of $\mathscr{F}(I)$.

Thus, in the framework of evolution equations, the α-solution concept is an extension of the classical solution concept, and may be also considered as a new type of weak solution provided by distribution theory in the nonlinear setting.

6. The Riemann problem (1.1), (1.2)

Let us consider the equation (1.1) with $(x,t)\in\mathbb{R}\times\mathbb{R}$ (we could also have considered $(x,t)\in\mathbb{R}\times[0,+\infty[$), ϕ an entire function taking real values on the real axis, and the unknown u subjected to the initial condition (1.2) with $u_1,u_2\in\mathbb{R}$. When we read this problem in $\mathscr{F}(\mathbb{R})$ having in mind the identification $u\mapsto\tilde{u}$, we must replace the equation (1.1) by the equation (5.1) and the initial condition (1.2) by the following one

$$\tilde{u}(0) = u_1 + (u_2 - u_1)H. \tag{6.1}$$

Theorem 6.1 concerns the α-solutions $\tilde{u}$ of the problem (5.1), (6.1) in the time interval $I = \mathbb{R}$, which belong to a convenient space $\tilde{W} \subset \mathscr{F}(\mathbb{R})$, defined the following way: $\tilde{u} \in \tilde{W}$ if and only if

$$\tilde{u}(t) = f(t) + g(t)\tau_{vt}H + h(t)\tau_{vt}\delta, \tag{6.2}$$

for certain C^1-functions $f, g, h\colon \mathbb{R} \to \mathbb{R}$ and all $t \in \mathbb{R}$.

Theorem 6.1. *Let $A = k + v(u_2 - u_1) - [\phi(u_2) - \phi(u_1)]$. Then, given α, the problem* (5.1), (6.1) *has α-solution $\tilde{u} \in \tilde{W}$ if and only if one of the following four conditions is satisfied:*

(I) $A = 0$*;*
(II) $A \neq 0,\ \phi' = 0$ *and* $v = 0$*;*
(III) $A \neq 0,\ \phi' \neq 0,\ u_1 = u_2$ *and* $v = \phi'(u_1)$*;*
(IV) $A \neq 0,\ \phi' \neq 0,\ u_1 \neq u_2$ *and* $v = \dfrac{\phi(u_2) - \phi(u_1)}{u_2 - u_1}$.

In anyone of these four cases the α-solution is independent on α, is unique in $\tilde{W}$, and is given by

$$\tilde{u}(t) = u_1 + (u_2 - u_1)\tau_{vt}H + At\tau_{vt}\delta. \tag{6.3}$$

Proof. Let us suppose $\tilde{u} \in \tilde{W}$. Then, from (6.2), we have, for each t,

$$\begin{aligned}\frac{d\tilde{u}}{dt}(t) &= f'(t) + g'(t)\tau_{vt}H - vg(t)\tau_{vt}\delta + h'(t)\tau_{vt}\delta - vh(t)\tau_{vt}D\delta\\ &= f'(t) + g'(t)\tau_{vt}H + [h'(t) - vg(t)]\tau_{vt}\delta - vh(t)\tau_{vt}D\delta.\end{aligned} \tag{6.4}$$

On the other hand, from theorem 4.1 with $a = f(t)$, $b = f(t) + g(t)$ and $m = h(t)$, we have, for each t,

$$\phi \circ \tilde{u}(t) = \phi \circ [f(t) + g(t)\tau_{vt}H + h(t)\tau_{vt}\delta]$$

$$= \tau_{vt}\{\phi \circ [f(t) + g(t)H + h(t)\delta]$$

$$= \tau_{vt}\{\phi[f(t)] + [\phi(f(t) + g(t)) - \phi(f(t)]H + h(t)W_\phi[f(t) + \alpha(0)h(t) + qg(t)]\delta\}$$

$$= \phi[f(t)] + [\phi(f(t) + g(t)) - \phi(f(t)]\tau_{vt}H + h(t)W_\phi[f(t) + \alpha(0)h(t) + qg(t)]\tau_{vt}\delta,$$

$$D[\phi \circ \tilde{u}(t)] = [\phi(f(t) + g(t)) - \phi(f(t)]\tau_{vt}\delta + h(t)W_\phi[f(t) + \alpha(0)h(t) + qg(t)]\tau_{vt}D\delta. \tag{6.5}$$

Thus, using (6.4) and (6.5), (5.1) turns out to be

$$f'(t) + g'(t)\tau_{vt}H + [h'(t) - vg(t) + \phi(f(t) + g(t)) - \phi(f(t)) - k]\tau_{vt}\delta$$

$$+h(t)\{W_\phi[f(t) + \alpha(0)h(t) + qg(t)] - v\}\tau_{vt}D\delta = 0.$$

Therefore, (6.2) is an α-solution of (5.1) if and only if, for each t, the following four equations are satisfied

$$f'(t) = 0, \quad g'(t) = 0, \tag{6.6}$$

$$h'(t) = k + vg(t) - \phi(f(t) + g(t)) + \phi(f(t)), \tag{6.7}$$

$$h(t)\{W_\phi[f(t) + \alpha(0)h(t) + qg(t)] - v\} = 0. \tag{6.8}$$

Also from (6.1) and (6.2) we can write

$$f(0) + g(0)H + h(0)\delta = u_1 + (u_2 - u_1)H,$$

and $f(0) = u_1$, $g(0) = u_2 - u_1$ and $h(0) = 0$ follows. Thus, applying (6.6), (6.7) and (6.8) we conclude that (6.2) satisfies the problem (5.1), (6.1) if and only if for each t, the following four equations are satisfied

$$f(t) = u_1, \quad g(t) = u_2 - u_1, \tag{6.9}$$

$$h'(t) = k + v(u_2 - u_1) - [\phi(u_2) - \phi(u_1)] = A, \tag{6.10}$$

$$h(t)\{W_\phi[pu_1 + qu_2 + \alpha(0)h(t)] - v\} = 0. \tag{6.11}$$

From (6.10) we have $h(t) = At$, and so, (6.2) is an α-solution of (5.1), (6.1) if and only if, for all t,

$$At\{W_\phi[pu_1 + qu_2 + \alpha(0)At] - v\} = 0. \tag{6.12}$$

(a) If $A = 0$, (6.12) is satisfied and (I) follows.
(b) If $A \neq 0$ and $\phi' = 0$, then, by Lemma 4.1 (c) with $a = u_1$ and $b = u_2$, $W_\phi = 0$ and (6.12) is satisfied if and only if $v = 0$. Hence, (II) follows.
(c) If $A \neq 0$, and $\phi' \neq 0$, (6.12) is satisfied if and only if, for all $t \neq 0$,

$$W_\phi[pu_1 + qu_2 + \alpha(0)At] = v. \tag{6.13}$$

Let us suppose $\alpha(0) = 0$. Then by Lemma 4.1 (b), (6.13) is satisfied if and only if

$$v = \begin{cases} \dfrac{\phi(u_2) - \phi(u_1)}{u_2 - u_1} & \text{if } u_2 \neq u_1, \\ \phi'(u_1) & \text{if } u_2 = u_1. \end{cases} \tag{6.14}$$

Thus, if $\alpha(0) = 0$, (III) and (IV) follows.
Let us suppose $\alpha(0) \neq 0$. Then, from (6.13) we have necessarily

$$W'_\phi[pu_1 + qu_2 + \alpha(0)At]\alpha(0)A = 0,$$

which means that, for all $t \neq 0$,

$$W'_\phi[pu_1 + qu_2 + \alpha(0)At] = 0.$$

Then, we conclude that $W'_\phi = 0$: actually, if $W'_\phi \neq 0$, since W'_ϕ is an entire function (see Lemma 4.1 (a)), for each $t \neq 0$ the number $pu_1 + qu_2 + \alpha(0)At$ would be a zero of W'_ϕ, which is impossible because the zeros of an entire function that does not vanishes are isolated points. As a consequence, W_ϕ is a constant function and applying Lemma 4.1 (b), we have

$$W_\phi = W_\phi(pu_1+qu_2) = \begin{cases} \dfrac{\phi(u_2)-\phi(u_1)}{u_2-u_1} & \text{if } u_1 \neq u_2, \\ \phi'(u_1) & \text{if } u_1 = u_2. \end{cases}$$

Therefore, (6.13) turns out to be (6.14).

Thus, if $\alpha(0) \neq 0$, (III) and (IV) follows again.

(d) Clearly, in all cases, the α-solution (6.3) is unique in $\tilde{W}$ and independent on α.

□

As a consequence of definition 5.2, we can say that the problem (1.1), (1.2) has an unique α-solution within W, independent on α and given by (1.3).

Corollary 6.1. *If the Riemann problem* (5.1), (6.1) *with* $k=0$ *has an* α*-solution* $\tilde{u} \in \tilde{W}$, *then* $A=0$.

Proof. Suppose $k=0$. Then $A = v(u_2-u_1) - [\phi(u_2)-\phi(u_1)]$, and if $u_1 = u_2$, $A=0$ follows. If $u_1 \neq u_2$, the following cases of theorem 6.1 cannot be applied:

- case (II), because if $\phi' = 0$ and $v=0$ then ϕ is a constant function and $A=0$ follows;
- case (III); because, in this case, $u_1 = u_2$;
- case (IV); because if $v = \frac{\phi(u_2)-\phi(u_1)}{u_2-u_1}$,then $A=0$ follows.

Thus, by theorem 6.1, only case (I) can be applied and $A=0$ follows again. □

As a consequence of definition 5.2, we can say that if the Riemann problem (1.5), (1.2) has an α-solution in W, this α-solution is given by the constant state $u(x,t) = u_1$, if $u_1 = u_2$, or by the travelling shock wave $u(x,t) = u_1 + (u_2-u_1)H(x-vt)$, if $u_1 \neq u_2$. Thus, as we said in the introduction, the Riemann problem (1.5), (1.2) cannot develop nor delta waves neither delta shock waves in W.

7. Examples

(a) Let us consider the following Riemann problem

$$u_t + \left(\frac{u^2}{2}\right)_x = \delta(x-vt), \tag{7.1}$$

$$u(x,0) = 1 - H(x).$$

Since $\phi(u) = \frac{u^2}{2}$, $k=1$, $u_1 = 1$ and $u_2 = 0$, by theorem 6.1 it follows $A = \frac{3}{2} - v$, and only cases (I) and (IV) of this theorem can be applied. Hence, this problem has an α-solution in W if and only if $v = \frac{3}{2}$ or $v = \frac{1}{2}$.

If $v = \frac{3}{2}$, then $A=0$ and the unique α-solution in W is the travelling shock wave

$$u(x,t) = 1 - H\left(x - \frac{3}{2}t\right), \tag{7.2}$$

propagating with speed $\frac{3}{2}$. Thus, as we said in the introduction, (7.2) shows that the impulsive source does not necessarily imply the appearing of an impulse in the solution. Also recalling that the unique solution of the problem $u_t + \left(\frac{u^2}{2}\right)_x = 0$, $u(x,0) = 1 - H(x)$ is the travelling wave

$$u(x,t) = 1 - H\Big(x - \frac{1}{2}t\Big), \tag{7.3}$$

it is interesting to note that, among all impulse sources of the form $\delta(x - vt)$, only the one with speed $v = \frac{3}{2}$ is able to modify the speed of the travelling wave (7.3).

If $v = \frac{1}{2}$, we have $A = 1$ and the unique α-solution in W is the delta shock wave

$$u(x,t) = 1 - H\Big(x - \frac{1}{2}t\Big) + t\,\delta\Big(x - \frac{1}{2}t\Big). \tag{7.4}$$

(b) For the problem

$$\begin{aligned} u_t + \Big(\frac{u^2}{2}\Big)_x &= \delta(x - vt), \\ u(x,0) &= 1, \end{aligned}$$

since $u_1 = u_2 = 1$, we have $A = 1$ and only case (III) of theorem 6.1 can be applied. We conclude that this problem has an α-solution in W if and only if $v = 1$; this α-solution is unique, independent on α, and is given by the delta wave

$$u(x,t) = 1 + t\,\delta(x - t). \tag{7.5}$$

(c) For the problem

$$\begin{aligned} u_t + \Big(\frac{u^2}{2} + u\Big)_x &= \delta(x - vt), \\ u(x,0) &= 0, \end{aligned}$$

we have $\phi(u) = \frac{u^2}{2} + u$, $k = 1$, $u_1 = u_2 = 0$, $A = 1$. Thus, this problem has an α-solution in W if and only if $v = \phi'(0) = 1$. This α-solution is unique in W, is independent of α, and is given by

$$u(x,t) = t\delta(x - t). \tag{7.6}$$

Final comment: Interpreting u as a density of mass, any of the solutions (7.4), (7.5) or (7.6) shows concentration of matter in a moving point. The more interesting one is perhaps the solution (7.6) where this concentration is obtained from a vanishing initial condition! Such concentrations of matter in a single point may be associated to the formation of galaxies in one-dimensional models of universe. Recall that the Burgers equation can be seen as a drastic simplification of the Navier-Stokes equation.

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References

[1] A. Bressan and F. Rampazzo, On differential systems with vector valued impulsive controls, *Bull. Un. Mat. Ital.* 2B, **7** (1988), 641–656.

[2] J.F. Colombeau and A. Le Roux, Multiplication of distributions in elasticity and hydrodynamics, *J. Math. Phys.*, **29** (1988), 315–319.
[3] G. Dal Maso, P. LeFloch and F. Murat, Definitions and weak stability of nonconservative products, *J. Math. Pure Appl.*, **74** (1995), 483–548.
[4] V.G. Danilov and D. Mitrovic, Delta shock wave formation in the case of triangular hyperbolic system of conservation laws, *J. Differ. Equations*, **245** (2008), 3704–3734.
[5] H. Kalisch and D. Mitrovic, Singular solutions of a Fully Nonlinear 2×2 System of Conservation Laws, *Proc. Edinb. Math. Soc.* (2), **55** (3) (2012), 711–729.
[6] D. Mitrovic, V. Bojkovic and V.G. Danilov, Linearization of the Riemann problem for a triangular system of conservation laws and delta shock wave formation process, *Math. Method. Appl. Sci.*, **33** (2010), 904–921..
[7] C.O.R. Sarrico, About a family of distributional products important in the applications, *Port. Math.*, **45** (1988), 295–316.
[8] C.O.R. Sarrico, Distributional products and global solutions for nonconservative inviscid Burgers equation, *J. Math. Anal. Appl.*, **281** (2003), 641–656..
[9] C.O.R. Sarrico, The multiplication of distributions and the Tsodyks model of synapses dynamics, *Int. J. Math. Anal.* **6** (21) (2012), 999–1014.
[10] C.O.R. Sarrico, A distributional product approach to δ-shock wave solutions for a generalized pressureless gas dynamics system, *Int. J. Math.*, **25** (1) (2014), 1450007 (12 pages).
[11] C.O.R. Sarrico, The Riemann problem for the Brio system: A solution containing a Dirac mass obtained via a distributional product, *Russ. J. Math. Phys.* **22** (4) (2015), 518–527.
[12] C.O.R. Sarrico and A. Paiva, The multiplication of distributions in the study of a Riemann problem in fluid dynamics, *J. Nonlinear Math. Phy.* **24** (3) (2017), 328–345.
[13] C.O.R. Sarrico and A. Paiva, Newton second law and the multiplication of distributions, *J. Math. Phys.* **59**, 013505 (2018).
[14] C.O.R. Sarrico and A. Paiva, Delta shock waves in the shallow water system, *J. Dynam. Differential Equations*, **30** (3) (2018), 1187–1198.
[15] C.O.R. Sarrico, Multiplication of distributions and a nonlinear model in elastodynamics, *Pacific Journal of Mathematics*, **294** (1) (2018).
[16] S.F. Schandarin and Y.B. Zeldovich, The Large scale structure of the universe: Turbulence, intermittency, structures in a selfgravitating medium, *Rev. Mod. Phys.*, **61** (1989), 185–220.
[17] Chun Shen and Meina Sun, A distributional product approach to the delta shock wave solution for the one-dimensional zero-pressure gas dynamic system, *International Journal of Nonlinear Mechanics*, **105** (2018), 105–112.
[18] L. Schwartz, *Théorie des distributions*, Hermann, Paris (1965).