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## Quasiperiodic Solutions of the Heisenberg Ferromagnet Hierarchy

Peng Zhao

*College of Arts and Sciences, Shanghai Maritime University, Shanghai, 201306, People's Republic of China<sup>1</sup>*  
*College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, 266590, People's Republic of China<sup>2</sup>*  
*School of Mathematical Sciences, Qufu Normal University, Qufu, 273165, People's Republic of China<sup>3</sup>*  
*pengzhao@shmtu.edu.cn*

Engui Fan\*

*School of Mathematical Sciences, Institute of Mathematics and Key Laboratory of Mathematics for Nonlinear Science, Fudan University, Shanghai, 200433, People's Republic of China*  
*faneg@fudan.edu.cn*

Temuerchaolu

*College of Arts and Sciences, Shanghai Maritime University, Shanghai, 201306, People's Republic of China*  
*tmchaolu@shmtu.edu.cn*

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We present quasi-periodic solutions in terms of Riemann theta functions of the Heisenberg ferromagnet hierarchy by using algebro-geometric method. Our main tools include algebraic curve and Riemann surface, polynomial recursive formulation and a special meromorphic function.

**Keywords:** Heisenberg ferromagnet hierarchy; spectral curve; Riemann theta function; quasiperiodic solution.

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### 1. Introduction

Integrable discretizations of soliton equations have attracted much attention during the past few years. The celebrated Landau-Lifshits equation [2]

$$s_t = [s, s_{xx} + Js], \quad s \in \mathbb{R}^3, \quad J = \text{diag}(J_1, J_2, J_3), \quad J_1 + J_2 + J_3 = 0, \quad (1.1)$$

has two well-known integrable discretizations. One of them is the Sklyanin lattice which is determined by the Poisson brackets algebra (see (13) in [7] or [1])

$$\{s_\alpha^{(n)}, s_0^{(n)}\} = J_{\beta\gamma} s_\beta^{(n)} s_\gamma^{(n)}, \quad \{s_\alpha^{(n)}, s_\beta^{(n)}\} = -s_0^{(n)} s_\gamma^{(n)}, \quad (1.2)$$

and the discrete variant of the Landau-Lifshits Hamiltonian (see (23) in [7])

$$H^{(0)} = \sum_n \ln \left( s_0^{(n+1)} s_0^{(n)} + \sum_{\alpha=1}^3 \left( \frac{K_1}{K_0} - J_\alpha \right) s_\alpha^{(n+1)} s_\alpha^{(n)} \right). \quad (1.3)$$

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\*Corresponding author

Another discretization of the Landau-Lifshits equation (1.1) is the so-called Shabat-Yamilov lattice [5, 6]

$$u_t = \frac{2h}{u^+ - v} + h_v, \quad v_t = \frac{2h}{u - v^-} - h_u, \quad (1.4)$$

where  $u = u(n, t)$ ,  $v = v(n, t)$ ,  $u_t = \frac{du(n, t)}{dt}$ ,  $v_t = \frac{dv(n, t)}{dt}$ ,  $u^+ = E^+ u = u(n+1, t)$ ,  $v^- = E^- v = v(n-1, t)$  ( $E^\pm$  are the shift operators and  $(n, t) \in \mathbb{Z} \times \mathbb{R}$ ),  $h = h(u, v)$  is a symmetric polynomial of  $u, v$  degree no higher than two with respect to each of the variables and  $h_{uuu} = 0$ . By introducing the complexified stereographic projection

$$S = S(u, v) = \frac{1}{u - v}(1 - uv, i + iuv, u + v), \quad i = \sqrt{-1}, \quad (1.5)$$

one may relate the Shabat-Yamilov lattice (1.4) to the Sklyanin lattice by a special transformation of variables [1]. The linear combination of the flows of (1.4) and its shift

$$u_t = \frac{2h}{u^- - v} + h_v, \quad v_t = \frac{2h}{u - v^+} - h_u, \quad (1.6)$$

give rise to [5]

$$\begin{aligned} u_t &= \rho_1 \left( \frac{2h}{u^+ - v} + h_v \right) + \rho_2 \left( \frac{2h}{u^- - v} + h_v \right), \\ v_t &= \rho_1 \left( \frac{2h}{u - v^-} - h_u \right) + \rho_2 \left( \frac{2h}{u - v^+} - h_u \right), \end{aligned} \quad (1.7)$$

where  $\rho_1, \rho_2$  are real constants. By specifying  $\rho_1 = 1, \rho_2 = 0, h = \frac{1}{2}(u - v)^2$ , the lattice (1.7) reduces to the so-called Heisenberg ferromagnet (HF) lattice

$$\begin{cases} u_t = (u - v)(u - u^+)(u^+ - v)^{-1}, \\ v_t = (u - v)(v^- - v)(u - v^-)^{-1}. \end{cases} \quad (1.8)$$

In the spin variables  $S$  (see (1.5)), the system (1.7) reads [1]

$$\begin{aligned} S_t &= \rho_1 \langle S, KS \rangle \left( \frac{[S, S^+]}{1 + [S, S^+]} + \frac{[S, S^-]}{1 + [S, S^-]} \right) - 2\rho_1 [S, KS] \\ &\quad + \rho_2 \langle S, KS \rangle \left( \frac{S + S^+}{1 + [S, S^+]} - \frac{S + S^-}{1 + [S, S^-]} \right), \end{aligned} \quad (1.9)$$

where  $K = \text{diag}(K_1, K_2, K_3)$  and  $|S| = 1$ . It is well-known that the lattice (1.9) is integrable with a zero-curvature representation and can be reduced to the well-known Heisenberg lattice (see [1] and references therein).

In this paper, we study quasi-periodic solutions of the whole Heisenberg ferromagnet hierarchy by using algebro-geometric method. In section 2, we construct the stationary and time-dependent HF hierarchy from its zero-curvature representation. Then spectral curves and an auxiliary function  $\phi$  are introduced in section 3. In section 4, based on analytic and asymptotic properties of  $\phi$ , we derive theta function representations for the entire HF hierarchy.

## 2. Heisenberg ferromagnet hierarchy

In this section, we construct the Heisenberg ferromagnet hierarchy by developing zero-curvature formulism of the HF lattice. For later use we denote by  $E^\pm$  the shift operators acting on  $\psi = \{\psi(n, t)\}_{n=-\infty}^{+\infty} \in \mathbb{C}^{\mathbb{Z}}$  according to  $(E^\pm \psi)(n, t) = \psi(n \pm 1, t)$ , or  $E^\pm \psi = \psi^\pm$  for convenience. These notations are followed by [4].

It is well-known that the HF lattice has symplectic operator, Hamiltonian structure, recursion operator, nontrivial generalised symmetry and Lax representation [5]. To construct the HF hierarchy we start from the following two  $2 \times 2$  Lax matrices

$$U = U(\lambda, u, v) = \begin{pmatrix} \lambda - 2u(u-v)^{-1} & -2(u-v)^{-1} \\ 2uv(u-v)^{-1} & \lambda + 2v(u-v)^{-1} \end{pmatrix}, \quad (2.1)$$

$$V = V(\lambda, u, v) = \lambda^{-1}(u-v^-)^{-1} \begin{pmatrix} u+v^- & 2 \\ -2uv^- & -(u+v^-) \end{pmatrix}, \quad (2.2)$$

where  $\lambda \in \mathbb{C}$  is a spectral parameter and  $u, v$  are functions of the lattice variable  $n$  and the time variable  $t$ .

In the following we temporarily view  $u, v$  as functions of only  $n$  and define the sequence  $\{a_\ell, b_\ell, c_\ell\}_{\ell \in \mathbb{N}_0}$  recursively by

$$2\delta(a_\ell + a_\ell^+ - ub_\ell - vb_\ell^+) = b_{\ell-1}^+ - b_{\ell-1}, \quad \ell \in \mathbb{N}_0, \quad (2.3)$$

$$2uv\delta(a_\ell + a_\ell^+ + c_\ell/u + c_\ell^+/v) = c_{\ell-1}^+ - c_{\ell-1}, \quad \ell \in \mathbb{N}_0, \quad (2.4)$$

$$2v\delta(a_\ell - a_\ell^+) - 2uv\delta b_\ell - 2\delta c_\ell^+ = -a_{\ell-1} + a_{\ell-1}^+, \quad \ell \in \mathbb{N}_0, \quad (2.5)$$

where  $a_\ell, b_\ell, c_\ell$  are polynomials of  $u, v$  and their shifts, and  $\delta = (u-v)^{-1}$ . In matrix form, the relations (2.3)-(2.5) can be written as

$$\begin{pmatrix} I + E^+ & -u - vE^+ & 0 \\ I + E^+ & 0 & u^{-1} + v^{-1}E^+ \\ I - E^+ & -u & v^{-1}E^+ \end{pmatrix} \begin{pmatrix} a_\ell \\ b_\ell \\ c_\ell \end{pmatrix} = \begin{pmatrix} 0 & (2\delta)^{-1}(E^+ - I) & 0 \\ 0 & 0 & (2uv\delta)^{-1}(E^+ - I) \\ E^+ - I & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{\ell-1} \\ b_{\ell-1} \\ c_{\ell-1} \end{pmatrix}, \quad (2.6)$$

where  $I$  is the identity operator from  $\mathbb{C}^{\mathbb{Z}}$  to  $\mathbb{C}^{\mathbb{Z}}$ . Apparently, once the initial value  $(a_0, b_0, c_0)$  is given,  $\{a_\ell, b_\ell, c_\ell\}_{\ell \in \mathbb{N}}$  can be recursively determined by the relations (2.3)-(2.5) or (2.6). However, the calculations involved are rather big and therefore it is uneasy to obtain explicit form of  $\{a_\ell, b_\ell, c_\ell\}_{\ell \in \mathbb{N}}$ . In spite of difficulties, we can still get the following result.

**Theorem 2.1.** *Solutions of the system (2.3)-(2.5) are explicitly given by the following recursion relations*

$$a_\ell = \frac{u+v^-}{u-v^-}(E^+ - I)^{-1}S_{1,\ell-1}, \quad \ell \in \mathbb{N}_0, \quad (2.7)$$

$$b_\ell = S_{2,\ell-1} + \frac{2}{u-v^-}(E^+ - I)^{-1}S_{1,\ell-1}, \quad \ell \in \mathbb{N}_0, \quad (2.8)$$

$$c_\ell = S_{3,\ell-1} - \frac{2uv^-}{u-v^-}(E^+ - I)^{-1}S_{1,\ell-1}, \quad \ell \in \mathbb{N}_0, \quad (2.9)$$

where  $S_{1,\ell-1}$ ,  $S_{2,\ell-1}$ ,  $S_{3,\ell-1}$  are defined by

$$\begin{aligned} S_{1,\ell-1} &= \frac{u}{u^2 - (v^-)^2} \left( \frac{v(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+}{2\delta} \right)^- - \frac{v}{(u^+)^2 - v^2} \left( \frac{u(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+}{2\delta} \right)^+ \\ &\quad - \frac{u}{u^2 - (v^-)^2} \frac{u(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+}{2\delta} + \frac{v}{(u^+)^2 - v^2} \frac{v(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+}{2\delta} \\ &\quad + \frac{b_{\ell-1}^+ - b_{\ell-1}}{2\delta}, \\ S_{2,\ell-1} &= \frac{1}{u^2 - (v^-)^2} \left( \left( \frac{v(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+}{2\delta} \right)^- - \frac{u(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+}{2\delta} \right), \\ S_{3,\ell-1} &= \frac{uv^-}{u^2 - (v^-)^2} \left( \frac{u}{v^-} \left( \frac{v(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+}{2\delta} \right)^- - \frac{v^-}{u} \frac{v(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+}{2\delta} \right). \end{aligned}$$

*Proof.* First, from (2.3)-(2.5) it follows

$$2u\delta(a_\ell - a_\ell^+) = -2uv\delta b_\ell^+ - 2\delta c_\ell + a_{\ell-1} - a_{\ell-1}^+. \quad (2.10)$$

Eliminating  $b_\ell^+$  from (2.3) and (2.10), we have

$$2u\delta(2a_\ell - ub_\ell + \frac{c_\ell}{u}) = u(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+. \quad (2.11)$$

Then insertion of (2.3) into (2.5) yields

$$4v\delta a_\ell^+ - 2v^2\delta b_\ell^+ + 2\delta c_\ell^+ = v(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+, \quad (2.12)$$

or equivalently,

$$2a_\ell - v^-b_\ell + \frac{c_\ell}{v^-} = \left( \frac{v(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+}{2v\delta} \right)^-. \quad (2.13)$$

Combining (2.11) with (2.13), we obtain

$$b_\ell + \frac{1}{uv^-}c_\ell = \left( \frac{v(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+}{2v\delta} \right)^- - \frac{u(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+}{2u\delta} \quad (2.14)$$

$$\begin{aligned} b_\ell - \frac{2a_\ell}{u + v^-} &= \frac{1}{u^2 - (v^-)^2} \left( \frac{v(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+}{2\delta} \right)^- - \frac{1}{u^2 - (v^-)^2} \\ &\quad \times \frac{u(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+}{2\delta}. \end{aligned} \quad (2.15)$$

Thus, inserting (2.15) into (2.3), we conclude (2.7) and (2.8) hold. Finally, the formula (2.9) can be derived from the following two equalities

$$2ua_\ell - uv^-b_\ell + \frac{u}{v^-}c_\ell = u \left( \frac{v(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+}{2v\delta} \right)^-, \quad (2.16)$$

$$2v^-a_\ell - uv^-b_\ell + \frac{v^-}{u}c_\ell = v^- \left( \frac{v(b_{\ell-1}^+ - b_{\ell-1}) + a_{\ell-1} - a_{\ell-1}^+}{2u\delta} \right). \quad (2.17)$$

by eliminating  $b_\ell$ . □

Next we compute the stationary HF hierarchy. To this end we start from the stationary zero-curvature equation

$$0 = UW - (EW)U, \quad W = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \quad (2.18)$$

where  $A, B, C, D$  are polynomials of  $\lambda$ :

$$A = \sum_{\ell=1}^{N+1} a_{N+1-\ell} \lambda^{-\ell}, \quad B = \sum_{\ell=1}^{N+1} b_{N+1-\ell} \lambda^{-\ell}, \quad C = \sum_{\ell=1}^{N+1} c_{N+1-\ell} \lambda^{-\ell}, \quad N \in \mathbb{N}. \quad (2.19)$$

Using (2.1), one may rewrite Eq. (2.18) as

$$(\lambda - 2u\delta)A - 2\delta C = (\lambda - 2u\delta)A^+ + 2uv\delta B^+, \quad (2.20)$$

$$(\lambda - 2u\delta)B + 2\delta A = -2\delta A^+ + (\lambda + 2v\delta)B^+, \quad (2.21)$$

$$2uv\delta A + (\lambda + 2v\delta)C = (\lambda - 2u\delta)C^+ - 2uv\delta A^+, \quad (2.22)$$

$$2uv\delta B - (\lambda + 2v\delta)A = -2\delta C^+ - (\lambda + 2uv)\delta A^+. \quad (2.23)$$

Inserting (2.19) into (2.20)-(2.23) and taking into account (2.3)-(2.5), one derives the stationary HF hierarchy

$$\begin{aligned} a_N - a_N^+ &= 0, \\ b_N - b_N^+ &= 0, \\ c_N - c_N^+ &= 0. \end{aligned} \quad (2.24)$$

In the case  $N = 0$ , the system (2.24) is reduced to the stationary HF lattice

$$\begin{aligned} (u - v)(u - u^+)(u^+ - v)^{-1} &= 0, \\ (u - v)(v^- - v)(u - v^-)^{-1} &= 0. \end{aligned} \quad (2.25)$$

The time-dependent HF hierarchy can be derived as follows. First,  $a_\ell, b_\ell, c_\ell$  are considered as functions of both  $n$  and  $t$ . To distinguish different higher order flows that are related to different time variables, we replace  $t$  by  $t_r$ . Then inserting (2.1), (2.3)-(2.5) and (2.19) into the time-dependent zero-curvature equation

$$0 = U_{t_r} + UW - (E^+W)U, \quad (2.26)$$

we obtain the  $r$ -th equation in the HF hierarchy

$$(2u\delta)_{t_r} = -a_r + a_r^+, \quad (2.27)$$

$$(2\delta)_{t_r} = -b_r + b_r^+, \quad (2.28)$$

$$(2uv\delta)_{t_r} = c_r - c_r^+, \quad (2.29)$$

$$(2v\delta)_{t_r} = -a_r + a_r^+. \quad (2.30)$$

The system (2.27)-(2.30) is overdetermined and can be simplified into the  $r$ -th HF lattice.

**Theorem 2.2.** *The  $r$ -th HF lattice has the form of*

$$u_{t_r} = \frac{u-v}{2} (u(b_r - b_r^+) - a_r + a_r^+), \quad (2.31)$$

$$v_{t_r} = \frac{u-v}{2} (v(b_r - b_r^+) - a_r + a_r^+), \quad r \in \mathbb{N}_0. \quad (2.32)$$

*Proof.* First, using (2.27), we obtain

$$2u_{t_r} \delta + 2u \delta_{t_r} = -a_r + a_r^+. \quad (2.33)$$

Then the relations (2.28) and (2.33) give rise to

$$2u_{t_r} \delta = u(b_r - b_r^+) - a_r + a_r^+, \quad (2.34)$$

and hence (2.31) holds. Using (2.28) and (2.30), we arrive at (2.32). To complete the proof, it remains to show that (2.28), (2.29) are compatible with (2.31), (2.32). Actually, using (2.31) and (2.32), we obtain

$$\begin{aligned} (2\delta)_{t_r} &= \left( \frac{2}{u-v} \right)_{t_r} \\ &= -2\delta^2(u_{t_r} - v_{t_r}) \\ &= -\delta(u(b_r - b_r^+) - a_r + a_r^+) + \delta(v(b_r - b_r^+) - a_r + a_r^+) \\ &= -b_r + b_r^+. \end{aligned} \quad (2.35)$$

Then by (2.31), (2.32) and (2.35), it follows

$$\begin{aligned} 2u_{t_r} v \delta &= uv(b_r - b_r^+) - v(a_r - a_r^+), \\ 2uv_{t_r} \delta &= uv(b_r - b_r^+) - u(a_r - a_r^+), \\ 2uv \delta_{t_r} &= -uv(b_r - b_r^+), \end{aligned}$$

and consequently,

$$\begin{aligned} (2uv\delta)_{t_r} &= 2u_{t_r} v \delta + 2uv_{t_r} \delta + 2uv \delta_{t_r} \\ &= uv(b_r - b_r^+) - (u+v)(a_r - a_r^+) \\ &= c_r - c_r^+. \end{aligned} \quad (2.36)$$

Here we use the recursion relations

$$\begin{aligned} 2v\delta(a_\ell - a_\ell^+) - 2uv\delta b_\ell - 2\delta c_\ell^+ &= -a_{\ell-1} + a_{\ell-1}^+, \quad \ell \in \mathbb{N}_0, \\ 2u\delta(a_\ell - a_\ell^+) + 2uv\delta b_\ell + 2\delta c_\ell^+ &= a_{\ell-1} - a_{\ell-1}^+, \quad \ell \in \mathbb{N}_0 \end{aligned}$$

in the last equality of (2.36). □

### 3. Spectral curves and a basic meromorphic function

In this section, we first introduce the spectral curves associated with the HF hierarchy. Then we introduce a basic meromorphic function  $\varphi$  and study its analytic and asymptotic properties.

Let  $(\psi_1, \psi_2)^T$  and  $(\phi_1, \phi_2)^T$  be two fundamental solutions of auxiliary linear problem

$$\begin{aligned} E^+ \Phi(\lambda, n, t_r) &= U(\lambda, n, t_r) \Phi(\lambda, n, t_r), \\ \Phi_{t_r}(\lambda, n, t_r) &= W(\lambda, n, t_r) \Phi(\lambda, n, t_r), \end{aligned} \quad (3.1)$$

where  $\Phi(\lambda, n, t_r) = (\Phi_1(\lambda, n, t_r), \Phi_2(\lambda, n, t_r))^T$ . Moreover, we define

$$f(\lambda, n, t_r) = -(\lambda^2 - 2\lambda)^{-n} \psi_1(\lambda, n, t_r) \phi_1(\lambda, n, t_r), \quad (3.2)$$

$$h(\lambda, n, t_r) = (\lambda^2 - 2\lambda)^{-n} \psi_2(\lambda, n, t_r) \phi_2(\lambda, n, t_r), \quad (3.3)$$

$$g(\lambda, n, t_r) = 2^{-1}(\lambda^2 - 2\lambda)^{-n} (\psi_1(\lambda, n, t_r) \phi_2(\lambda, n, t_r) + \phi_1(\lambda, n, t_r) \psi_2(\lambda, n, t_r)). \quad (3.4)$$

**Theorem 3.1.** *The functions  $f, g, h$  defined in (3.2)-(3.4) satisfy*

$$\begin{pmatrix} g & f \\ h & -g \end{pmatrix}^+ = U \begin{pmatrix} g & f \\ h & -g \end{pmatrix} U^{-1}, \quad (3.5)$$

$$\begin{pmatrix} g & f \\ h & -g \end{pmatrix}_{t_r} = \left[ \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \begin{pmatrix} g & f \\ h & -g \end{pmatrix} \right], \quad (3.6)$$

where  $[\cdot, \cdot]$  is the Lie bracket of two matrices  $P, Q$

$$[P, Q] = PQ - QP$$

and  $g^2 + fh$  is independent of  $x$  and  $t_r$ .

*Proof.* The relation (3.5) can be derived from (3.1), (3.2), (3.3) and (3.4). Indeed, we have

$$\begin{aligned} g^+ &= 2^{-1}(\lambda^2 - 2\lambda)^{-(n+1)} \psi_1^+ \phi_2^+ + \phi_1^+ \psi_2^+ \\ &= 2^{-1}(\lambda^2 - 2\lambda)^{-(n+1)} ((\lambda - 2u\delta)\psi_1 - 2\delta\psi_2)(2uv\delta\phi_1 + (\lambda + 2v\delta)\phi_2) \\ &\quad + ((\lambda - 2u\delta)\phi_1 - 2\delta\phi_2)(2uv\delta\psi_1 + (\lambda + 2v\delta)\psi_2) \\ &= -(\lambda^2 - 2\lambda)^{-1} (2uv\delta(\lambda - 2u\delta)f - (\lambda - 2u\delta)(\lambda + 2v\delta)g \\ &\quad + 4uv\delta^2g + 2\delta(\lambda + 2v\delta)h), \\ f^+ &= -(\lambda^2 - 2\lambda)^{-(n+1)} \psi_1^+ \phi_1^+ \\ &= -(\lambda^2 - 2\lambda)^{-(n+1)} ((\lambda - 2u\delta)\psi_1 - 2\delta\psi_2)((\lambda - 2u\delta)\phi_1 - 2\delta\phi_2) \\ &= (\lambda^2 - 2\lambda)^{-1} ((\lambda - 2u\delta)(\lambda - 2u\delta)f + 2\delta(\lambda - 2u\delta)2g - (-2\delta)^2h), \\ h^+ &= (\lambda^2 - 2\lambda)^{-(n+1)} \psi_2^+ \phi_2^+, \\ &= (\lambda^2 - 2\lambda)^{-(n+1)} (2uv\delta\psi_1 + (\lambda + 2v\delta)\psi_2)(2uv\delta\phi_1 + (\lambda + 2v\delta)\phi_2) \\ &= (\lambda^2 - 2\lambda)^{-1} (-(2uv\delta)^2f + 2uv\delta(\lambda + 2v\delta)2g + (\lambda + 2v\delta)^2h). \end{aligned}$$



Similarly, according to the definition of  $f, g, h, \phi_1, \phi_2, \psi_1, \psi_2$ , it follows that

$$\begin{aligned}
 g_{t_r} &= 2^{-1}(\lambda^2 - 2\lambda)^{-n}(\psi_{1,t_r}\phi_2 + \psi_1\phi_{2,t_r} + \phi_{1,t_r}\psi_2 + \phi_1\psi_{2,t_r}) \\
 &= 2^{-1}(\lambda^2 - 2\lambda)^{-n}((A\psi_1 + B\psi_2)\phi_2 + \psi_1(C\phi_1 - A\phi_2) \\
 &\quad + (A\phi_1 + B\phi_2)\psi_2 + \phi_1(C\psi_1 - A\psi_2)) \\
 &= Bh - Cf, \\
 f_{t_r} &= -(\lambda^2 - 2\lambda)^{-n}(\psi_1\phi_1)_{t_r} = -(\lambda^2 - 2\lambda)^{-n}(\psi_{1,t_r}\phi_1 + \psi_1\phi_{1,t_r}) \\
 &= -(\lambda^2 - 2\lambda)^{-n}((A\psi_1 + B\psi_2)\phi_1 + \psi_1(A\phi_1 + B\phi_2)) \\
 &= 2Af - 2Bg \\
 h_{t_r} &= (\lambda^2 - 2\lambda)^{-n}(\psi_2\phi_2)_{t_r} = (\lambda^2 - 2\lambda)^{-n}(\psi_{2,t_r}\phi_2 + \psi_2\phi_{2,t_r}) \\
 &= (\lambda^2 - 2\lambda)^{-n}((C\psi_1 - A\psi_2)\phi_2 + \psi_2(C\phi_1 - A\phi_2)) \\
 &= 2Cg - 2Ah,
 \end{aligned}$$

which indicates (3.6) holds. Finally, using

$$\left( \det \begin{pmatrix} g & f \\ h & -g \end{pmatrix} \right)_{t_r} = \det \begin{pmatrix} g & f \\ h & -g \end{pmatrix} \operatorname{tr} \left( \begin{pmatrix} g & f \\ h & -g \end{pmatrix}_{t_r} \begin{pmatrix} g & f \\ h & -g \end{pmatrix}^{-1} \right), \quad (3.7)$$

$$\det \begin{pmatrix} g & f \\ h & -g \end{pmatrix} = -g^2 - fh, \quad (3.8)$$

we conclude that  $g^2 + fh$  is independent of  $n$  and  $t_r$ .  $\square$

In what follows it is convenient to introduce

$$g_M = \sum_{\ell=1}^{M+1} a_{M+1-\ell} \lambda^{-\ell}, \quad f_M = \sum_{\ell=1}^{M+1} b_{M+1-\ell} \lambda^{-\ell}, \quad h_M = \sum_{\ell=1}^{M+1} c_{M+1-\ell} \lambda^{-\ell}, \quad M \in \mathbb{N}_0. \quad (3.9)$$

Then one may obtain the following result.

**Theorem 3.2.** Assume  $u, v$  are solutions of the  $r$ -th HF lattice and the  $M$ -th stationary HF lattice (2.24). Then for fixed  $M \in \mathbb{N}_0$ ,

$$g = g_M, \quad f = f_M, \quad h = h_M, \quad (3.10)$$

solve the system (3.6).

*Proof.* First we show that (3.10) satisfy the first equation in (3.6). To this end we make the ansatz

$$g = \sum_{\ell=1}^{M+1} \check{a}_{M+1-\ell} \lambda^{-\ell}, \quad f = \sum_{\ell=1}^{M+1} \check{b}_{M+1-\ell} \lambda^{-\ell}, \quad h = \sum_{\ell=1}^{M+1} \check{c}_{M+1-\ell} \lambda^{-\ell}, \quad (3.11)$$

where  $\check{a}_j, \check{b}_j, \check{c}_j, j = 1, \dots, M+1$  are polynomials with respect to  $u, v$  and their shift. Inserting (3.11) into (3.6), one derives

$$\check{a}_j = a_j, \quad \check{b}_j = b_j, \quad \check{c}_j = c_j, \quad j = 1, \dots, M+1. \quad (3.12)$$

and hence (3.10) and (3.9) hold. To complete the proof, one has to prove that  $g, f, h$  also satisfy (3.6). Since  $u, v$  are both solutions of the  $M$ -th time-independent lattice, there exists a common

eigenfunction  $\chi = (\chi_1, \chi_2)$  for the following two linear problems

$$\chi^+ = U\chi, \quad V_M\chi = \lambda^{-(M+1)}y\chi, \quad (3.13)$$

where the  $2 \times 2$  matrix

$$V_M = \begin{pmatrix} g_M & f_M \\ h_M & -g_M \end{pmatrix}. \quad (3.14)$$

The relation (3.13) implies

$$\lambda^{M+1}(g_M^2 + f_M h_M)^{1/2} = y.$$

Next we introduce a new function

$$\varphi' = \frac{\chi_2}{\chi_1} = \frac{\lambda^{-(M+1)}y - g_M}{f_M} = \frac{h_M}{\lambda^{-(M+1)}y + g_M}. \quad (3.15)$$

A direct computation shows that  $\varphi'$  satisfies

$$(\lambda - 2u\delta)\varphi'^+ - 2\delta\varphi'\varphi' = 2uv\delta + (\lambda + 2v\delta)\varphi'. \quad (3.16)$$

Differentiating (3.16) with respect to  $t_N$  then yields

$$((\lambda - 2u\delta - 2\delta\varphi')E^+ - 2\delta\varphi'^+ - (\lambda + 2v\delta))\varphi'_{t_r} = (2uv\delta)_{t_r} + (2v\delta)_{t_r}\varphi' + (\lambda + 2v\delta)\varphi'_{t_r}. \quad (3.17)$$

On the other hand, we have

$$\begin{aligned} & ((\lambda - 2u\delta - 2\delta\varphi')E^+ - 2\delta\varphi'^+ - (\lambda + 2v\delta))(C - 2A\varphi' - B(\varphi')^2) \\ &= (\lambda - 2u\delta - 2\delta\varphi')C^+ - 2\delta\varphi'^+C - (\lambda + 2v\delta)C + \left(\frac{2uv\delta + (\lambda + 2v\delta)\varphi'}{\varphi'^+}E^+ \right. \\ & \quad \left. + \frac{2uv\delta - (\lambda - 2u\delta)\varphi'^+}{\varphi'}\right)(-2A\varphi' - B(\varphi')^2) \\ &= (2uv\delta)_{t_r} + (2v\delta)_{t_r}\varphi' + (\lambda + 2v\delta)\varphi'_{t_r}. \end{aligned} \quad (3.18)$$

Combining (3.17) with (3.18), one obtains

$$(\lambda - 2u\delta - 2\delta\varphi')E^+ - 2\delta\varphi'^+ - (\lambda + 2v\delta))(\varphi'_{t_r} - C + 2A\varphi' + B(\varphi')^2) = 0, \quad (3.19)$$

and consequently,

$$\varphi'_{t_r} = C - 2A\varphi' - B(\varphi')^2. \quad (3.20)$$

On the other hand, from the definition of  $\psi_i$ ,  $\phi_i$ , one infers that the function  $\varphi$  (see (3.32)) satisfies

$$(\lambda - 2u\delta)\varphi^+ - 2\delta\varphi^+\varphi - 2uv\delta - (\lambda + 2v\delta)\varphi = 0. \quad (3.21)$$

Then by (3.1), we have

$$\begin{aligned} \left(\frac{\psi_2}{\psi_1}\right)_{t_r} &= \frac{\psi_{2,t_N}\psi_1 - \psi_2\psi_{1,t_N}}{\psi_1^2} \\ &= \frac{C\psi_1 - A\psi_2}{\psi_1} - \frac{\psi_2}{\psi_1} \frac{A\psi_1 + B\psi_2}{\psi_1} \\ &= C - 2A\frac{\psi_2}{\psi_1} - B\left(\frac{\psi_2}{\psi_1}\right)^2. \end{aligned} \quad (3.22)$$

Similarly one may derive

$$\left(\frac{\phi_2}{\phi_1}\right)_{t_r} = C - 2A\frac{\phi_2}{\phi_1} - B\left(\frac{\phi_2}{\phi_1}\right)^2. \quad (3.23)$$

Thus one arrives at

$$\phi_{t_r} = C - 2A\phi - B\phi^2. \quad (3.24)$$

A comparison of (3.17), (3.20), (3.21) and (3.24) yields

$$\phi = F(\lambda)\phi', \quad (3.25)$$

where  $F(\lambda)$  is an arbitrary function of  $\lambda$ . Without loss of generality, one can take  $F(\lambda) \equiv 1$ . Therefore, one infers from (3.25) that there exists common eigenfunctions of the following linear system

$$\chi^+ = U\chi, \quad (3.26)$$

$$\chi_{t_r} = V\chi, \quad (3.27)$$

$$V_M\chi = \lambda^{-(M+1)}y\chi. \quad (3.28)$$

From the compatibility condition of (3.27) and (3.28), we finally obtain

$$V_{M,t_r} = [V, V_M], \quad (3.29)$$

which completes the proof.  $\square$

Next let us define

$$\Delta_{\pm}(\lambda, n, t_r) = \pm 2^{-1}(\lambda^2 - 2\lambda)^{-n}(\psi_1(\lambda, n, t_r)\phi_2(\lambda, n, t_r) - \phi_1(\lambda, n, t_r)\psi_2(\lambda, n, t_r)). \quad (3.30)$$

It is not difficult to check that  $\Delta_{\pm}$  satisfy

$$\begin{aligned} \Delta_{\pm}^2(\lambda, n, t_r) &= g^2(\lambda, n, t_r) + f(\lambda, n, t_r)h(\lambda, n, t_r), \\ \Delta_+(\lambda, n, t_r) + g(\lambda, n, t_r) &= -\Delta_-(\lambda, n, t_r) + g(\lambda, n, t_r) = (\lambda^2 - 2\lambda)^{-n}\psi_1(\lambda, n, t_r)\phi_2(\lambda, n, t_r), \\ \phi_2(\lambda, n, t_r)/\phi_1(\lambda, n, t_r) &= (\Delta_-(\lambda, n, t_r) - g(\lambda, n, t_r))/f(\lambda, n, t_r), \\ \psi_2/\psi_1(\lambda, n, t_r) &= h(\lambda, n, t_r)/(\Delta_+(\lambda, n, t_r) + g(\lambda, n, t_r)) = (\Delta_+(\lambda, n, t_r) - g(\lambda, n, t_r))/f(\lambda, n, t_r). \end{aligned}$$

For fixed  $M \in \mathbb{N}$ , the spectral curve  $\mathcal{K}_{M-1}$  of genus  $M-1$  can be introduced as follows

$$\mathcal{K}_{M-1} : y^2 = \lambda^{2M+2}(f^2 + gh) = \prod_{j=0}^{2M-1} (\lambda - E_j), \quad E_j \in \mathbb{C}.$$

Throughout this paper, one assumes that

$$E_j \neq E_{\ell}, \quad j \neq \ell, \quad j = 0, 1, \dots, 2M-1. \quad (3.31)$$

The next step is crucial; we lift the functions  $\phi_2/\phi_1$  and  $\psi_2/\psi_1$  on the Riemann surface  $\mathcal{K}_{M-1}$  by

$$\varphi(P, x, t_r) = \begin{cases} \psi_2(\lambda, n, t_r)/\psi_1(\lambda, n, t_r), & P \in \mathcal{K}_{M-1}^+ \\ \phi_2(\lambda, n, t_r)/\phi_1(\lambda, n, t_r), & P \in \mathcal{K}_{M-1}^- \end{cases} \quad (3.32)$$

where  $\mathcal{K}_{M-1}^{\pm}$  are two sheets of  $\mathcal{K}_{M-1}$ . The spectral curve  $\mathcal{K}_{M-1}$  is compactified by the point at infinity  $P_{\infty}$ . A point on  $\mathcal{K}_{M-1}$  is denoted by  $P = (\lambda, y(P))$ , where  $\lambda \in \mathbb{C}$  and  $y(P)$  is a holomorphic

function defined on the two sheets  $\mathcal{K}_{M-1}^{\pm}$  of  $\mathcal{K}_{M-1}$ :

$$y(P) = \begin{cases} \sqrt{\prod_{j=0}^{2M-1} (\lambda - E_j)}, & P \in \mathcal{K}_{M-1}^+ \\ -\sqrt{\prod_{j=0}^{2M-1} (\lambda - E_j)}, & P \in \mathcal{K}_{M-1}^- \end{cases} \quad (3.33)$$

In particular, one introduces  $P_{0\pm} = (0, \pm(\prod_{j=0}^{2M-1} E_j)^{1/2})$ .

Based on above preparations, we turn to study the analytic and asymptotic properties of  $\varphi$ .

**Lemma 3.3.** *The function  $\varphi$  defined in (3.32) is meromorphic. Its divisor is given by <sup>a</sup>*

$$(\varphi(\cdot, n, t_r)) = \mathcal{D}_{\hat{\mathbf{v}}(n, t_r) \hat{\mathbf{v}}_M(n, t_r)} - \mathcal{D}_{\hat{\mathbf{u}}(n, t_r) \hat{\mathbf{u}}_M(n, t_r)}. \quad (3.34)$$

Here the notations  $\hat{\mathbf{v}}(n, t_r)$  and  $\hat{\mathbf{u}}(n, t_r)$  represent two vectors

$$\hat{\mathbf{v}}(n, t_r) = (\hat{v}_1(n, t_r), \dots, \hat{v}_{M-1}(n, t_r)), \quad \hat{\mathbf{u}}(n, t_r) = (\hat{u}_1(n, t_r), \dots, \hat{u}_{M-1}(n, t_r)).$$

**Lemma 3.4.** *Near the points  $P_{0\pm}$ , the function  $\varphi$  has the following expansions*

$$\varphi(\cdot, n, t_r) = c_{\infty\pm} + O(\zeta), \quad P \rightarrow P_{\infty\pm}, \quad \zeta = \lambda^{-1} \quad (3.35)$$

and

$$\varphi(\cdot, n, t_r) = \begin{cases} -v^- - E^{-1} \frac{(v - v^-)(u - v)}{2(u - v^-)} \zeta + O(\zeta^2), & P \rightarrow P_{0+}, \quad \zeta = \lambda, \\ -u + \frac{(u^+ - u)(u - v)}{2(u^+ - v)} \zeta + O(\zeta^2). & P \rightarrow P_{0-}, \quad \zeta = \lambda, \end{cases} \quad (3.36)$$

where  $c_{\infty\pm}$  are lattice constants.

*Proof.* First one observes that the expansion (3.35) can be easily derived from the definition of  $\varphi$ . Thus it suffices to compute the expansions of  $\varphi$  near  $P_{0\pm}$ . Using the relation

$$\begin{aligned} \Delta_{\pm}^2 = g^2 + fh &= \lambda^{-2(M+1)} \left( \left( \sum_{i=0}^N g_i \lambda^i \right)^2 - \left( \sum_{i=0}^N f_i \lambda^i \right) \left( \sum_{i=0}^N h_i \lambda^i \right) \right) \\ &= \lambda^{-2(M+1)} \sum_{i=0}^N \left( \sum_{k=0}^i (g_k g_{i-k} - f_k h_{i-k}) \right) \lambda^i, \end{aligned}$$

one infers

$$\Delta_{\pm} \stackrel{\zeta \rightarrow 0}{=} \pm \zeta^{-(M+1)} \sqrt{g_0^2 + f_0 h_0} \left( 1 + \frac{1}{2} \frac{2g_0 g_1 + f_0 h_1 + f_1 h_0}{g_0^2 + f_0 h_0} \zeta + O(\zeta^2) \right), \quad (3.37)$$

<sup>a</sup>For the meaning of these notations one may refer to [4].

as  $P \rightarrow P_{0\pm}$ . Then from the relation  $g_0^2 + f_0 h_0 = 1$  and (3.32), it follows

$$\begin{aligned} \varphi(\cdot, n, t_r) &= \begin{cases} \frac{1 - \frac{u+v^-}{u-v^-}}{\frac{2}{u-v^-}} + O(\zeta), & P \rightarrow P_{0+}, \lambda = \zeta, \\ \frac{-1 - \frac{u+v^-}{u-v^-}}{\frac{2}{u-v^-}} + O(\zeta), & P \rightarrow P_{0-}, \lambda = \zeta, \end{cases} \\ &= \begin{cases} -v^- + O(\zeta), & P \rightarrow P_{0+}, \lambda = \zeta, \\ -u + O(\zeta), & P \rightarrow P_{0-}, \lambda = \zeta. \end{cases} \end{aligned} \quad (3.38)$$

To get the coefficient of  $\zeta$  in (3.36), we use the equation (see (3.16))

$$(\lambda - 2u\delta)\varphi^+ - 2\delta\varphi^+\varphi = 2uv\delta + (\lambda + 2v\delta)\varphi. \quad (3.39)$$

Inserting the ansatz

$$\varphi = \varphi_{0\pm} + \varphi_{1\pm}\zeta + \varphi_{2\pm}\zeta^2 + O(\zeta^3), \text{ as } P \rightarrow P_{0\pm} \quad (3.40)$$

into (3.39) and comparing the coefficients of  $\zeta$  lead to

$$\varphi_{0\pm}^+ - 2u\delta\varphi_{1\pm}^+ - 2\delta\varphi_{0\pm}^+\varphi_{1\pm} - 2\delta\varphi_{0\pm}\varphi_{1\pm}^+ = \varphi_{0\pm} + 2v\delta\varphi_{1\pm}. \quad (3.41)$$

Explicitly, Eq. (3.41) can be equivalently written as

$$-v - 2u\delta\varphi_{1+}^+ + 2v\delta\varphi_{1+} + 2v^-\delta\varphi_{1+}^+ = -v^- + 2v\delta\varphi_{1+}, \quad (3.42)$$

$$-u^+ - 2u\delta\varphi_{1-}^+ + 2u^+\delta\varphi_{1-} + 2u\delta\varphi_{1-}^+ = -u + 2v\delta\varphi_{1-}, \quad (3.43)$$

which implies

$$\varphi_{1+} = E^{-1} \frac{(v - v^-)(u - v)}{2(v^- - u)}, \quad \varphi_{1-} = \frac{(u^+ - u)(u - v)}{2(u^+ - v)}. \quad (3.44)$$

Using similar method we can prove (3.35).  $\square$

#### 4. Quasi-periodic solutions

In this section, we obtain theta function representations for the meromorphic function  $\varphi$  and quasi-periodic solutions  $u, v$ .

First, we choose a convenient base point  $Q_0 \in K_{M-1} \setminus \{P_{\infty\pm}, P_{0\pm}\}$ . The Abel maps  $\underline{A}_{Q_0}(\cdot), \underline{\alpha}_{Q_0}(\cdot)$  are defined by

$$\underline{A}_{Q_0} : X \rightarrow J(X) = \mathbb{C}^{M-1}/L_{M-1},$$

$$\begin{aligned} P \mapsto \underline{A}_{Q_0}(P) &= (A_{Q_0,1}(P), \dots, A_{Q_0,M-1}(P)) \\ &= \left( \int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_{M-1} \right) \pmod{L_{M-1}}, \end{aligned}$$

and

$$\begin{aligned}\underline{\alpha}_{Q_0} : \text{Div}(X) &\rightarrow J(X), \\ \mathcal{D} &\mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_{M-1}} \mathcal{D}(P) \underline{A}_{Q_0}(P),\end{aligned}$$

where  $L_{M-1} = \{\underline{z} \in \mathbb{C}^{M-1} \mid \underline{z} = \underline{N} + \Gamma \underline{M}, \underline{N}, \underline{M} \in \mathbb{Z}^{M-1}\}$ , and  $\Gamma, \underline{\Xi}_{Q_0}$  are the Riemann matrix and the vector of Riemann constants, respectively. Moreover, we choose a homology basis  $\{a_j, b_j\}_{j=1}^{M-1}$  on  $X$  in such a way that the intersection matrix of the cycles satisfies

$$a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j, k = 1, \dots, n. \quad (4.1)$$

For brevity, we introduce

$$\begin{aligned}\underline{z}(P, \underline{Q}) &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \\ P &\in \mathcal{K}_{M-1}, \quad \underline{Q} = (Q_1, \dots, Q_{M-1}) \in \sigma^{M-1} \mathcal{K}_{M-1},\end{aligned} \quad (4.2)$$

where  $\underline{z}(\cdot, \underline{Q})$  is independent of the choice of  $Q_0$ . The Riemann theta function  $\theta(\underline{z})$  associated with  $X$  and the homology basis  $\{a_j, b_j\}_{j=1}^{M-1}$  is defined by

$$\theta(\underline{z}) = \sum_{\underline{n} \in \mathbb{Z}} \exp(2\pi i \langle \underline{n}, \underline{z} \rangle + \pi i \langle \underline{n}, \underline{n} \Gamma \rangle), \quad \underline{z} \in \mathbb{C}^{M-1},$$

where  $\langle \underline{B}, \underline{C} \rangle = \sum_{j=1}^{M-1} \overline{B}_j C_j$  is the scalar product in  $\mathbb{C}^{M-1}$ .

Next, we introduce the differential of the third kind with simple zero and pole respectively at  $\hat{v}_M(n, t_r)$  and  $\hat{\mu}_M(n, t_r)$  by

$$\omega_{\hat{v}_M(n, t_r) \hat{\mu}_M(n, t_r)}^{(3)}(P) = \left( \frac{y + y(\hat{v}_M(n, t_r))}{\lambda - v_M(n, t_r)} - \frac{y + y(\hat{\mu}_M(n, t_r))}{\lambda - \mu_M(n, t_r)} \right) \frac{d\lambda}{2y} + \frac{\lambda_0}{y} \prod_{i=1}^{M-2} (\lambda - \lambda_i) d\lambda,$$

where  $\lambda_j, j = 0, \dots, M-2$ , are constants that are uniquely determined by the requirement of vanishing  $a$ -period, i.e.

$$\int_{a_j} \omega_{\hat{v}_M(n, t_r) \hat{\mu}_M(n, t_r)}^{(3)}(P) = 0, \quad j = 1, \dots, M-1. \quad (4.3)$$

A simple computation shows

$$\int_{Q_0}^P \omega_{\hat{v}_M(n, t_r) \hat{\mu}_M(n, t_r)}^{(3)}(P) = \begin{cases} \ln \zeta + d_0(n, t_r) + O(\zeta), & P \rightarrow \hat{v}_M(n, t_r), \\ -\ln \zeta + d_1(n, t_r) + O(\zeta), & P \rightarrow \hat{\mu}_M(n, t_r), \\ d_{P_{0\pm}} + O(\zeta) & P \rightarrow P_{0\pm}, \\ d_{P_{\infty\pm}} + d_2(n, t_r)\zeta + O(\zeta^2), & P \rightarrow P_{\infty\pm}. \end{cases}$$

Here  $d_0(n, t_r), d_1(n, t_r), d_2(n, t_r)$  are functions of variables  $n, t_r$  and  $d_{P_{\infty\pm}}, d_{P_{0\pm}}$  are integration constants.

Theta function representations for quasiperiodic solutions of the  $r$ th HF lattice can be obtained as follows.

**Theorem 4.1.** *The function  $\varphi$  defined in (3.32) is meromorphic on  $\mathcal{K}_{M-1}$  and has the following theta function representation*

$$\varphi(P, n, t_r) = c_{P_{\infty+}} \frac{\theta(\underline{\lambda}(P_{\infty+}, \underline{\hat{\mu}}(n, t_r)))}{\theta(\underline{\lambda}(P_{\infty+}, \underline{\hat{\nu}}(n, t_r)))} \frac{\theta(\underline{\lambda}(P, \underline{\hat{\nu}}(n, t_r)))}{\theta(\underline{\lambda}(P, \underline{\hat{\mu}}(n, t_r)))} \exp \left( \int_{Q_0}^P \omega_{\hat{\nu}_M(n, t_r) \hat{\mu}_M(n, t_r)}^{(3)} - d_{P_{\infty+}} \right). \quad (4.4)$$

*Proof.* To derive the formula (4.4), one has to use Riemann vanishing theorem and Riemann-Roch theorem in standard literature [3]. From Lemma 2.3, one infers that the function  $\varphi(P, n, t_N)$  take the form

$$\varphi(P, n, t_r) = C(n, t_r) \frac{\theta(\underline{\lambda}(P, \underline{\hat{\nu}}(n, t_r)))}{\theta(\underline{\lambda}(P, \underline{\hat{\mu}}(n, t_r)))} \exp \left( \int_{Q_0}^P \omega_{\hat{\nu}_M(n, t_r) \hat{\mu}_M(n, t_r)}^{(3)} \right), \quad (4.5)$$

where  $C(n, t_r)$  is an undetermined function. Then taking the limit  $P \rightarrow P_{\infty+}$  on both sides of (4.5), one derives

$$c_{P_{\infty+}} = C(n, t_N) \frac{\theta(\underline{\lambda}(P_{\infty+}, \underline{\hat{\nu}}(n, t_r)))}{\theta(\underline{\lambda}(P_{\infty+}, \underline{\hat{\mu}}(n, t_r)))} \exp(d_{P_{\infty+}}). \quad (4.6)$$

Inserting (4.6) into (4.5) and eliminating  $C(n, t_r)$ , one finally obtains (4.4).  $\square$

Next, we obtain theta function representations for solutions  $u, v$  with the help of (4.4).

**Theorem 4.2.** *Quasiperiodic solutions of the  $r$ -th HF lattice have the following Riemann theta function representation*

$$u(n, t_r) = -c_{P_{\infty+}} \frac{\theta(\underline{\lambda}(P_{\infty+}, \underline{\hat{\mu}}(n, t_r)))}{\theta(\underline{\lambda}(P_{\infty+}, \underline{\hat{\nu}}(n, t_r)))} \frac{\theta(\underline{\lambda}(P_{0-}, \underline{\hat{\nu}}(n, t_r)))}{\theta(\underline{\lambda}(P_{0-}, \underline{\hat{\mu}}(n, t_r)))} \exp(d_{P_{0-}} - d_{P_{\infty+}}), \quad (4.7)$$

$$v(n, t_r) = -c_{P_{\infty+}} \frac{\theta(\underline{\lambda}(P_{\infty+}, \underline{\hat{\mu}}^+(n, t_r)))}{\theta(\underline{\lambda}(P_{\infty+}, \underline{\hat{\nu}}^+(n, t_r)))} \frac{\theta(\underline{\lambda}(P_{0+}, \underline{\hat{\nu}}^+(n, t_r)))}{\theta(\underline{\lambda}(P_{0+}, \underline{\hat{\mu}}^+(n, t_r)))} \exp(d_{P_{0+}} - d_{P_{\infty+}}). \quad (4.8)$$

*Proof.* Using the coordinate  $\lambda = \zeta$  near  $P_{0\pm}$ , we have

$$\begin{aligned} \varphi(n, x, t_r) &= c_{P_{\infty+}} \frac{\theta(\underline{\lambda}(P_{\infty+}, \underline{\hat{\mu}}(n, t_r)))}{\theta(\underline{\lambda}(P_{\infty+}, \underline{\hat{\nu}}(n, t_r)))} \frac{\theta(\underline{\lambda}(P, \underline{\hat{\nu}}(x, t_r)))}{\theta(\underline{\lambda}(P, \underline{\hat{\mu}}(x, t_r)))} \\ &\quad \times \exp \left( \int_{Q_0}^P \omega_{\hat{\nu}_M(x, t_r) \hat{\mu}_M(x, t_r)}^{(3)} - d_{P_{\infty+}} \right) \\ &\stackrel{\zeta \rightarrow 0}{=} c_{P_{\infty+}} \frac{\theta(\underline{\lambda}(P_{\infty+}, \underline{\hat{\mu}}(n, t_r)))}{\theta(\underline{\lambda}(P_{\infty+}, \underline{\hat{\nu}}(n, t_r)))} \frac{\theta(\underline{\lambda}(P_{0\pm}, \underline{\hat{\nu}}(x, t_r)))}{\theta(\underline{\lambda}(P_{0\pm}, \underline{\hat{\mu}}(x, t_r)))} \\ &\quad \times \exp(d_{P_{0\pm}} - d_{P_{\infty+}}) (1 + O(\zeta)). \end{aligned} \quad (4.9)$$

Comparing the leading coefficients of (3.36) and (4.9) naturally gives rise to (4.7) and (4.8).  $\square$

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## References

- [1] V.E. Adler, Discretizations of the Landau-Lifshits equation, *Theor. Math. Phys.*, **124** (2000) 897–908.
- [2] L. Faddeev, L. Takhtajan, *Hamiltonian methods in the theory of solitons* (Springer Science & Business Media, Berlin, Heidelberg, 2007).
- [3] H.M. Farkas, I. Kra, *Riemann surfaces* (Springer, Berlin, 1992).
- [4] F. Gesztesy, H. Holden, J. Michor, G. Teschl, *Soliton equations and their algebro-geometric solutions: Volume II, (1+1)-dimensional discrete models* (Cambridge University Press, Cambridge, 2008).
- [5] F. Khanizadeh, A.V. Mikhailov, J.P. Wang, Darboux transformations and recursion operators for differential-difference equations, *Theor. Math. Phys.*, **177** (2013) 1606–1654.
- [6] A.B. Shabat, R.I. Yamilov, Symmetries of nonlinear chains, *Leningrad Math. J.* **2** (1991) 377–400.
- [7] E. K. Sklyanin, Some algebraic structures connected with the Yang-Baxter equation, *Funct. Anal. Appl.* **16** (1982) 263–270.