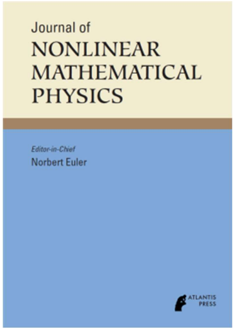

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Generalized Solvable Structures and First Integrals for ODEs Admitting an $\mathfrak{sl}(2,\mathbb{R})$ Symmetry Algebra

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The notion of solvable structure is generalized in order to exploit the presence of an $\mathfrak{sl}(2,\mathbb{R})$ algebra of symmetries for a kth-order ordinary differential equation $\mathscr{E}$ with $k > 3$. In this setting, the knowledge of a generalized solvable structure for $\mathscr{E}$ allows us to reduce $\mathscr{E}$ to a family of second-order linear ordinary differential equations depending on $k-3$ parameters. Examples of explicit integration of fourth and fifth order equations are provided in order to illustrate the procedure.

Keywords: Generalized solvable structure; first integral; nonsolvable symmetry algebra.

2000 Mathematics Subject Classification: 34C14, 34A26, 34A05

1. Introduction

It is well known that the existence of a k-dimensional solvable Lie symmetry algebra for a kth-order ordinary differential equation (ODE) ensures the (local) integrability of the equation by quadratures. However, it is not difficult to provide examples of equations lacking of Lie point symmetries but such that their solutions can be computed by quadratures [4, 15, 22, 23, 27]. This fact triggered, in recent decades, a number of generalizations of the classical Lie reduction method such as hidden symmetries [1, 2], nonlocal symmetries [3, 18], λ-symmetries [20, 21], μ-symmetries [12, 13], σ-symmetries [14, 19], and solvable structures [6–8, 11, 16, 26].

In this paper we focus on the concept of solvable structure, which provides a useful tool for integrating by quadratures ODEs lacking of local symmetries or admitting a nonsolvable symmetry algebra [23]. Although, under some regularity assumptions, the local existence of a solvable structure for any given ODE is guaranteed, its explicit determination is, in general, a quite difficult task. A noteworthy simplification may be obtained by looking for solvable structures which are adapted to some admitted symmetry algebra. For example, in [9, 10] solvable structures adapted to local and nonlocal symmetry algebras are considered, while in [24] a solvable structure for any third-order ODE admitting a nonsolvable symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ is constructed exploiting the symmetry generators. In this paper we address a generalization of the results of [24], considering ODEs of arbitrary order $k > 3$ admitting a nonsolvable symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. For this class of equations we introduce the notion of generalized solvable structure and we prove that the knowledge of a generalized solvable structure allows the construction of a complete set of first integrals for the equation, given in terms of two independent solutions to a corresponding second-order linear ODE. As a consequence, the general solution to the equation can be expressed in parametric form in terms of a fundamental set of solutions to a $(k-3)$-parameter family of second-order linear equations. It is important to remark that, despite the symmetry algebra is nonsolvable, the first integrals can be computed by quadratures. Moreover, the generators of the symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$ and the vector fields belonging to the generalized solvable structure do not form a (standard) solvable structure.

The paper is organized as follows. We start Section 2 briefly recalling the main definitions and facts about distributions of vector fields and their symmetries and the concept of solvable structure, as well as its role in the integration by quadratures of ODEs. Afterwards we define generalized solvable structures for kth-order ODEs admitting an $\mathfrak{sl}(2,\mathbb{R})$ symmetry algebra and we prove a result which allows us to exploit the vector fields belonging to the generalized solvable structure for computing by quadratures $k-3$ functionally independent first integrals shared by the equation and the symmetry generators of $\mathfrak{sl}(2,\mathbb{R})$. In particular, it is proved that the restriction of the ODE to the generic leaf defined by these first integrals provides a third-order equation which inherits $\mathfrak{sl}(2,\mathbb{R})$ as Lie symmetry algebra. Since the classical Lie reduction of order for this kind of equations ends with a Riccati-type equation that does not permit to recover a closed form expression for the general solution to the third-order ODE [5, 17]. We exploit the recent results on the first integrals and general solutions to this class of equations [24, 25], in order to complete the set of first integrals. In particular we obtain the remaining three functionally independent first integrals in terms of the solutions to a related second-order ODE. The complete set of k independent first integrals can be used to provide the general solution to the original kth-order equation, expressed, in explicit or parametric form, in terms of a fundamental set of solutions to a $(k-3)$-parameter family of second-order linear equations.

In Section 3 our results are applied to fourth and fifth order ODEs, whose symmetry algebra is three-dimensional and isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Since the symmetry algebra is nonsolvable and its dimension is strictly lower than the order of the ODE, the integrability by quadratures cannot be guaranteed by the classical Lie theory. For each one of the two examples of fourth-order equations, the construction of a generalized solvable structure requires the determination of a single vector field, and allows us to provide the general solutions to the ODEs (in explicit and parametric form, respectively) in terms of the solutions to a one-parameter family of second-order linear equations. For the fifth-order ODE presented in Section 3.3, the generalized solvable structure is formed by two vector fields and the general solution to the equation is expressed in parametric form in terms of the solutions to a two-parameter family of Schrödinger-type equations.

2. Solvable Structures and Their Generalizations

2.1. *Distributions of vector fields and their symmetries*

Given a set of vector fields $\{A_1,\ldots,A_{n-k}\}$ defined on an n-dimensional manifold N, we denote by $\mathscr{A} := \langle A_1,\ldots,A_{n-k}\rangle$ the distribution generated by $\{A_1,\ldots,A_{n-k}\}$. Moreover, given a set of one-forms $\{\beta_1,\ldots,\beta_k\}$, we denote by $\langle\beta_1,\ldots,\beta_k\rangle$ the corresponding Pfaffian system (*i.e.* the sub-module over $C^\infty(N)$) generated by $\{\beta_1,\ldots,\beta_k\}$.

The distribution $\mathscr{A}$ is *integrable* (in Frobenius sense) if $[A,B]\in\mathscr{A}$, for any $A,\ B\in\mathscr{A}$. If U is an open domain of N where the vector fields $\{A_1,\ldots,A_{n-k}\}$ are pointwise linearly independent, we say that $\mathscr{A}$ is a distribution of *maximal rank* $n-k$ (or of *codimension* k) on U. It is well known that any integrable distribution $\mathscr{A}$ of maximal rank on $U\subseteq N$ determines a $(n-k)$-dimensional foliation of U. If this foliation is described through the vanishing of k functions of the form I_h-c_h, where $I_h\in\mathscr{C}^\infty(U)$ and $c_h\in\mathbb{R}$, we can chose $\langle dI_1,\ldots,dI_k\rangle$ as generators for the Pfaffian system annihilating the distribution $\mathscr{A}$. A submanifold $S\subset N$ is an integral manifold for $\mathscr{A}$ if $\mathscr{A}|_S\subseteq TS$. If in particular $\mathscr{A}|_S=TS$ we say that S is a maximal integral manifold for $\mathscr{A}$.

Given a distribution $\mathscr{A}$, a vector field X is a symmetry of $\mathscr{A}$ if $[X,A]\in\mathscr{A}$, for any $A\in\mathscr{A}$. Let $\mathscr{A}$ and $\mathscr{B}$ be two distributions on N. We say that $\mathscr{A}$ and $\mathscr{B}$ are *transversal* at $p\in N$ if they do not vanish at p and $\mathscr{A}(p)\cap\mathscr{B}(p)=\{0\}$. Analogously, $\mathscr{A}$ and $\mathscr{B}$ are transversal in U if they are transversal at any point of U. An algebra $\mathscr{G}$ of symmetries for a distribution $\mathscr{A}$ is *nontrivial* if $\mathscr{G}$ generates a distribution which is transversal to $\mathscr{A}$.

2.2. *Solvable structures for ODEs*

It is well known that, given a $(n-k)$-dimensional integrable distribution $\mathscr{A}$ on an n-dimensional manifold N, the knowledge of a solvable k-dimensional algebra $\mathscr{G}$ of nontrivial symmetries for $\mathscr{A}$ guarantees that a complete set of first integrals for $\mathscr{A}$ can be found by quadratures. Solvable structures provide an extension of this classical result, significantly enlarging the class of vector fields which can be used to integrate by quadratures a distribution of vector fields.

In this section we recall some basic definitions and facts on solvable structures. The interested reader is referred to [6,8,16,26] for further details.

Definition 2.1. Let $\mathscr{A}$ be a $(n-k)$-dimensional distribution on an n-dimensional manifold N. A set of vector fields $\{Y_1,\ldots,Y_k\}$ is a *solvable structure* for $\mathscr{A}$ in an open domain $U\subseteq N$ if, denoting by $\mathscr{A}_0=\mathscr{A}$ and $\mathscr{A}_h=\mathscr{A}\oplus\langle Y_1,\ldots,Y_h\rangle$ $(h\le k)$, the following conditions hold:

(1) The distribution $\langle Y_1,Y_2,\ldots,Y_h\rangle$ has maximal rank h and is transversal to $\mathscr{A}$ in U, for any $h\le k$;
(2) $\mathscr{A}_h$ has maximal rank $(n-k+h)$ in U;
(3) $\mathscr{L}_{Y_h}\mathscr{A}_{h-1}\subseteq\mathscr{A}_{h-1}$, for $1\le h\le k$.

Theorem 2.1. *Let $\mathscr{A}=\langle A_1,\ldots,A_{n-k}\rangle$ be an integrable $(n-k)$-dimensional distribution defined on an orientable n-dimensional manifold N and let $\{Y_1,\ldots,Y_k\}$ be a solvable structure for $\mathscr{A}$. Denoting by Ω a volume form on N and by α the k-form $A_1\lrcorner\ldots\lrcorner A_{n-k}\lrcorner\Omega$, the distribution $\mathscr{A}$ can be described as the annihilator of the Pfaffian system generated by*

$$\omega_i=\frac{1}{\Delta}(Y_1\lrcorner\ldots\lrcorner\widehat{Y}_i\lrcorner\ldots\lrcorner Y_k\lrcorner\alpha),\qquad (i=1,\ldots,k)\tag{2.1}$$

where the hat denotes omission of the corresponding vector field and Δ *is the function on* N *defined by*

$$\Delta = Y_1 \lrcorner Y_2 \lrcorner \ldots \lrcorner Y_k \lrcorner \alpha.$$

Moreover, the forms ω_i *satisfy*

$$d\omega_k = 0, \\ d\omega_i = 0 \ \ mod\ \{\omega_{i+1}, \ldots, \omega_k\}$$

for $i \in \{1, \ldots, k-1\}$*. Therefore the integral manifolds of the distribution* $\mathscr{A}$ *can be described in implicit form as the level manifolds* $I_1 = c_1$*,* $I_2 = c_2$*, …,* $I_k = c_k$*,* $c_i \in \mathbb{R}$*, where*

$$\omega_k = dI_k, \ \ \omega_{k-1}|_{\{I_k=c_k\}} = dI_{k-1}, \ \ldots \ , \omega_1|_{\{I_k=c_k, I_{k-1}=c_{k-1}, \ldots, I_2=c_2\}} = dI_1.$$

Proof. The interested reader is referred to the original papers [6–8, 16, 26] for a proof of this theorem. □

We remark that, if the distribution $\mathscr{A}$ admits an Abelian Lie algebra of symmetries generated by the vector fields $Y_1, \ldots, Y_k$, then all the 1-forms ω_i are closed, i.e. the function $1/\Delta$ provides an integrating factor for all the 1-forms $(Y_1 \lrcorner \ldots \lrcorner \widehat{Y_i} \lrcorner \ldots \lrcorner Y_k \lrcorner \alpha)$, for $1 \leq i \leq k$.

The main difference between a solvable structure and a solvable symmetry algebra for a completely integrable distribution $\mathscr{A}$ is that the fields belonging to a solvable structure do not need to be symmetries of $\mathscr{A}$. This, of course, gives more freedom in the choice of the vector fields which can be exploited to find integral manifolds of $\mathscr{A}$ by quadratures.

Let $(x, u^{(k-1)}) = (x, u, u_1, \ldots, u_{k-1})$ denote the coordinates of the jet space $J^{k-1}(\mathbb{R}, \mathbb{R})$, for $k \geq 2$. In order to apply Theorem 2.1 to the integration of ODEs, we recall that, with any kth-order ODE in normal form

$$u_k = F(x, u^{(k-1)}), \tag{2.2}$$

we can associate the vector field A defined on a suitable domain $U \subset J^{k-1}(\mathbb{R}, \mathbb{R})$ and given by

$$A = \partial_x + u_1 \partial_u + \cdots + F(x, u^{(k-1)}) \partial_{u_{k-1}}. \tag{2.3}$$

Definition 2.2. If $\mathscr{E}$ is a kth-order ODE of the form (2.2) and A is the corresponding vector field given by (2.3), the vector fields $\{Y_1, Y_2, \ldots, Y_k\}$ defined on a domain $U \subset J^{k-1}(\mathbb{R}, \mathbb{R})$ are a *solvable structure* for $\mathscr{E}$ if they are a solvable structure for the one-dimensional distribution $\mathscr{A} = \langle A \rangle$.

Corollary 2.1. *The knowledge of a k-dimensional solvable structure for a kth-order ODE, allows us to obtain the solution to the ODE by quadratures.*

Proof. The distribution $\mathscr{A}$ is obviously integrable, being one-dimensional. Moreover the functions I_h of Theorem 2.1 provide a complete set of first integrals for A. □

2.3. *Generalized solvable structures for ODEs*

In this section we generalize the notion of solvable structure for ODEs, in order to exploit the knowledge of a symmetry algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ for kth-order ODEs with $k > 3$. In particular we give the following

Definition 2.3. Let $\mathscr{E}$ be a kth-order ODE of the form (2.2) with $k > 3$ and let A be the corresponding vector field defined by (2.3). If $\mathscr{E}$ admits a symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ with generators $\{X_1, X_2, X_3\}$, we call *generalized solvable structure* for $\mathscr{E}$ a set of vector fields $\{Y_1, \ldots, Y_{k-3}\}$ which form a standard solvable structure for the integrable distribution $\mathscr{A} = \langle A, X_1, X_2, X_3 \rangle$.

We remark that the k vector fields $\{X_1, X_2, X_3, Y_1, \ldots, Y_{k-3}\}$ do not form a solvable structure for the distribution generated by A, due to the commutation relations between X_i. Despite this fact we have the following result.

Theorem 2.2. *Let $\mathscr{E}$ be a kth-order ODE with $k > 3$ admitting a symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and let A be the corresponding vector field defined by* (2.3). *The knowledge of a generalized solvable structure $\{Y_1, \ldots, Y_{k-3}\}$ for $\mathscr{E}$ allows us to construct k first integrals for the ODE* (2.2) *in terms of two independent solutions to a linear second order ODE.*

Proof. Since $\{Y_1, \ldots, Y_{k-3}\}$ provide a solvable structure for the integrable distribution $\mathscr{A} = \langle A, X_1, X_2, X_3 \rangle$, we can use Theorem 2.1 in order to find suitable functions $\{I_1, \ldots, I_{k-3}\}$ such that $\Sigma := \{I_i = c_i\}_{i=1,\ldots,k-3}$ is an integrable manifold for $\mathscr{A}$. Since in this setting the manifold $N = J^{k-1}(\mathbb{R},\mathbb{R})$ has dimension $(k+1)$, we have that the dimension of Σ is $k+1-(k-3) = 4$. Moreover, since the vector fields A, X_1, X_2, X_3 are tangent to Σ, equation (2.2) restricted to Σ turns out to be a third-order ODE admitting $\mathfrak{sl}(2,\mathbb{R})$ as symmetry algebra. Therefore, we can exploit the results of [24,25] in order to obtain the remaining three first integrals. In particular, starting from the restriction of equation (2.2) to the 4-dimensional submanifold Σ, we get I_{k-2}, I_{k-1} and I_k in terms of two dependent solutions to a second-order linear ODE and of the real constants c_i, $i = 1, \ldots, k-3$. □

3. Examples

3.1. *Example 1*

Let us consider the fourth-order equation

$$u_4 u_1^2 - 2(3u_2 u_3 u_1 - 3u_2^2 - u u_1^6) = 0 \tag{3.1}$$

which corresponds, when $u_1 \neq 0$, to the vector field

$$A = \partial_x + u_1 \partial_u + u_2 \partial_{u_1} + u_3 \partial_{u_2} + \frac{2(3u_2 u_3 u_1 - 3u_2^2 - u u_1^6)}{u_1^2} \partial_{u_3}.$$

The symmetry algebra of equation (3.1) is spanned by

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = x^2 \partial_x, \qquad \mathbf{v}_3 = x \partial_x, \tag{3.2}$$

and hence it is three-dimensional and isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Since equation (3.1) lacks of further symmetries, we can look for a generalized solvable structure. In particular we have to find a vector field Y on a suitable domain $U \subset J^3(\mathbb{R},\mathbb{R})$ such that Y is a symmetry of the distribution $\mathscr{A} =$

$\langle A, X_1, X_1, X_3\rangle$, where X_i stands for the third-order prolongation of $\mathbf{v}_i$, for $i = 1, 2, 3$, i.e.

$$\begin{aligned} X_1 &= \mathbf{v}_1^{(3)} = \partial_x, \\ X_2 &= \mathbf{v}_2^{(3)} = x^2\partial_x - 2xu_1\partial_{u_1} - (2u_1 + 4u_2 x)\partial_{u_2} - (6u_2 + 6u_3 x)\partial_{u_3}, \\ X_3 &= \mathbf{v}_3^{(3)} = x\partial_x - u_1\partial_{u_1} - 2u_2\partial_{u_2} - 3u_3\partial_{u_3}. \end{aligned} \tag{3.3}$$

It is worthwhile to remark that finding a symmetry for the distribution $\mathscr{A} = \langle A, X_1, X_1, X_3\rangle$ is definitely simpler than finding a symmetry for the vector field A. For instance, it is easy to check that the vector field

$$Y = u_1^3\partial_{u_3}, \tag{3.4}$$

provides a generalized solvable structure for ODE (3.1), due to the following commutation relations

$$\begin{aligned} &[X_1, Y] = [X_2, Y] = [X_3, Y] = 0, \\ &[A, Y] = \left(\frac{1}{2}u_1^2x^2\right) X_1 + \left(\frac{1}{2}u_1^2\right) X_2 - u_1^2 x X_3. \end{aligned} \tag{3.5}$$

Once we have got Y we can apply Theorem 2.2, by considering the volume form $\Omega = dx \wedge du \wedge du_1 \wedge du_2 \wedge du_3$ and the differential 1-form $\alpha = A \lrcorner X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner \Omega$ given by

$$\alpha = 4uu_1^6 du - (6u_3u_1^2 - 12u_2^2u_1)du_1 - 6u_1^2u_2du_2 + 2u_1^3du_3.$$

Since Y is a symmetry of α, the function $1/\Delta$, with

$$\Delta = Y \lrcorner \alpha = 2u_1^6,$$

is an integrating factor of α. Therefore, the one-form

$$\omega = \frac{1}{\Delta}\alpha = 2udu - \frac{3(u_3u_1 - 2u_2^2)}{u_1^5}du_1 - \frac{3u_2}{u_1^4}du_2 + \frac{1}{u_1^3}du_3$$

is closed and so (locally) exact, i.e. $\omega = dI$, where the function $I = I(x, u^{(3)})$ is given (up to an additive constant) by:

$$I = u^2 + \frac{u_3}{u_1^3} - \frac{3u_2^2}{2u_1^4}. \tag{3.6}$$

Function (3.6) provides a first integral for the integrable distribution $\langle A, X_1, X_2, X_3\rangle$ and the restriction of (3.1) to the submanifold $I = c_1$, where $c_1 \in \mathbb{R}$, leads to the reduced equation

$$2u^2u_1^4 + 2u_1u_3 - 3u_2^2 = 2u_1^4c_1. \tag{3.7}$$

Equation (3.7) inherits a symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and spanned by the restrictions of the vector fields (3.2) to $I = c_1$. Moreover (3.7) corresponds to the equation presented in [25], Case 1 in Table 2, for the function $C(u) = u^2 - c_1$. By Proposition 4.1 in [25], a complete set of first integrals for equation (3.7) can be expressed in terms of two independent solutions to the second-order linear ODE (see Case 1, Table 6 in [25]):

$$\Psi''_{c_1}(u) + (u^2 - c_1)\Psi_{c_1}(u) = 0. \tag{3.8}$$

Let $\Psi_{c_1;1}(u)$ and $\Psi_{c_1;2}(u)$ denote two linearly independent solutions to (3.8), such that the corresponding Wronskian is equal to 1 (note that the Wronskian is constant by the Liouville's formula).

According to Table 7 (Case 1) in [25], three independent first integrals for equation (3.7) are given by

$$\begin{aligned} I_1(x,u,u_1,u_2;c_1) &= \frac{u_2\Psi_{c_1;1}(u) - 2u_1^2\Psi'_{c_1;1}(u)}{u_2\Psi_{c_1;2}(u) - 2u_1^2\Psi'_{c_1;2}(u)}, \\ I_2(x,u,u_1,u_2;c_1) &= \frac{(2u_1+xu_2)\Psi_{c_1;1}(u) - 2xu_1^2\Psi'_{c_1;1}(u)}{(2u_1+xu_2)\Psi_{c_1;2}(u) - 2xu_1^2\Psi'_{c_1;2}(u)}, \\ I_3(x,u,u_1,u_2;c_1) &= \frac{((2u_1+xu_2)\Psi_{c_1;2}(u) - 2xu_1^2\Psi'_{c_1;2}(u))^2}{4u_1^3}. \end{aligned} \tag{3.9}$$

If we put $c_1 = u^2 + \frac{u_3}{u_1^3} - \frac{3u_2^2}{2u_1^4}$ in (3.9), the resulting functions I_1, I_2, I_3 provide, together with (3.6), a complete set of first integrals for equation (3.1). Therefore, the general solution to (3.1) is given by (see Section 5 in [25]):

$$x = c_4(c_2 - c_3)\frac{c_2\Psi_{c_1;1}(u) - \Psi_{c_1;2}(u)}{c_3\Psi_{c_1;1}(u) - \Psi_{c_1;2}(u)}, \tag{3.10}$$

where $c_i \in \mathbb{R}$ for $i = 1, 2, 3, 4$, $c_4 \neq 0$, $c_2 \neq c_3$.

We finally remark that the solutions to (3.8) can be expressed in terms of Whittaker functions: if $M_{\mu,\nu}(z)$ and $W_{\mu,\nu}$(z) denote the Whittaker functions of parameters $\mu = \frac{i}{4}c_1$ and $\nu = \frac{1}{4}$, then two linearly independent solutions to (3.8) are given by

$$\Psi_{c_1;1}(u) = \frac{1}{\sqrt{u}}M_{\mu,\nu}(iu^2) \ \text{ and } \ \Psi_{c_1;2}(u) = \frac{1}{\sqrt{u}}W_{\mu,\nu}(iu^2).$$

Consequently, the general solution (3.10) can be expressed as follows:

$$x = c_4(c_2 - c_3)\,\frac{c_2W_{\mu,\nu}(iu^2) - M_{\mu,\nu}(iu^2)}{c_3W_{\mu,\nu}(iu^2) - M_{\mu,\nu}(iu^2)}.$$

3.2. *Example 2*

Let us consider the fourth-order ODE

$$u_1(x-u)u_4 - 2xu_2u_3 - 6u_2^2u_1 + 4u_1^2u_3 + 2u_2u_3u - 12u_2^2 + 8u_3u_1 = 0 \tag{3.11}$$

and let A denote the corresponding vector field of the form (2.3) on a suitable domain $U \subset J^4(\mathbb{R},\mathbb{R})$ such that $u_1(x-u) \neq 0$.

The Lie symmetry algebra of equation (3.11) is three-dimensional and isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. The third-order prolongations of the corresponding Lie symmetry generators are given by

$$\begin{aligned} X_1 &= \partial_x + \partial_u, \\ X_2 &= x^2\partial_x + u^2\partial_u - 2u_1(x-u)\partial_{u_1} - (2u_1 + 4xu_2 - 2u_1^2 - 2uu_2)\partial_{u_2} \\ &\quad -(6u_2 + 6xu_3 - 6u_2u_1 - 2u_3u)\partial_{u_3}, \\ X_3 &= x\partial_x + u\partial_u - u_2\partial_{u_2} - 2u_3\partial_{u_3}. \end{aligned}$$

A symmetry of the integrable distribution $\langle A, X_1, X_2, X_3 \rangle$ can be easily found searching, for instance, a symmetry of the form $\eta(x, u^{(3)})\partial_u$. In particular, if we consider the vector field

$$Y = \left(\frac{u_1^2}{u_3(x-u) + 3u_2(u_1+1)} \right) \partial_u, \tag{3.12}$$

we have the following commutation relations

$$\begin{aligned} &[X_1, Y] = [X_3, Y] = 0, \\ &[X_2, Y] = \mu_0 A + \mu_1 X_1 + \mu_2 X_2 + \mu_3 X_3, \\ &[A, Y] = \rho_0 A + \rho_1 X_1 + \rho_2 X_2 + \rho_3 X_3, \end{aligned} \tag{3.13}$$

for suitable functions μ_i, ρ_i $(i = 0,1,2,3)$ whose expressions are omitted, not being involved in the following discussion. Since the vector field (3.12) provides a generalized solvable structure for equation (3.11), an integrating factor for the 1-form $\alpha = A \lrcorner X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner \Omega$ is given by $1/\Delta$, where

$$\Delta = A \lrcorner X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner Y \lrcorner \Omega = 2u_1(3u_2^2 - 2u_3u_1)$$

and we assume that $\Delta \neq 0$.

Therefore, the differential one-form $\omega = \frac{1}{\Delta}\alpha$ is closed and, locally, exact. A corresponding first integral is given by

$$I_1 = \frac{1}{2u_1^2}\left(6u_2(u-x) - u_3(x-u)^2 + 6u_2u_1(u-x) - 6u_1 - 6u_1^3\right). \tag{3.14}$$

The restriction of equation (3.11) to a generic leaf $I_1 = c_1$, $c_1 \in \mathbb{R}$, provides the following third-order ODE:

$$u_3 = \frac{-2(c_1u_1^2 + 3u_1^3 + 3u_1 - 3uu_1u_2 + 3u_1u_2x - 3u_2u + 3xu_2)}{(x-u)^2}. \tag{3.15}$$

Equation (3.15) inherits the symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$ and corresponds to the third-order ODE appearing in [25, Case 3 in Table 1], for the particular case of

$$C(s) = \frac{1}{3s^2 + 4c_1 - 24}, \quad \text{where } s = \frac{2u_1 + 2u_1^2 + u_2(x-u)}{u_1^{3/2}}.$$

According to [25], a complete set of first integrals to (3.15) can be expressed in terms of a system of solutions to the second-order linear equation

$$\Psi''_{c_1}(s) + \frac{7s}{3s^2 + 4c_1 - 24}\Psi'_{c_1}(s) + \frac{4}{(3s^2 + 4c_1 - 24)}\Psi_{c_1}(s) = 0. \tag{3.16}$$

In particular, if $\Psi_{c_1;1} = \Psi_{c_1;1}(s), \Psi_{c_1;2} = \Psi_{c_1;2}(s)$ denote two linearly independent solutions to (3.16) and $W = W(\Psi_{c_1;1}, \Psi_{c_1;2})(s)$ denotes the corresponding Wronskian, a complete system of first integrals for equation (3.15) is given by (see [25, Case 3 in Table 8]):

$$\begin{aligned} I_2(x,u,u_1,u_2;c_1) &= \frac{2\sqrt{u_1}\Psi_{c_1;1}(s) + (3s^2+4c_1-24)\Psi'_{c_1;1}(s)}{2\sqrt{u_1}\Psi_{c_1;2}(s) + (3s^2+4c_1-24)\Psi'_{c_1;2}(s)}, \\ I_3(x,u,u_1,u_2;c_1) &= \frac{2x\sqrt{u_1}\Psi_{c_1;1}(s) + u(3s^2+4c_1-24)\Psi'_{c_1;1}(s)}{2x\sqrt{u_1}\Psi_{c_1;2}(s) + u(3s^2+4c_1-24)\Psi'_{c_1;2}(s)}, \\ I_4(x,u,u_1,u_2;c_1) &= \frac{\left(2x\sqrt{u_1}\Psi_{c_1;2}(s) + 3u(3s^2+4c_1-24)\Psi'_{c_1;2}(s)\right)^2}{6\sqrt{u_1}(u-x)(3s^2+4c_1-24)W}. \end{aligned} \tag{3.17}$$

By setting $J_i = J_i(x,u^{(3)}) = I_i(x,u,u_1,u_2;I_1)$, for $i = 2,3,4$, we finally obtain that $\{I_1,J_2,J_3,J_4\}$ is a complete set of first integrals for the fourth-order equation (3.11). Hence, the general solution to (3.15), and therefore to the original equation (3.11), is implicitly defined by

$$\begin{aligned} I_2(x,u,u_1,u_2;c_1) &= c_2, \\ I_3(x,u,u_1,u_2;c_1) &= c_3, \\ I_4(x,u,u_1,u_2;c_1) &= c_4, \end{aligned} \tag{3.18}$$

where $c_i \in \mathbb{R}$ for $i = 1, 2, 3, 4$.

In order to obtain a parametric expression for the implicit solution (3.18), we consider a new parameter t such that $s = s(t)$ is determined as follows (see [25], Section 5 for further details):

$$s'(t) = \frac{1}{C(s(t))} = 3s(t)^2 + 4c_1 - 24. \tag{3.19}$$

Three possible cases arise:

- If $c_1 > 6$ then (3.19) yields

$$s(t) = \sqrt{2c-12}\tan(2t\sqrt{2c_1-12}),$$

  and the second-order equation (3.16) is transformed by means of the change $\phi_{c_1}(t) = \Psi_{c_1}(s(t))$ into

$$\phi''_{c_1}(t) + \sqrt{2c-12}\tan(2t\sqrt{2c_1-12})\phi'_{c_1}(t) + 4\phi_{c_1}(t) = 0. \tag{3.20}$$

- If $c_1 < 6$ then (3.19) yields

$$s(t) = -\sqrt{12-2c_1}\tanh(2t\sqrt{12-2c_1}),$$

  and through the transformation $\phi_{c_1}(t) = \Psi_{c_1}(s(t))$ the second-order equation (3.16) becomes

$$\phi''_{c_1}(t) - \sqrt{12-2c}\tanh(2t\sqrt{12-2c})\phi'_{c_1}(t) + 4\phi_{c_1}(t) = 0. \tag{3.21}$$

- If $c_1 = 6$ then a solution to (3.19) is locally given by

$$s(t) = -\frac{1}{2t}$$

and by means of the transformation $\phi_{c_1}(t) = \Psi_{c_1}(s(t))$, equation (3.16) is mapped into

$$\phi''_{c_1}(t) - \frac{1}{2t}\phi'_{c_1}(t) + 4\phi_{c_1}(t) = 0. \tag{3.22}$$

For each one of the three cases considered above, let $\phi_{1;c_1}(t) = \Psi_{c_1;1}(s(t))$ and $\phi_{2;c_1}(t) = \Psi_{c_1;2}(s(t))$ denote two linearly independent solutions to the second-order linear equations (3.20), (3.21), and (3.22), respectively. In terms of these functions, the general solution to equation (3.11) can be expressed in parametric as follows [25]:

$$\begin{cases} x(t) = c_4(c_2 - c_3)\dfrac{c_3\phi'_{2;c_1}(t) - \phi'_{1;c_1}(t)}{c_2\phi'_{2;c_1}(t) - \phi'_{1;c_1}(t)}, \\ u(t) = c_4(c_2 - c_3)\dfrac{c_3\phi_{2;c_1}(t) - \phi_{1;c_1}(t)}{c_2\phi_{2;c_1}(t) - \phi_{1;c_1}(t)}. \end{cases} \tag{3.23}$$

3.3. *Example 3*

In this section we consider the fifth-order ODE

$$uu_3u_5 + u_1u_3u_4 + 2u_2u_3^2 - uu_4^2 + 4u^3u_3^4 = 0 \tag{3.24}$$

on a suitable domain $U \subset J^5(\mathbb{R},\mathbb{R})$ such that $uu_3 \neq 0$ and its associated vector field

$$A = \partial_x + u_1\partial_u + u_2\partial_{u_1} + u_3\partial_{u_2} + u_4\partial_{u_3} - \left(\frac{u_4u_1u_3 + 2u_2u_3^2 - uu_4^2 + 4u^3u_3^4}{uu_3}\right)\partial_{u_4}.$$

The Lie symmetry algebra of equation (3.24) is three dimensional and isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. The corresponding fourth-order prolongations of the symmetry generators are

$$\begin{aligned} X_1 &= \partial_x, \\ X_2 &= x^2\partial_x + 2xu\partial_u + 2u\partial_{u_1} - (2xu_2 - 2u_1)\partial_{u_2} - 4xu_3\partial_{u_3} - (4u_3 + 6xu_4)\partial_{u_4}, \\ X_3 &= x\partial_x + u\partial_u - u_2\partial_{u_2} - 2u_3\partial_{u_3} - 3u_4\partial_{u_4}. \end{aligned}$$

A symmetry Y_1 of the integrable distribution $\langle A, X_1, X_2, X_3\rangle$ can be easily determined by searching, for instance, a vector field of the form $\eta(x, u^{(4)})\partial_{u_2}$. For example, the vector field

$$Y_1 = \frac{1}{u}\partial_{u_2},$$

satisfies the following commutation relations:

$$\begin{aligned} [X_1, Y_1] &= [X_2, Y_1] = [X_3, Y_1] = 0, \\ [A, Y_1] &= -\frac{x^2}{2u^2}X_1 - \frac{1}{2u^2}X_2 + \frac{x}{u^2}X_3. \end{aligned} \tag{3.25}$$

The next step in order to obtain a generalized solvable structure for equation (3.24) is looking for a symmetry Y_2 of the integrable distribution $\langle A, X_1, X_2, X_3, Y_1\rangle$. It is easy to check that the vector field

$$Y_2 = -2u\partial_u - 3u_1\partial_{u_1} + u_3\partial_{u_3}$$

satisfies

$$\begin{aligned}
&[X_1,Y_2] = [X_3,Y_2] = 0, [X_2,Y_2] = -x^2X_1 - X_2 + 2xX_3 + 8uu_1Y_1,\\
&[Y_1,Y_2] = -2Y_1,\\
&[A,Y_2] = A - \frac{(u+2x^2u_2)}{u}X_1 - \frac{2u_2}{u}X_2 + \frac{4xu_2}{u}X_3 + (4u_2u_1 - 2uu_3)Y_1.
\end{aligned}$$

Therefore $\langle Y_1, Y_2\rangle$ provides a generalized solvable structure for equation (3.24).

In order to apply Theorem 2.2, we consider the volume form $\Omega = dx\wedge du\wedge du_1\wedge du_2\wedge du_3\wedge du_4$ and the two one-forms

$$\alpha_2 = A\lrcorner X_1\lrcorner X_2\lrcorner X_3\lrcorner Y_1\lrcorner\Omega \ \text{ and } \ \alpha_1 = A\lrcorner X_1\lrcorner X_2\lrcorner X_3\lrcorner Y_2\lrcorner\Omega.$$

An integrating factor for α_2 is given by $1/\Delta$, where

$$\Delta = Y_2\lrcorner\alpha_2 = \frac{6(4u^4u_3^4 + u^2u_4^2 + 4u_4u_1u_3u + 4u_3^2u_1^2)}{u}$$

and we assume that $\Delta \neq 0$. Therefore we can chose as first integral for the (locally) exact form $\omega_2 = \frac{1}{\Delta}\alpha_2$ the function $\ln(I_2)/6$, where

$$I_2 = \frac{u_3^2}{4u^4u_3^4 + u^2u_4^2 + 4u_4u_1u_3u + 4u_3^2u_1^2} \tag{3.26}$$

is a first integral for the distribution $\langle A, X_1, X_2, X_3, Y_1\rangle$.

Since $\omega_1 = \frac{1}{\Delta}\alpha_1$ is closed on each leaf $I_2 = c_2$, with $c_2 \in \mathbb{R}$, $c_2 > 0$, we have that ω_1 is locally exact module ω_2 and a corresponding first integral is given by

$$I_1 = \frac{1}{2}u_1^2 - u_2u + \frac{1}{2}\arctan\left(\frac{2u_3^2u^2}{2u_3u_1 + uu_4}\right). \tag{3.27}$$

Hence, by restricting to a generic leaf $\{I_1 = c_1,\ I_2 = c_2\}$, we get the reduced equation

$$u_3 = \frac{-\sin(2c_1 + 2uu_2 - u_1^2)}{2u^2\sqrt{c_2}}, \tag{3.28}$$

which is a third-order ODE admitting $\mathfrak{sl}(2,\mathbb{R})$ as Lie symmetry algebra. Equation (3.28) corresponds to the third-order ODE given in [25, Case 2 in Table 2] for

$$C(s) = \frac{\sqrt{c_2}}{4\sin(2c_1 - s)}, \ \text{ where } s = u_1^2 - 2uu_2.$$

According to the results obtained in [25], if $\Psi_{c_1,c_2;1} = \Psi_{c_1,c_2;1}(s)$ and $\Psi_{c_1,c_2;2} = \Psi_{c_1,c_2;2}(s)$ are two linearly independent solutions to the linear second-order equation [25, Case 2 in Table 6],

$$-4\sin^2(2c_1 - s)\Psi''(s) + 2\sin(4c_1 - 2s)\Psi'(s) + c_2 s\Psi(s) = 0, \tag{3.29}$$

three functionally independent first integrals for equation (3.28) are given by

$$I_3(x,u^{(2)};c_1,c_2) = \frac{\sqrt{c_2}u_1\Psi_{c_1,c_2;1}(s)+2\sin(2c_1-s)\Psi'_{c_1,c_2;1}(s)}{\sqrt{c_2}u_1\Psi_{c_1,c_2;2}(s)+2\sin(2c_1-s)\Psi'_{c_1,c_2;2}(s)},$$

$$I_4(x,u^{(2)};c_1,c_2) = \frac{\sqrt{c_2}(-2u+u_1x)\Psi_{c_1,c_2;1}(s)+2x\sin(2c_1-s)\Psi'_{c_1,c_2;1}(s)}{\sqrt{c_2}(-2u+u_1x)\Psi_{c_1,c_2;2}(s)+2x\sin(2c_1-s)\Psi'_{c_1,c_2;2}(s)},$$

$$I_5(x,u^{(2)};c_1,c_2) = \frac{\left(\sqrt{c_2}(-2u+u_1x)\Psi_{c_1,c_2;2}(s)+2x\sin(2c_1-s)\Psi'_{c_1,c_2;2}(s)\right)^2}{4\sqrt{c_2}u\sin(2c_1-s)W},$$

where $W = W(\Psi_{c_1,c_2;1},\Psi_{c_1,c_2;2})(s)$ denotes the Wronskian corresponding to the solutions $\Psi_{c_1,c_2;1}$ and $\Psi_{c_1,c_2;2}$. Therefore, as in the previous examples, the general solution to the original fifth-order equation (3.24) is implicitly given by

$$\begin{aligned} I_3(x,u^{(2)};c_1,c_2) &= c_3, \\ I_4(x,u^{(2)};c_1,c_2) &= c_4, \\ I_5(x,u^{(2)};c_1,c_2) &= c_5. \end{aligned} \tag{3.30}$$

In order to obtain a parametric general solution from (3.30), we consider a new parameter t such that $s = s(t)$ is determined as follows:

$$s'(t) = \frac{1}{C(s(t))} = \frac{4\sin(2c_1-s(t))}{\sqrt{c_2}},$$

which yields

$$s(t) = -\arctan\left(\frac{2e^{-4t/\sqrt{c_2}}}{e^{-8t/\sqrt{c_2}}-1}\right)+2c_1. \tag{3.31}$$

By means of the transformation $\phi(t) = \Psi(s(t))$ equation (3.29) is transformed into the Schrödinger-type equation

$$\phi''(t)+4\left(\arctan\left(\frac{2e^{-4t/\sqrt{c_2}}}{e^{-8t/\sqrt{c_2}}-1}\right)-2c_1\right)\phi(t) = 0. \tag{3.32}$$

If $\phi_1 = \phi_1(t;c_1,c_2)$ and $\phi_2 = \phi_2(t;c_1,c_2)$ are two linearly independent solutions to (3.32) such that $W(\phi_1,\phi_2)(t;c_1,c_2) = 1$, then the parametric general solution to equation (3.24) can be expressed as follows

$$\begin{cases} x(t) = c_5(c_3-c_4)\dfrac{\phi_1(t;c_1,c_2)-c_4\phi_2(t;c_1,c_2)}{\phi_1(t;c_1,c_2)-c_3\phi_2(t;c_1,c_2)}, \\ u(t) = \dfrac{c_5(c_3-c_4)^2}{4\big(\phi_1(t;c_1,c_2)-c_3\phi_2(t;c_1,c_2)\big)^2}, \end{cases} \tag{3.33}$$

where $c_i \in \mathbb{R}$ for $i = 1, 2, 3, 4, 5$, $c_2 > 0$, $c_3 \neq c_4$, $c_5 \neq 0$.

4. Conclusions

Generalized solvable structures for ODEs of order $k > 3$ admitting a symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ have been introduced, providing a powerful tool for the explicit determination of the general solutions to these equations.

A generalized solvable structure is formed by $k-3$ vector fields, which, in general, are not symmetries of the equation. Moreover, the symmetry generators of $\mathfrak{sl}(2,\mathbb{R})$ and the vector fields of the generalized solvable structure do not form a solvable structure. Nevertheless, the knowledge of a generalized solvable structure allows the determination by quadratures of a complete set of first integrals, and hence of the general solution to the equation, in terms of two independent solutions to an associated family of second-order linear ODEs. This link between equations admitting $\mathfrak{sl}(2,\mathbb{R})$ and second-order linear ODEs is, to the best of our knowledge, new in the literature and generalizes the recent results obtained for third-order equations.

The concept of generalized solvable structure gives more freedom in the choice of the vector fields which can be used to integrate the equation under study. The effectiveness of the proposed method is illustrated by several examples of equations admitting only an $\mathfrak{sl}(2,\mathbb{R})$ algebra of Lie symmetries. In these cases, after finding a generalized solvable structure, we provide a complete set of first integrals as well as the general solutions in parametric or explicit forms.

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