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The Riemann–Hilbert problem to coupled nonlinear Schrödinger equation: Long-time dynamics on the half-line

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We derive the long-time asymptotics for the solution of initial-boundary value problem of coupled nonlinear Schrödinger equation whose Lax pair involves 3×3 matrix in present paper. Based on a nonlinear steepest descent analysis of an associated 3×3 matrix Riemann–Hilbert problem, we can give the precise asymptotic formulas for the solution of the coupled nonlinear Schrödinger equation on the half-line.

Keywords: Coupled nonlinear Schrödinger equation; Riemann–Hilbert problem; Initial-boundary value problem; Long-time asymptotics.

2000 Mathematics Subject Classification: 35K20, 37K40, 35Q15

1. Introduction

The well-known nonlinear steepest descent method first introduced by Deift and Zhou in [14] provides a powerful technique for determining asymptotics of solutions of nonlinear integrable evolution equations. This approach since then has been successfully applied in analyzing the long-time asymptotics of initial-value problems (IVPs) for a number of nonlinear integrable partial differential equations (PDEs) associated with 2×2 matrix spectral problems including the modified Korteweg-de Vries (mKdV) equation [14], the defocusing nonlinear Schrödinger (NLS) equation [15], the KdV equation [19], the derivative NLS equation [25], the Fokas–Lenells equation [28], the short pulse equation [10] and the Kundu–Eckhaus equation [27]. Moreover, by combining the ideas of [14] with the so-called "g-function mechanism" [13], it is also possible to study asymptotics of solutions of the IVPs with shock-type oscillating initial data [11], nondecaying step-like initial data [6, 22], nonzero boundary conditions at infinity [4] for various integrable equations. There also exists some meaningful papers [8, 9, 17] about the study of long-time asymptotics for the IVPs of integrable nonlinear evolution equations associated with 3×3 matrix spectral problems. For the large-time asymptotic analysis of the initial-boundary value problems (IBVPs) of integrable nonlinear PDEs, Lenells et al. derived some interesting asymptotic formulas for the solutions of mKdV equation [23] and derivative NLS equation [3] by using the steepest descent method. Furthermore, the long-time asymptotics for the focusing NLS equation with t-periodic boundary condition on the half-line is analyzed in [5]. We also have done some work about determining the long-time asymptotics for integrable equations on the half-line, see [20, 21]. However, there is only a little of literature [7] to consider the asymptotic behaviors for integrable nonlinear PDEs with Lax

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pairs involving 3×3 matrices on the half-line. Thus, it is necessary and important to consider the large-time asymptotic behaviors for the IBVPs of integrable equations with 3×3 Lax pairs on the half-line.

In particular, the purpose of this paper is aim to consider the long-time asymptotics for the IBVP of the coupled nonlinear Schrödinger (CNLS) equation

$$\begin{cases} \mathrm{i}u_t + u_{xx} + 2(|u|^2 + |v|^2)u = 0, \\ \mathrm{i}v_t + v_{xx} + 2(|u|^2 + |v|^2)v = 0, \end{cases} \tag{1.1}$$

posed in the quarter-plane domain

$$\Omega = \{0 \le x < \infty,\ 0 \le t < \infty\}.$$

We will denote the initial data, Dirichlet and Neumann boundary values of (1.1) as follows:

$$\begin{aligned} &u(x,0) = u_0(x),\ \ v(x,0) = v_0(x),\ \ x \ge 0, \\ &u(0,t) = g_0(t),\ \ u_x(0,t) = g_1(t),\ \ v(0,t) = h_0(t),\ \ v_x(0,t) = h_1(t),\ \ t \ge 0. \end{aligned} \tag{1.2}$$

We also suppose that $\{u_0(x), v_0(x)\}$ and $\{g_j(t), h_j(t)\}_0^1$ belong to the Schwartz class $S(\mathbb{R}_+)$. Equation (1.1) was also called Manakov model, which can be used to describe the propagation of an optical pulse in a birefringent optical fiber [26]. Subsequently, this system also arises in the context of multi-component Bose-Einstein condensates [12]. Due to its physical interest, equation (1.1) has been widely studied. It is noted that the IBVP for (1.1) on the half-line has been investigated via the Fokas method [16], where it was shown that the solution $\{u(x,t), v(x,t)\}$ can be expressed in terms of the unique solution of a 3×3 matrix Riemann–Hilbert (RH) problem formulated in the complex k-plane (see [18]). Meanwhile, the leading-order long-time asymptotics of the Cauchy problem of equation (1.1) was obtained in [17].

Our goal here is to derive the long-time asymptotics of the solution of (1.1) on the half-line by performing a nonlinear steepest descent analysis of the associated RH problem. Compared with other integrable equations, the asymptotic analysis of (1.1) presents some distinctive features: (1) Since the RH problem associated with (1.1) involves 3×3 jump matrix $J(x,t,k)$, we first introduce two 2×1 vector-valued spectral functions $r_1(k)$, $h(k)$ and a 1×2 vector-valued spectral function $r(k)$, and then rewrite our main RH problem as a 2×2 block one. This procedure is more convenient for the following long-time asymptotic analysis. (2) As we all known, the important step of the steepest descent method is to split the jump matrix $J(x,t,k)$ into an appropriate upper/lower triangular form. This immediately leads to construct a $\delta(k)$ function to remove the middle matrix term, however, the function δ satisfies a 2×2 matrix RH problem in our present problem. The unsolvability of the 2×2 matrix function $\delta(k)$ is a challenge when we perform the scaling transformation to reduce the RH problem to a model RH problem. Fortunately, we can follow the idea introduced in [18] to use the available function $\det\delta(k)$ which can be explicitly solved by the Plemelj formula to approximate the function $\delta(k)$ by error control. (3) The relevant RH problem for the Cauchy problem (1.1) considered in [17] only has a jump across $\mathbb{R}$, whereas the RH problem for the IBVP also has a jump across $\mathrm{i}\mathbb{R}$, and the jump across this line involves the spectral function $h(k)$. Moreover, during the asymptotic analysis, one should find an analytic approximation $h_a(t,k)$ of $h(k)$. (4) Recalling the meaningful work about analyzing the long-time asymptotics for the Degasperis–Procesi equation on the half-line, the analysis presented in [7] shows that the structure of the jump matrix, which is 3×3, is essentially 2×2 (under an appropriate change of basis), whereas the analysis given in our present paper is more general.

The main result of this paper is stated as the following theorem.

Theorem 1.1. *Assume the assumption 1 be valid. Then, for any positive constant N, as $t\to\infty$, the solution $\big(u(x,t)\ v(x,t)\big)$ of the IBVP for the CNLS equation* (1.1) *on the half-line satisfies the following asymptotic formula*

$$\big(u(x,t)\ v(x,t)\big)=\frac{\big(u_a(x,t)\ v_a(x,t)\big)}{\sqrt{t}}+O\left(\frac{\ln t}{t}\right),\quad t\to\infty,\ 0\le x\le Nt, \tag{1.3}$$

where the error term is uniform with respect to x in the given range, and the leading-order coefficient $\big(u_a(x,t)\ v_a(x,t)\big)$ is defined by

$$\big(u_a(x,t)\ v_a(x,t)\big)=\frac{e^{-\frac{\pi\nu}{2}}\nu\Gamma(-i\nu)r(k_0)}{2\sqrt{\pi}}e^{i\alpha(\xi,t)}, \tag{1.4}$$

where the 1×2 *vector-valued spectral function $r(k)$ is defined by* (3.3)*, which determined by all the initial and boundary values,*

$$\xi=\frac{x}{t},\ \ \nu=\frac{1}{2\pi}\ln(1+r(k_0)r^{\dagger}(k_0)),\ \ k_0=-\frac{\xi}{4},$$

and

$$\alpha(\xi,t)=-\frac{\pi}{4}+\nu\ln(8t)+4k_0^2t+\frac{1}{\pi}\int_{-\infty}^{k_0}\ln(k_0-s)d\ln(1+r(s)r^{\dagger}(s)).$$

Remark 1.1. The asymptotic formula for the half-line problem obtained in Theorem 1.1 has the exact same functional form for the pure initial value problem. The only difference is that the definition of the spectral function $r(k)$ for the half-line problem, which enters the asymptotic formula, involves not only the initial data but also the boundary values. In other words, the only effect of the boundary is to modify $r(k)$. We can understand this as follows: since x grows faster than t in the given region, the distance to the boundary eventually gets so big that what happens at the boundary has a small effect on the solution (recall that we assume that the boundary values decay as $t\to\infty$). Therefore, the boundary values and the initial data play similar roles and it is natural that they enter the asymptotic formula in similar ways.

The outline of the paper is following. In Section 2, we recall how the solution of CNLS equation (1.1) on the half-line can be expressed in terms of the solution of a 3×3 matrix RH problem. In section 3, we present the detailed derivation of the long-time asymptotics for the solution of CNLS equation, that is, we prove Theorem 1.1.

2. Preliminaries

In this section, we give a short review of the RH problem for (1.1) on the half-line, see [18] for further details. The Lax pair of equation (1.1) is

$$\begin{cases}\mu_x(x,t,k)-\mathrm{i}k[\sigma,\mu(x,t,k)]=U(x,t)\mu(x,t,k),\\ \mu_t(x,t,k)-2\mathrm{i}k^2[\sigma,\mu(x,t,k)]=V(x,t,k)\mu(x,t,k).\end{cases} \tag{2.1}$$

where $\mu(x,t,k)$ is a 3×3 matrix-valued eigenfunction, $k\in\mathbb{C}$ is the spectral parameter, and

$$\sigma=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad U(x,t)=\mathrm{i}\begin{pmatrix}0&u(x,t)&v(x,t)\\\overline{u}(x,t)&0&0\\\overline{v}(x,t)&0&0\end{pmatrix},\tag{2.2}$$

$$V(x,t,k)=2kU(x,t)+\mathrm{i}U_x(x,t)\sigma+\mathrm{i}U^2(x,t)\sigma.\tag{2.3}$$

Let $\{\gamma_j\}_1^3$ denote contours in the (x,t)-plane connecting (x_j,t_j) with (x,t), and $(x_1,t_1)=(0,\infty)$, $(x_2,t_2)=(0,0)$, $(x_3,t_3)=(\infty,t)$. The contours can be chosen to consist of straight line segments parallel to the x- or t-axis. We define three eigenfunctions $\{\mu_j\}_1^3$ of the Lax pair (2.1) by the solutions of the following integral equations

$$\mu_j(x,t,k)=I+\int_{\gamma_j}\mathrm{e}^{\mathrm{i}(kx+2k^2t)\widehat{\sigma}}w_j(x',t',k),\quad j=1,2,3,\tag{2.4}$$

where the closed one-form $w_j(x,t,k)$ is defined by

$$w_j(x,t,k)=\mathrm{e}^{-\mathrm{i}(kx+2k^2t)\widehat{\sigma}}[U(x,t)\mathrm{d}x+V(x,t,k)\mathrm{d}t]\mu_j(x,t,k),\tag{2.5}$$

and $\widehat{\sigma}$ denote the operators which act on a 3×3 matrix X by $\widehat{\sigma}X=[\sigma,X]$, then $\mathrm{e}^{\widehat{\sigma}}X=\mathrm{e}^{\sigma}X\mathrm{e}^{-\sigma}$. It then can be shown that the functions $\{\mu_j\}_1^3$ are bounded and analytical for $k\in\mathbb{C}$ while k belongs to

$$\mu_1:(D_2,D_3,D_3),\ \ \mu_2:(D_1,D_4,D_4),\ \ \mu_3:(\mathbb{C}_-,\mathbb{C}_+,\mathbb{C}_+),\tag{2.6}$$

where D_n denotes nth quadrant, $1\le n\le 4$, $\mathbb{C}_+$ and $\mathbb{C}_-$ denote the upper and lower half complex k-plane, respectively.

We define the matrix-valued spectral functions $s(k)$ and $S(k)$ by the relations

$$\begin{aligned}\mu_3(x,t,k)&=\mu_2(x,t,k)\,\mathrm{e}^{\mathrm{i}(kx+2k^2t)\widehat{\sigma}}s(k),\\\mu_1(x,t,k)&=\mu_2(x,t,k)\,\mathrm{e}^{\mathrm{i}(kx+2k^2t)\widehat{\sigma}}S(k).\end{aligned}\tag{2.7}$$

Evaluation of (2.7) at $(x,t)=(0,0)$ gives the following expressions

$$s(k)=\mu_3(0,0,k),\ \ S(k)=\mu_1(0,0,k).\tag{2.8}$$

Hence, the functions $s(k)$ and $S(k)$ can be obtained respectively from the evaluations at $x=0$ and at $t=0$ of the functions $\mu_3(x,0,k)$ and $\mu_1(0,t,k)$.

On the other hand, we can deduce from the properties of μ_j that $s(k)$ and $S(k)$ have the following properties:

(i) $s(k)$ is bounded and analytic for $k\in(D_3\cup D_4,D_1\cup D_2,D_1\cup D_2)$, $S(k)$ is bounded and analytic for $k\in(D_2\cup D_4,D_1\cup D_3,D_1\cup D_3)$;

(ii) $\det s(k)=1$ for $k\in\mathbb{R}$, $\det S(k)=1$ for $k\in\mathbb{R}\cup\mathrm{i}\mathbb{R}$;

(iii) $s(k)=I+O(k^{-1})$ and $S(k)=I+O(k^{-1})$ uniformly as $k\to\infty$;

(iv)

$$\overline{s}(k)=(s^{-1})^T(k),\quad \overline{S}(k)=(S^{-1})^T(k),\tag{2.9}$$

where $\overline{s}(k)=\overline{s(\overline{k})}$ and $\overline{S}(k)=\overline{S(\overline{k})}$ denote the Schwartz conjugates.

The initial and boundary values of a solution of the CNLS equation (1.1) are not independent. It turns out that the spectral functions $s(k)$ and $S(k)$ must satisfy a surprisingly simple relation

$$S^{-1}(k)s(k) = 0, \quad k \in (\overline{D}_4, \overline{D}_1, \overline{D}_1), \tag{2.10}$$

which called the global relation.

For each $n = 1, \ldots, 4$, we define solution $M_n(x,t,k)$ of (2.1) by the solution of following Fredholm integral equation:

$$(M_n)_{jl}(x,t,k) = \delta_{jl} + \int_{\gamma^n_{jl}} \left(\mathrm{e}^{\mathrm{i}(kx+2k^2t)\widehat{\sigma}} w_n(x',t',k) \right)_{jl}, \quad k \in D_n, \quad j,l = 1,2,3, \tag{2.11}$$

where w_n is given by (2.5), and the contours γ^n_{jl}, $n = 1, 2, 3, 4$, $j,l = 1,2,3$ are defined by

$$\gamma^n_{jl} = \begin{cases} \gamma_1 \text{ if } \mathrm{Re}y_j < \mathrm{Re}y_l \text{ and } \mathrm{Re}z_j \geq \mathrm{Re}z_l, \\ \gamma_2 \text{ if } \mathrm{Re}y_j < \mathrm{Re}y_l \text{ and } \mathrm{Re}z_j < \mathrm{Re}z_l, \\ \gamma_3 \text{ if } \mathrm{Re}y_j \geq \mathrm{Re}y_l, \end{cases} \tag{2.12}$$

and we denote $\mathrm{i}k\sigma = \mathrm{diag}(y_1,y_2,y_3)$, $2\mathrm{i}k^2\sigma = \mathrm{diag}(z_1,z_2,z_3)$. For each $n = 1,\ldots,4$, we define spectral functions $S_n(k)$ by

$$M_n(x,t,k) = \mu_2(x,t,k)\,\mathrm{e}^{\mathrm{i}(kx+2k^2t)\widehat{\sigma}} S_n(k), \quad k \in D_n. \tag{2.13}$$

According to (2.11), the $\{S_n(k)\}_1^4$ can be computed from initial and boundary values alone as well as the spectral functions $s(k)$ and $S(k)$ (the calculation results can see [18]).

Then, equation (2.13) can be rewritten in the form of a 3×3 RH problem as follows:

$$M_+(x,t,k) = M_-(x,t,k)J(x,t,k), \quad k \in \mathbb{R} \cup \mathrm{i}\mathbb{R}, \tag{2.14}$$

where the matrices $M_+(x,t,k)$, $M_-(x,t,k)$, and $J(x,t,k)$ are defined by

$$M(x,t,k) = \begin{cases} M_+(x,t,k), & k \in D_1 \cup D_3, \\ M_-(x,t,k), & k \in D_2 \cup D_4, \end{cases}$$

and

$$\begin{aligned}
J_1 &= \begin{pmatrix} 1 & 0 & 0 \\ -\frac{s_{33}S_{21}-s_{23}S_{31}}{\bar{s}_{11}\overline{W}_{11}}\mathrm{e}^{2\mathrm{i}\theta} & 1 & 0 \\ -\frac{s_{22}S_{31}-s_{32}S_{21}}{\bar{s}_{11}\overline{W}_{11}}\mathrm{e}^{2\mathrm{i}\theta} & 0 & 1 \end{pmatrix}, \\
J_2 &= J_1 J_4^{-1} J_3, \\
J_3 &= \begin{pmatrix} 1 & -\frac{\bar{s}_{33}\overline{S}_{21}-\bar{s}_{23}\overline{S}_{31}}{s_{11}W_{11}}\mathrm{e}^{-2\mathrm{i}\theta} & -\frac{\bar{s}_{22}\overline{S}_{31}-\bar{s}_{32}\overline{S}_{21}}{s_{11}W_{11}}\mathrm{e}^{-2\mathrm{i}\theta} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
J_4 &= \begin{pmatrix} 1+\frac{|s_{12}|^2}{|s_{11}|^2}+\frac{|s_{13}|^2}{|s_{11}|^2} & \frac{s_{12}}{s_{11}}\mathrm{e}^{-2\mathrm{i}\theta} & \frac{s_{13}}{s_{11}}\mathrm{e}^{-2\mathrm{i}\theta} \\ \frac{\bar{s}_{12}}{\bar{s}_{11}}\mathrm{e}^{2\mathrm{i}\theta} & 1 & 0 \\ \frac{\bar{s}_{13}}{\bar{s}_{11}}\mathrm{e}^{2\mathrm{i}\theta} & 0 & 1 \end{pmatrix},
\end{aligned} \tag{2.15}$$

$$W_{11} = \overline{S}_{11}s_{11} + \overline{S}_{21}s_{21} + \overline{S}_{31}s_{31}, \quad \theta(k) = kx + 2k^2t. \tag{2.16}$$

The contour for this RH problem is depicted in Fig. 1.

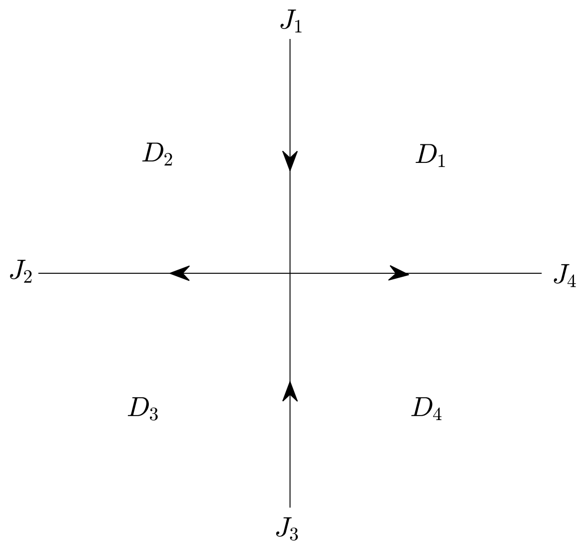

Fig. 1. The contour for the RH problem.

In what follows, we will make the following simple assumptions.

Assumption 1. We assume that the following conditions hold:

- the initial and boundary values lie in the Schwartz class.
- the spectral functions $s(k), S(k)$ defined in (2.7) satisfy the global relation (2.10).
- $s_{11}(k)$ and $W_{11}(k)$ have no zeros in $\overline{D}_3 \cup \overline{D}_4$ and $\overline{D}_3$, respectively.
- all the initial and boundary values are compatible with equation (1.1) to all orders at $x = t = 0$.

We have the following representation theorem (the proof can be found in [18]).

Theorem 2.1. *Let $u_0(x), g_0(t), g_1(t), v_0(x), h_0(t), h_1(t)$ be functions in the Schwartz class $\mathscr{S}([0,\infty))$, and the assumption 1 is satisfied. Then the RH problem* (2.14) *with the jump matrices given by* (2.15) *and the following asymptotics:*

$$M(x,t,k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty \tag{2.17}$$

admits a unique solution $M(x,t,k)$ for each $(x,t) \in \Omega$. Define $\{u(x,t), v(x,t)\}$ in terms of $M(x,t,k)$ by

$$\begin{aligned} u(x,t) &= 2\lim_{k\to\infty}(kM(x,t,k))_{12}, \\ v(x,t) &= 2\lim_{k\to\infty}(kM(x,t,k))_{13}. \end{aligned} \tag{2.18}$$

Then $\{u(x,t),v(x,t)\}$ *satisfies the CNLS equation* (1.1)*. Furthermore,* $\{u(x,t),v(x,t)\}$ *satisfies the initial and boundary value conditions*

$$\begin{aligned} &u(x,0)=u_0(x),\ \ u(0,t)=g_0(t),\ \ u_x(0,t)=g_1(t),\\ &v(x,0)=v_0(x),\ \ v(0,t)=h_0(t),\ \ v_x(0,t)=h_1(t). \end{aligned} \tag{2.19}$$

Remark 2.1. In the case when $s_{11}(k)$ and $W_{11}(k)$ have no zeros, the unique solvability of above RH problem (2.14) is a consequence of the 'vanishing' lemma (the proof can be found in the appendix of [18]).

Remark 2.2. For a well-posed problem, only a subset of the initial and boundary values can be independently prescribed. However, all boundary values are needed for the definition of $S(k)$, and hence for the formulation of the main RH problem as well as the spectral function $r(k)$ defined in (3.3). In general, the computation of the unknown boundary values, namely, the construction of the generalized Dirichlet-to-Neumann map, involves the solution of a nonlinear Volterra integral equation. We do not consider this parts in present paper since the detailed analysis has been studied in the paper [18], (see the Sections 5 and 6 in [18]). Our main concern in present paper is the derivation of the long-time asymptotics for the solution of the IBVP of the CNLS equation (1.1).

3. Long-time asymptotic analysis

Before we proceed to the following analysis, we will follow the ideas used in [2, 17] to rewrite our main 3×3 matrix RH problem as a 2×2 block one. This procedure is more convenient for the following long-time asymptotic analysis. More precisely, we rewrite a 3×3 matrix A as a block form

$$A=\begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix},$$

where A_{11} is scalar. Define the vector-valued spectral functions $r_1(k)$ and $h(k)$ by

$$r_1(k)=\begin{pmatrix} \dfrac{\bar{s}_{12}}{\bar{s}_{11}}\\ \dfrac{\bar{s}_{13}}{\bar{s}_{11}}\end{pmatrix}, \qquad\qquad k\in\mathbb{R}, \tag{3.1}$$

$$h(k)=\begin{pmatrix} \dfrac{s_{33}S_{21}-s_{23}S_{31}}{\bar{s}_{11}\overline{W}_{11}}\\ \dfrac{s_{22}S_{31}-s_{32}S_{21}}{\bar{s}_{11}\overline{W}_{11}}\end{pmatrix}, \quad k\in\overline{D}_2, \tag{3.2}$$

and let $r(k)$ denote the sum given by

$$r(k)=r_1^{\dagger}(\bar{k})+h^{\dagger}(\bar{k}),\quad k\in\mathbb{R}_-. \tag{3.3}$$

Then we can rewrite the RH problem (2.14) as

$$\begin{cases} M_+(x,t,k)=M_-(x,t,k)J(x,t,k), & k\in\Sigma=\mathbb{R}\cup\mathrm{i}\mathbb{R},\\ M(x,t,k)\to I, & k\to\infty, \end{cases} \tag{3.4}$$

with the jump matrix $J(x,t,k)$ is given by

$$J(x,t,k)=\begin{cases}\begin{pmatrix}1 & 0\\ -h(k)\mathrm{e}^{t\Phi(\xi,k)} & I\end{pmatrix}, & k\in\mathrm{i}\mathbb{R}_+,\\ \begin{pmatrix}1 & -r(k)\mathrm{e}^{-t\Phi(\xi,k)}\\ -r^\dagger(\bar{k})\mathrm{e}^{t\Phi(\xi,k)} & I+r^\dagger(\bar{k})r(k)\end{pmatrix}, & k\in\mathbb{R}_-,\\ \begin{pmatrix}1 & -h^\dagger(\bar{k})\mathrm{e}^{-t\Phi(\xi,k)}\\ 0 & I\end{pmatrix}, & k\in\mathrm{i}\mathbb{R}_-,\\ \begin{pmatrix}1+r_1^\dagger(\bar{k})r_1(k) & r_1^\dagger(\bar{k})\mathrm{e}^{-t\Phi(\xi,k)}\\ r_1(k)\mathrm{e}^{t\Phi(\xi,k)} & I\end{pmatrix}, & k\in\mathbb{R}_+,\end{cases} \tag{3.5}$$

where

$$\Phi(\xi,k)=4\mathrm{i}k^2+2\mathrm{i}\xi k,\quad \xi=\frac{x}{t}, \tag{3.6}$$

and $\mathbb{R}_+=[0,\infty)$ and $\mathbb{R}_-=(-\infty,0]$ denote the positive and negative halves of the real axis. In view of (2.18), we have

$$\big(u(x,t)\ \ v(x,t)\big)=2\lim_{k\to\infty}(kM(x,t,k))_{12}. \tag{3.7}$$

Moreover, we can deduce from the properties of $s(k)$ and $S(k)$ that the functions $r_1(k),\ h(k),\ r(k)$ defined by (3.1), (3.2) and (3.3) possess the following properties:

- $r_1(k)$ is smooth and bounded on $\mathbb{R}$;
- $h(k)$ is smooth and bounded on $\bar{D}_2$ and analytic in D_2;
- $r(k)$ is smooth and bounded on $\mathbb{R}_-$;
- There exist complex constants $\{r_{1,j}\}_{j=1}^\infty$ and $\{h_j\}_{j=1}^\infty$ such that, for any $N\geq 1$,

$$r_1(k)=\sum_{j=1}^N\frac{r_{1,j}}{k^j}+O\left(\frac{1}{k^{N+1}}\right),\quad |k|\to\infty,\ k\in\mathbb{R}, \tag{3.8}$$

$$h(k)=\sum_{j=1}^N\frac{h_j}{k^j}+O\left(\frac{1}{k^{N+1}}\right),\quad k\to\infty,\quad k\in\bar{D}_2. \tag{3.9}$$

3.1. *Transformations of the RH problem*

Let $N>1$ be given, and let $\mathscr{I}$ denote the interval $\mathscr{I}=(0,N]$. The jump matrix J defined in (3.5) involves the exponentials $\mathrm{e}^{\pm t\Phi}$. It follows that there is a single stationary point located at the point where $\frac{\partial\Phi}{\partial k}=0$, i.e., at $k=k_0=-\frac{\xi}{4}$. By performing a number of transformations in the following, we can bring the RH problem (3.4) to a form suitable for determining the long-time asymptotics.

The first transformation is to deform the vertical part of Σ so that it passes through the critical point k_0. Letting U_1 and U_2 denote the triangular domains shown in Fig. 2. Then the first transform

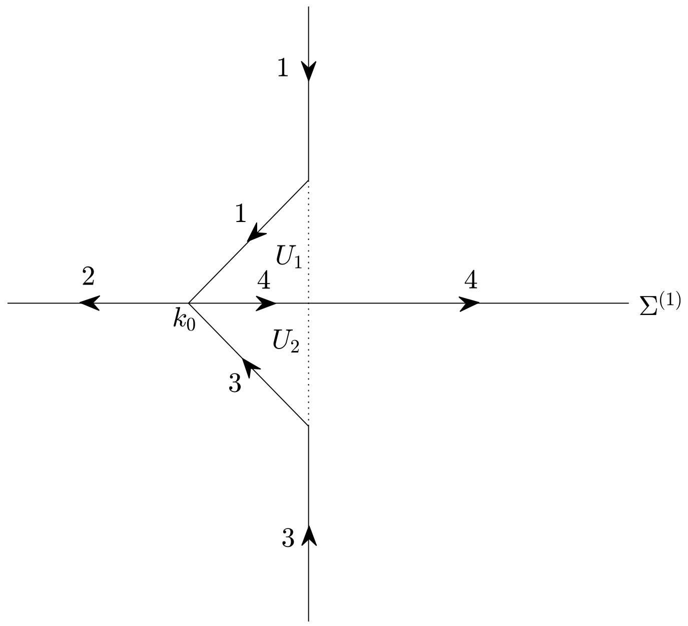

Fig. 2. The contour $\Sigma^{(1)}$ in the complex k-plane.

is as follows:

$$M^{(1)}(x,t,k) = M(x,t,k) \times \begin{cases} \begin{pmatrix} 1 & 0 \\ -h(k)\,\mathrm{e}^{t\Phi(\xi,k)} & I \end{pmatrix}, & k \in U_1, \\ \begin{pmatrix} 1 & h^{\dagger}(\bar{k})\,\mathrm{e}^{-t\Phi(\xi,k)} \\ 0 & I \end{pmatrix}, & k \in U_2, \\ I, & \text{elsewhere.} \end{cases} \tag{3.10}$$

Then we obtain the RH problem

$$\begin{cases} M_+^{(1)}(x,t,k) = M_-^{(1)}(x,t,k) J^{(1)}(x,t,k), & k \in \Sigma^{(1)}, \\ M^{(1)}(x,t,k) \to I, & k \to \infty. \end{cases} \tag{3.11}$$

The jump matrix $J^{(1)}(x,t,k)$ is given by

$$\begin{aligned} J_1^{(1)} &= \begin{pmatrix} 1 & 0 \\ -h\mathrm{e}^{t\Phi} & I \end{pmatrix}, & J_2^{(1)} &= \begin{pmatrix} 1 & -r\mathrm{e}^{-t\Phi} \\ -r^{\dagger}\mathrm{e}^{t\Phi} & I + r^{\dagger} r \end{pmatrix}, \\ J_3^{(1)} &= \begin{pmatrix} 1 & -h^{\dagger}\mathrm{e}^{-t\Phi} \\ 0 & I \end{pmatrix}, & J_4^{(1)} &= \begin{pmatrix} 1 + r_1^{\dagger} r_1 & r_1^{\dagger}\mathrm{e}^{-t\Phi} \\ r_1 \mathrm{e}^{t\Phi} & I \end{pmatrix}, \end{aligned} \tag{3.12}$$

where $J_i^{(1)}$ denotes the restriction of $J^{(1)}$ to the contour labeled by i in Fig. 2. The next transformation is:

$$M^{(2)}(x,t,k) = M^{(1)}(x,t,k)\Delta(k), \tag{3.13}$$

where

$$\Delta(k)=\begin{pmatrix}\frac{1}{\det\delta(k)} & 0\\ 0 & \delta(k)\end{pmatrix}, \tag{3.14}$$

and $\delta(k)$ satisfies the following 2×2 matrix RH problem across $(-\infty,k_0)$ oriented in Fig. 2:

$$\begin{cases}\delta_-(k)=(I+r^\dagger(k)r(k))\delta_+(k), & k<k_0,\\ \delta(k)\to I, & k\to\infty.\end{cases} \tag{3.15}$$

Furthermore,

$$\begin{cases}\det\delta_-(k)=(1+r(k)r^\dagger(k))\det\delta_+(k), & k<k_0,\\ \det\delta(k)\to 1, & k\to\infty.\end{cases} \tag{3.16}$$

Remark 3.1. It is noted that we introduce a 2×2 matrix-valued function $\delta(k)$ to remove the middle matrix term while we split the jump matrix $J_2^{(1)}(x,t,k)$ into an appropriate upper/lower triangular form. This may be a main difference compared with the long-time asymptotic analysis of integrable nonlinear evolution equations associated with 2×2 matrix spectral problems on the half-line [3,20, 21].

Since the jump matrix $I+r^\dagger(k)r(k)$ is positive definite, the vanishing lemma [1] yields the existence and uniqueness of the function $\delta(k)$. By Plemelj formula, $\det\delta(k)$ can be solved by

$$\begin{aligned}\det\delta(k)&=\exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{k_0}\frac{\ln(1+r(s)r^\dagger(s))}{s-k}\mathrm{d}s\right\}\\ &=(k-k_0)^{-\mathrm{i}\nu}\mathrm{e}^{\chi(k)},\quad k\in\mathbb{C}\setminus(-\infty,k_0],\end{aligned} \tag{3.17}$$

where

$$\nu=\frac{1}{2\pi}\ln(1+r(k_0)r^\dagger(k_0))>0, \tag{3.18}$$

$$\chi(k)=-\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{k_0}\ln(k-s)\mathrm{d}\ln(1+r(s)r^\dagger(s)). \tag{3.19}$$

On the other hand, a direct calculation as in [17] shows that

$$|\delta(k)|\le \text{const}<\infty,\quad |\det\delta(k)|\le\text{const}<\infty, \tag{3.20}$$

for all k, where we define

$$\begin{aligned}|A|&=(\mathrm{tr}A^\dagger A)^{\frac12}, && \text{for any matrix } A,\\ \|B(\cdot)\|_{L^p}&=\||B(\cdot)|\|_{L^p}, && \text{for any matrix function } B(\cdot).\end{aligned} \tag{3.21}$$

Then we find that $M^{(2)}(x,t,k)$ satisfies the following RH problem

$$\begin{cases}M_+^{(2)}(x,t,k)=M_-^{(2)}(x,t,k)J^{(2)}(x,t,k), & k\in\Sigma^{(2)},\\ M^{(2)}(x,t,k)\to I, & k\to\infty.\end{cases} \tag{3.22}$$

and the contour $\Sigma^{(2)}=\Sigma^{(1)}$, the jump matrix $J^{(2)}=\Delta_-^{-1}J^{(1)}\Delta_+$, namely,

$$J_1^{(2)}=\begin{pmatrix}1 & 0\\ -\delta^{-1}h(\det\delta)^{-1}\mathrm{e}^{t\Phi} & I\end{pmatrix},\quad J_2^{(2)}=\begin{pmatrix}1 & -r_2\delta_-\det\delta_-\mathrm{e}^{-t\Phi}\\ 0 & I\end{pmatrix}\begin{pmatrix}1 & 0\\ -\delta_+^{-1}r_2^\dagger(\det\delta_+)^{-1}\mathrm{e}^{t\Phi} & I\end{pmatrix},$$
$$J_3^{(2)}=\begin{pmatrix}1 & -h^\dagger\delta\det\delta\mathrm{e}^{-t\Phi}\\ 0 & I\end{pmatrix},\qquad J_4^{(2)}=\begin{pmatrix}1 & r_1^\dagger\delta\det\delta\mathrm{e}^{-t\Phi}\\ 0 & I\end{pmatrix}\begin{pmatrix}1 & 0\\ \delta^{-1}r_1(\det\delta)^{-1}\mathrm{e}^{t\Phi} & I\end{pmatrix},$$

where we define $r_2(k)$ by

$$r_2(k)=\frac{r(k)}{1+r(k)r^\dagger(\bar{k})}. \tag{3.23}$$

Before processing the next deformation, we will follow the idea of [3, 23, 24] and decompose each of the functions h, r_1, r_2 into an analytic part and a small remainder because the spectral functions have limited domains of analyticity. The analytic part of the jump matrix will be deformed, whereas the small remainder will be left on the original contour.

Lemma 3.1. *There exist a decomposition*

$$h(k)=h_a(t,k)+h_r(t,k),\quad t>0,\quad k\in i\mathbb{R}_+,$$

where the functions h_a and h_r have the following properties:

(i) *For each $t>0$, $h_a(t,k)$ is defined and continuous for $k\in\overline{D}_1$ and analytic for $k\in D_1$.*
(ii) *For each $\xi\in\mathscr{I}$ and each $t>0$, the function $h_a(t,k)$ satisfies*

$$\begin{aligned}|h_a(t,k)-h(0)|&\le Ce^{\frac{t}{4}|Re\Phi(\xi,k)|},\\ |h_a(t,k)|&\le\frac{C}{1+|k|^2}e^{\frac{t}{4}|Re\Phi(\xi,k)|},\end{aligned} \tag{3.24}$$

for $k\in\overline{D}_1$, where the constant C is independent of ξ, k, t.
(iii) *The L^1, L^2 and L^∞ norms of the function $h_r(t,\cdot)$ on $\mathrm{i}\mathbb{R}_+$ are $O(t^{-3/2})$ as $t\to\infty$.*

Proof. Since $h(k)\in C^5(\mathrm{i}\mathbb{R}_+)$, we find that

$$h^{(n)}(k)=\frac{\mathrm{d}^n}{\mathrm{d}k^n}\left(\sum_{j=0}^{4}\frac{h^{(j)}(0)}{j!}k^j\right)+O(k^{5-n}),\quad k\to0,\ k\in\mathrm{i}\mathbb{R}_+,\ n=0,1,2. \tag{3.25}$$

On the other hand, we have

$$h^{(n)}(k)=\frac{\mathrm{d}^n}{\mathrm{d}k^n}\left(\sum_{j=1}^{3}h_jk^{-j}\right)+O(k^{-4-n}),\quad k\to\infty,\ k\in\mathrm{i}\mathbb{R}_+,\ n=0,1,2. \tag{3.26}$$

Let

$$f_0(k)=\sum_{j=2}^{9}\frac{a_j}{(k+\mathrm{i})^j}, \tag{3.27}$$

where $\{a_j\}_2^9$ are complex such that

$$f_0(k)=\begin{cases}\sum_{j=0}^{4}\frac{h^{(j)}(0)}{j!}k^j+O(k^5), & k\to 0,\\ \sum_{j=1}^{3}h_jk^{-j}+O(k^{-4}), & k\to\infty.\end{cases}\tag{3.28}$$

It is easy to verify that (3.28) imposes eight linearly independent conditions on the a_j, hence the coefficients a_j exist and are unique. Letting $f=h-f_0$, it follows that

(1) $f_0(k)$ is a rational function of $k\in\mathbb{C}$ with no poles in $\overline{D}_1$;

(2) $f_0(k)$ coincides with $h(k)$ to four order at 0 and to third order at ∞, more precisely,

$$\frac{\mathrm{d}^n}{\mathrm{d}k^n}f(k)=\begin{cases}O(k^{5-n}), & k\to 0,\\ O(k^{-4-n}), & k\to\infty,\end{cases}\quad k\in\mathrm{i}\mathbb{R}_+,\ \ n=0,1,2.\tag{3.29}$$

The decomposition of $h(k)$ can be derived as follows. The map $k\mapsto\psi=\psi(k)$ defined by $\psi(k)=4k^2$ is a bijection $[0,\mathrm{i}\infty)\mapsto(-\infty,0]$, so we may define a function $F:\mathbb{R}\to\mathbb{C}$ by

$$F(\psi)=\begin{cases}(k+\mathrm{i})^2f(k), & \psi\le 0,\\ 0, & \psi>0.\end{cases}\tag{3.30}$$

$F(\psi)$ is C^5 for $\psi\neq 0$ and

$$F^{(n)}(\psi)=\left(\frac{1}{8k}\frac{\partial}{\partial k}\right)^n\Big((k+\mathrm{i})^2f(k)\Big),\quad \psi\le 0.$$

By (3.29), $F\in C^2(\mathbb{R})$ and $F^{(n)}(\psi)=O(|\psi|^{-1-n})$ as $|\psi|\to\infty$ for $n=0,1,2$. In particular,

$$\left\|\frac{\mathrm{d}^nF}{\mathrm{d}\psi^n}\right\|_{L^2(\mathbb{R})}<\infty,\quad n=0,1,2,\tag{3.31}$$

that is, F belongs to $H^2(\mathbb{R})$. By the Fourier transform $\widehat{F}(s)$ defined by

$$\widehat{F}(s)=\frac{1}{2\pi}\int_{\mathbb{R}}F(\psi)\mathrm{e}^{-\mathrm{i}\psi s}\mathrm{d}\psi\tag{3.32}$$

where

$$F(\psi)=\int_{\mathbb{R}}\widehat{F}(s)\mathrm{e}^{\mathrm{i}\psi s}\mathrm{d}s,\tag{3.33}$$

it follows from Plancherel theorem that $\|s^2\widehat{F}(s)\|_{L^2(\mathbb{R})}<\infty$. Equations (3.30) and (3.33) imply

$$f(k)=\frac{1}{(k+\mathrm{i})^2}\int_{\mathbb{R}}\widehat{F}(s)\mathrm{e}^{\mathrm{i}\psi s}\mathrm{d}s,\quad k\in\mathrm{i}\mathbb{R}_+.\tag{3.34}$$

Writing

$$f(k)=f_a(t,k)+f_r(t,k),\quad t>0,\ \ k\in\mathrm{i}\mathbb{R}_+,$$

where the functions f_a and f_r are defined by

$$f_a(t,k) = \frac{1}{(k+\mathrm{i})^2}\int_{-\frac{t}{4}}^{\infty}\widehat{F}(s)\,\mathrm{e}^{4\mathrm{i}k^2s}\mathrm{d}s, \quad t>0,\ \ k\in\overline{D}_1, \tag{3.35}$$

$$f_r(t,k) = \frac{1}{(k+\mathrm{i})^2}\int_{-\infty}^{-\frac{t}{4}}\widehat{F}(s)\,\mathrm{e}^{4\mathrm{i}k^2s}\mathrm{d}s, \quad t>0,\ \ k\in\mathrm{i}\mathbb{R}_+, \tag{3.36}$$

we infer that $f_a(t,\cdot)$ is continuous in $\overline{D}_1$ and analytic in D_1. Moreover, since $|\mathrm{Re}4\mathrm{i}k^2|\le|\mathrm{Re}\Phi(\xi,k)|$ for $k\in\overline{D}_1$ and $\xi\in\mathscr{I}$, we can get

$$\begin{aligned}|f_a(t,k)| &\le \frac{1}{|k+\mathrm{i}|^2}\|\widehat{F}(s)\|_{L^1(\mathbb{R})}\sup_{s\ge-\frac{t}{4}}\mathrm{e}^{s\mathrm{Re}4\mathrm{i}k^2}\\ &\le \frac{C}{1+|k|^2}\mathrm{e}^{\frac{t}{4}|\mathrm{Re}\Phi(\xi,k)|}, \quad t>0,\ \ k\in\overline{D}_1,\ \ \xi\in\mathscr{I}.\end{aligned} \tag{3.37}$$

Furthermore, we have

$$\begin{aligned}|f_r(t,k)| &\le \frac{1}{|k+\mathrm{i}|^2}\int_{-\infty}^{-\frac{t}{4}}s^2|\widehat{F}(s)|s^{-2}\mathrm{d}s\\ &\le \frac{C}{1+|k|^2}\|s^2\widehat{F}(s)\|_{L^2(\mathbb{R})}\sqrt{\int_{-\infty}^{-\frac{t}{4}}s^{-4}\mathrm{d}s},\\ &\le \frac{C}{1+|k|^2}t^{-3/2}, \quad t>0,\ \ k\in\mathrm{i}\mathbb{R}_+,\ \ \xi\in\mathscr{I}.\end{aligned} \tag{3.38}$$

Hence, the L^1, L^2 and L^∞ norms of f_r on $\mathrm{i}\mathbb{R}_+$ are $O(t^{-3/2})$. Letting

$$\begin{aligned}&h_a(t,k) = f_0(k)+f_a(t,k), && t>0,\ \ k\in\overline{D}_1,\\ &h_r(t,k) = f_r(t,k), && t>0,\ \ k\in\mathrm{i}\mathbb{R}_+,\end{aligned} \tag{3.39}$$

we find a decomposition of h with the properties listed in the statement of the lemma. □

We next introduce the open subsets $\{\Omega_j\}_1^8$, as displayed in Fig. 3. The following lemma describes how to decompose r_j, $j=1,2$ into an analytic part $r_{j,a}$ and a small remainder $r_{j,r}$.

Lemma 3.2. *There exist decompositions*

$$\begin{aligned}r_1(k) &= r_{1,a}(x,t,k)+r_{1,r}(x,t,k), \quad k>k_0,\\ r_2(k) &= r_{2,a}(x,t,k)+r_{2,r}(x,t,k), \quad k<k_0,\end{aligned} \tag{3.40}$$

where the functions $\{r_{j,a},\, r_{j,r}\}_1^2$ have the following properties:

(1) *For each $\xi\in\mathscr{I}$ and each $t>0$, $r_{j,a}(x,t,k)$ is defined and continuous for $k\in\overline{\Omega}_j$ and analytic for Ω_j, $j=1,2$.*
(2) *The functions $r_{1,a}$ and $r_{2,a}$ satisfy, for $\xi\in\mathscr{I}$, $t>0$, $j=1,2$,*

$$\left.\begin{aligned}|r_{j,a}(x,t,k)-r_j(k_0)| &\le C|k-k_0|e^{\frac{t}{4}|Re\Phi(\xi,k)|},\\ |r_{j,a}(x,t,k)| &\le \frac{C}{1+|k-k_0|^2}e^{\frac{t}{4}|Re\Phi(\xi,k)|},\end{aligned}\right\}\quad k\in\overline{\Omega}_j, \tag{3.41}$$

where the constant C is independent of ξ, k, t.

(3) *The L^1, L^2 and L^∞ norms of the function $r_{1,r}(x,t,\cdot)$ on (k_0,∞) are $O(t^{-3/2})$ as $t \to \infty$ uniformly with respect to $\xi \in \mathscr{I}$.*
(4) *The L^1, L^2 and L^∞ norms of the function $r_{2,r}(x,t,\cdot)$ on $(-\infty,k_0)$ are $O(t^{-3/2})$ as $t \to \infty$ uniformly with respect to $\xi \in \mathscr{I}$.*

Proof. Analogous to the proof of Lemma 3.1. One can also see [17, 24]. □

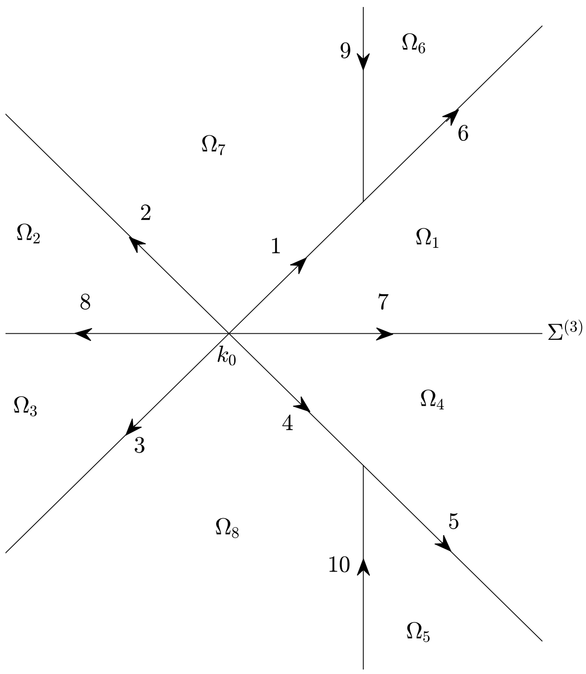

Fig. 3. The contour $\Sigma^{(3)}$ and the open sets $\{\Omega_j\}_1^8$ in the complex k-plane.

The purpose of the next transformation is to deform the contour so that the jump matrix involves the exponential factor $e^{-t\Phi}$ on the parts of the contour where $\mathrm{Re}\Phi$ is positive and the factor $e^{t\Phi}$ on the parts where $\mathrm{Re}\,\Phi$ is negative according to the signature table for $\mathrm{Re}\,\Phi$. More precisely, we put

$$M^{(3)}(x,t,k) = M^{(2)}(x,t,k)G(k), \tag{3.42}$$

where

$$G(k)=\begin{cases}\begin{pmatrix}1 & 0\\ -\delta^{-1}r_{1,a}(\det\delta)^{-1}\mathrm{e}^{t\Phi} & I\end{pmatrix}, & k\in\Omega_1,\\ \begin{pmatrix}1 & -r_{2,a}\delta\det\delta\mathrm{e}^{-t\Phi}\\ 0 & I\end{pmatrix}, & k\in\Omega_2,\\ \begin{pmatrix}1 & 0\\ \delta^{-1}r_{2,a}^{\dagger}(\det\delta)^{-1}\mathrm{e}^{t\Phi} & I\end{pmatrix}, & k\in\Omega_3,\\ \begin{pmatrix}1 & r_{1,a}^{\dagger}\delta\det\delta\mathrm{e}^{-t\Phi}\\ 0 & I\end{pmatrix}, & k\in\Omega_4,\\ \begin{pmatrix}1 & -h_a^{\dagger}\delta\det\delta\mathrm{e}^{-t\Phi}\\ 0 & I\end{pmatrix}, & k\in\Omega_5,\\ \begin{pmatrix}1 & 0\\ \delta^{-1}h_a(\det\delta)^{-1}\mathrm{e}^{t\Phi} & I\end{pmatrix}, & k\in\Omega_6,\\ I, & k\in\Omega_7\cup\Omega_8.\end{cases}\tag{3.43}$$

Then the matrix $M^{(3)}(x,t,k)$ satisfies the following RH problem

$$\begin{cases}M_+^{(3)}(x,t,k)=M_-^{(3)}(x,t,k)J^{(3)}(x,t,k), & k\in\Sigma^{(3)},\\ M^{(3)}(x,t,k)\to I, & k\to\infty.\end{cases}\tag{3.44}$$

with the jump matrix $J^{(3)}=G_-^{-1}(k)J^{(2)}G_+(k)$ given by

$$\begin{aligned}
&J_1^{(3)}=\begin{pmatrix}1 & 0\\ \delta^{-1}(r_{1,a}+h)(\det\delta)^{-1}\mathrm{e}^{t\Phi} & I\end{pmatrix}, \quad J_2^{(3)}=\begin{pmatrix}1 & -r_{2,a}\delta\det\delta\mathrm{e}^{-t\Phi}\\ 0 & I\end{pmatrix},\\
&J_3^{(3)}=\begin{pmatrix}1 & 0\\ -\delta^{-1}r_{2,a}^{\dagger}(\det\delta)^{-1}\mathrm{e}^{t\Phi} & I\end{pmatrix}, \quad J_4^{(3)}=\begin{pmatrix}1 & (r_{1,a}^{\dagger}+h^{\dagger})\delta\det\delta\mathrm{e}^{-t\Phi}\\ 0 & I\end{pmatrix},\\
&J_5^{(3)}=\begin{pmatrix}1 & (r_{1,a}^{\dagger}+h_a^{\dagger})\delta\det\delta\mathrm{e}^{-t\Phi}\\ 0 & I\end{pmatrix}, \quad J_6^{(3)}=\begin{pmatrix}1 & 0\\ \delta^{-1}(r_{1,a}+h_a)(\det\delta)^{-1}\mathrm{e}^{t\Phi} & I\end{pmatrix},\\
&J_7^{(3)}=\begin{pmatrix}1 & r_{1,r}^{\dagger}\delta\det\delta\mathrm{e}^{-t\Phi}\\ 0 & I\end{pmatrix}\begin{pmatrix}1 & 0\\ \delta^{-1}r_{1,r}(\det\delta)^{-1}\mathrm{e}^{t\Phi} & I\end{pmatrix},\\
&J_8^{(3)}=\begin{pmatrix}1 & -r_{2,r}\delta_-\det\delta_-\mathrm{e}^{-t\Phi}\\ 0 & I\end{pmatrix}\begin{pmatrix}1 & 0\\ -\delta_+^{-1}r_{2,r}^{\dagger}(\det\delta_+)^{-1}\mathrm{e}^{t\Phi} & I\end{pmatrix},\\
&J_9^{(3)}=\begin{pmatrix}1 & 0\\ -\delta^{-1}h_r(\det\delta)^{-1}\mathrm{e}^{t\Phi} & I\end{pmatrix}, \quad J_{10}^{(3)}=\begin{pmatrix}1 & -h_r^{\dagger}\delta\det\delta\mathrm{e}^{-t\Phi}\\ 0 & I\end{pmatrix},
\end{aligned}\tag{3.45}$$

with $J_i^{(3)}$ denoting the restriction of $J^{(3)}$ to the contour labeled by i in Fig. 3.

Remark 3.2. Recalling the meaningful work [3], begin performing the asymptotic analysis, the authors introduced a new spectral variable which led to a new phase function $\Phi(\zeta,\lambda)=4\mathrm{i}\lambda^2+2\mathrm{i}\zeta\lambda$

only possed a single critical point, which has the same form as our phase function (3.6). Thus, it is not surprising that the figures of deformation contours similar to [3]. However, our jump matrices involved are all 3×3 (that is, the spectral functions $r_1(k), r_2(k), h(k)$ are vectors) and the function $\delta(k)$ is a 2×2 matrix, which have the essential difference compared with [3]. As a result, the analysis in the next section will present some new skills, see Lemma 3.3.

3.2. *Local model near* k_0

It is easy to check that the jump matrix $J^{(3)}$ decays to identity matrix I as $t\to\infty$ everywhere except near k_0. Thus, the main contribution to the long-time asymptotics should come from a neighborhood of the stationary phase point k_0. To focus on k_0, we make a scaling transformation by

$$N_{k_0}: k \mapsto \frac{z}{\sqrt{8t}} + k_0. \tag{3.46}$$

Let $D_\varepsilon(k_0)$ denote the open disk of radius ε centered at k_0 for a small $\varepsilon > 0$. Then, the map $k\mapsto z$ is a bijection from $D_\varepsilon(k_0)$ to the open disk of radius $\sqrt{8t}\varepsilon$ centered at the origin. Since the function $\delta(k)$ satisfying a 2×2 matrix RH problem (3.15) can not be solved explicitly, to proceed the next step, we will follow the idea developed in [17] to use the available function $\det\delta(k)$ to approximate $\delta(k)$ by error estimate. More precisely, we rewrite the (12) entry of $J_2^{(3)}$ as

$$\left(r_{2,a}\delta\det\delta \mathrm{e}^{-t\Phi}\right)(k) = \left(r_{2,a}(\det\delta)^2\mathrm{e}^{-t\Phi}\right)(k) + \left(r_{2,a}(\delta - \det\delta I)\det\delta\mathrm{e}^{-t\Phi}\right)(k). \tag{3.47}$$

For the first part in the right-hand side of (3.47), we have

$$N_{k_0}\left(r_{2,a}(\det\delta)^2\mathrm{e}^{-t\Phi}\right)(z) = \eta^2\rho^2 r_{2,a}\left(\frac{z}{\sqrt{8t}} + k_0\right), \tag{3.48}$$

where

$$\eta = (8t)^{\frac{\mathrm{i}\nu}{2}}\mathrm{e}^{2\mathrm{i}k_0^2 t + \chi(k_0)},\quad \rho = z^{-\mathrm{i}\nu}\mathrm{e}^{-\frac{\mathrm{i}z^2}{4}}\mathrm{e}^{\chi(\frac{z}{\sqrt{8t}}+k_0)-\chi(k_0)}. \tag{3.49}$$

Let $\widetilde{\delta}(k) = r_{2,a}(k)(\delta(k) - \det\delta(k)I)\,\mathrm{e}^{-t\Phi(\xi,k)}$, then we have the following estimate.

Lemma 3.3. *For $z\in\widehat{L} = \{z = \alpha e^{\frac{3i\pi}{4}} : -\infty < \alpha < +\infty\}$, as $t\to\infty$, the following estimate for $\widetilde{\delta}(k)$ hold:*

$$|(N_{k_0}\widetilde{\delta})(z)| \le Ct^{-1/2}, \tag{3.50}$$

where the constant $C > 0$ independent of z, t.

Proof. The idea of the proof comes from [17]. It follows from (3.15) and (3.16) that $\widetilde{\delta}$ satisfies the following RH problem across $(-\infty, k_0)$ oriented in Fig. 2:

$$\begin{cases}\widetilde{\delta}_-(k) = (1 + r(k)r^\dagger(k))\widetilde{\delta}_+(k) + \mathrm{e}^{-t\Phi(\xi,k)}f(k), & k < k_0,\\ \widetilde{\delta}(k) \to 0, & k\to\infty,\end{cases} \tag{3.51}$$

where $f(k)=[r_{2,a}(r^\dagger r-rr^\dagger I)\delta_+](k)$. Then the function $\widetilde{\delta}(k)$ can be expressed by

$$\begin{aligned}&\widetilde{\delta}(k)=X(k)\int_{-\infty}^{k_0}\frac{\mathrm{e}^{-t\Phi(\xi,s)}f(s)}{X_-(s)(s-k)}\mathrm{d}s,\\&X(k)=\exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{k_0}\frac{\ln(1+r(s)r^\dagger(s))}{s-k}\mathrm{d}s\right\}.\end{aligned}\tag{3.52}$$

It follows from $r_{2,a}(r^\dagger r-rr^\dagger I)=r_{2,r}(rr^\dagger I-r^\dagger r)$ that $f(k)=O(t^{-3/2})$. Similar to Lemma 3.2, $f(k)$ can be decomposed into two parts: $f_a(x,t,k)$ which has an analytic continuation to Ω_2 and $f_r(x,t,k)$ which is a small remainder. In particular,

$$\begin{aligned}|f_a(x,t,k)|&\leq\frac{C}{1+|k-k_0+\frac{1}{t}|^2}\mathrm{e}^{\frac{t}{4}|\mathrm{Re}\Phi(\xi,k)|},\quad k\in L_t,\\|f_r(x,t,k|&\leq\frac{C}{1+|k-k_0+\frac{1}{t}|^2}t^{-3/2},\qquad k\in(-\infty,k_0),\end{aligned}\tag{3.53}$$

where $L_t=\{k=k_0-\frac{1}{t}+\alpha\mathrm{e}^{\frac{3i\pi}{4}}:0\leq\alpha<\infty\}$. Therefore, for $z\in\widehat{L}$, we can find

$$\begin{aligned}(N_{k_0}\widetilde{\delta})(z)&=X(k_0+\frac{z}{\sqrt{8t}})\int_{k_0-\frac{1}{t}}^{k_0}\frac{\mathrm{e}^{-t\Phi(\xi,s)}f(s)}{X_-(s)(s-k_0-\frac{z}{\sqrt{8t}})}\mathrm{d}s\\&\quad+X(k_0+\frac{z}{\sqrt{8t}})\int_{-\infty}^{k_0-\frac{1}{t}}\frac{\mathrm{e}^{-t\Phi(\xi,s)}f_a(x,t,s)}{X_-(s)(s-k_0-\frac{z}{\sqrt{8t}})}\mathrm{d}s\\&\quad+X(k_0+\frac{z}{\sqrt{8t}})\int_{-\infty}^{k_0-\frac{1}{t}}\frac{\mathrm{e}^{-t\Phi(\xi,s)}f_r(x,t,s)}{X_-(s)(s-k_0-\frac{z}{\sqrt{8t}})}\mathrm{d}s\\&=I_1+I_2+I_3.\end{aligned}$$

Then,

$$\begin{aligned}|I_1|&\leq\int_{k_0-\frac{1}{t}}^{k_0}\frac{|f(s)|}{|s-k_0-\frac{z}{\sqrt{8t}}|}\mathrm{d}s\leq Ct^{-3/2}\left|\ln\left|1-\frac{2\sqrt{2}}{zt^{1/2}}\right|\right|\leq Ct^{-2},\\|I_3|&\leq\int_{-\infty}^{k_0-\frac{1}{t}}\frac{|f_r(x,t,s)|}{|s-k_0-\frac{z}{\sqrt{8t}}|}\mathrm{d}s\leq Ct^{-1/2}.\end{aligned}$$

By the Cauchy's theorem, we can evaluate I_2 along the contour L_t instead of the interval $(-\infty,k_0-\frac{1}{t})$. Using the fact $\mathrm{Re}\Phi(\xi,k)>0$ in Ω_2, we can obtain $|I_2|\leq C\mathrm{e}^{-ct}$. This completes the proof of the lemma. $\square$

Remark 3.3. The estimate (3.50) also holds if $r_{2,a}$ is replaced with $r_{1,a}^\dagger+h^\dagger$ or $r_{1,a}^\dagger+h_a^\dagger$. There is a similar estimate

$$|(N_{k_0}\widehat{\delta})(z)|\leq Ct^{-1/2},\quad t\to\infty,\tag{3.54}$$

for $z\in\overline{\widehat{L}}$, where $\widehat{\delta}(k)=[\delta^{-1}(k)-(\det\delta)^{-1}I]\rho(k)\mathrm{e}^{t\Phi(\xi,k)}$, $\rho=r_{2,a}^\dagger,r_{1,a}+h$ or $r_{1,a}+h_a$.

Remark 3.4. As mentioned above, here we use function $\det\delta(k)$ which can be explicitly written down to approximate the unsolvable function $\delta(k)$ by error control. This procedure has never appeared in the long-time asymptotic analysis of the integrable nonlinear evolution equations associated with 2×2 matrix spectral problems.

In other words, we have the following important relation:

$$(N_{k_0}J_i^{(3)})(x,t,z) = \widetilde{J}(x,t,z) + O(t^{-1/2}), \;\; i = 1,\dots,6, \tag{3.55}$$

where $\widetilde{J}(x,t,z)$ is given by

$$\widetilde{J}(x,t,z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ (r_{1,a}+h_a)\eta^{-2}\rho^{-2} & I \end{pmatrix}, & k \in \mathscr{X}_1 \cap D_1, \\ \begin{pmatrix} 1 & 0 \\ (r_{1,a}+h)\eta^{-2}\rho^{-2} & I \end{pmatrix}, & k \in \mathscr{X}_1 \cap D_2, \\ \begin{pmatrix} 1 & -r_{2,a}\eta^2\rho^2 \\ 0 & I \end{pmatrix}, & k \in \mathscr{X}_2, \\ \begin{pmatrix} 1 & 0 \\ -r_{2,a}^{\dagger}\eta^{-2}\rho^{-2} & I \end{pmatrix}, & k \in \mathscr{X}_3, \\ \begin{pmatrix} 1 & (r_{1,a}^{\dagger}+h^{\dagger})\eta^2\rho^2 \\ 0 & I \end{pmatrix}, & k \in \mathscr{X}_4 \cap D_3, \\ \begin{pmatrix} 1 & (r_{1,a}^{\dagger}+h_a^{\dagger})\eta^2\rho^2 \\ 0 & I \end{pmatrix}, & k \in \mathscr{X}_4 \cap D_4, \end{cases} \tag{3.56}$$

where $\mathscr{X} = X + k_0$ denote the cross X defined by (3.57) centered at k_0, and $X = X_1 \cup X_2 \cup X_3 \cup X_4 \subset \mathbb{C}$ be the cross defined by

$$\begin{aligned} X_1 &= \big\{ l e^{\frac{i\pi}{4}} \mid 0 \le l < \infty \big\}, \quad & X_2 &= \big\{ l e^{\frac{3i\pi}{4}} \mid 0 \le l < \infty \big\}, \\ X_3 &= \big\{ l e^{-\frac{3i\pi}{4}} \mid 0 \le l < \infty \big\}, \quad & X_4 &= \big\{ l e^{-\frac{i\pi}{4}} \mid 0 \le l < \infty \big\}, \end{aligned} \tag{3.57}$$

and oriented as in Fig. 4.

For any fixed $z \in X$ and $\xi \in \mathscr{I}$, we have

$$\begin{aligned} &(r_{1,a}+h)\big(\frac{z}{\sqrt{8t}}+k_0\big) \to r^{\dagger}(k_0), \\ &\qquad r_{2,a}\big(\frac{z}{\sqrt{8t}}+k_0\big) \to \frac{r(k_0)}{1+r(k_0)r^{\dagger}(k_0)}, \\ &\rho^2 \to e^{-\frac{iz^2}{2}} z^{-2i\nu}, \end{aligned} \tag{3.58}$$

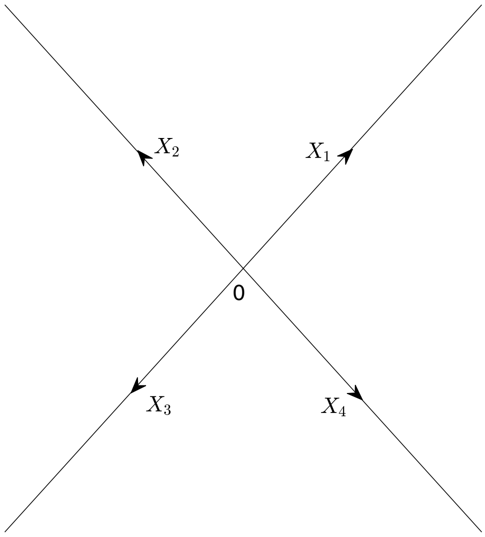

Fig. 4. The contour $X = X_1 \cup X_2 \cup X_3 \cup X_4$.

as $t \to \infty$. This implies that the jump matrix $\widetilde{J}$ tend to the matrix $J^{(k_0)}$ for large t, where

$$J^{(k_0)}(x,t,z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \eta^{-2} e^{\frac{iz^2}{2}} z^{2i\nu} r^\dagger(k_0) & I \end{pmatrix}, & z \in X_1, \\ \begin{pmatrix} 1 & -\eta^2 e^{-\frac{iz^2}{2}} z^{-2i\nu} \frac{r(k_0)}{1+r(k_0)r^\dagger(k_0)} \\ 0 & I \end{pmatrix}, & z \in X_2, \\ \begin{pmatrix} 1 & 0 \\ -\eta^{-2} e^{\frac{iz^2}{2}} z^{2i\nu} \frac{r^\dagger(k_0)}{1+r(k_0)r^\dagger(k_0)} & I \end{pmatrix}, & z \in X_3, \\ \begin{pmatrix} 1 & \eta^2 e^{-\frac{iz^2}{2}} z^{-2i\nu} r(k_0) \\ 0 & I \end{pmatrix}, & z \in X_4. \end{cases} \tag{3.59}$$

Theorem 3.1. *The following RH problem:*

$$\begin{cases} M_+^X(x,t,z) = M_-^X(x,t,z) J^X(x,t,z), & z \in X, \\ M^X(x,t,z) \to I, & z \to \infty, \end{cases} \tag{3.60}$$

with the jump matrix $J^X(x,t,z) = \eta^{\widehat{\sigma}} J^{(k_0)}(x,t,z)$ *has a unique solution* $M^X(x,t,z)$. *This solution satisfies*

$$M^X(x,t,z) = I - \frac{i}{z} \begin{pmatrix} 0 & \beta^X \\ (\beta^X)^\dagger & 0 \end{pmatrix} + O\left(\frac{1}{z^2}\right), \quad z \to \infty, \tag{3.61}$$

where the error term is uniform with respect to $\arg z \in [0, 2\pi]$ *and the function* β^X *is given by*

$$\beta^X = \frac{e^{\frac{i\pi}{4} - \frac{\pi\nu}{2}} \nu \Gamma(-i\nu)}{\sqrt{2\pi}} r(k_0), \tag{3.62}$$

where $\Gamma(\cdot)$ denotes the standard Gamma function. Moreover, for each compact subset $\mathscr{D}$ of $\mathbb{C}$,

$$\sup_{r(k_0)\in\mathscr{D}}\sup_{z\in\mathbb{C}\setminus X}|M^X(x,t,z)|<\infty. \tag{3.63}$$

Proof. The proof relies on deriving an explicit formula for the solution M^X in terms of parabolic cylinder functions. One can see [17]. □

Let $D_\varepsilon(k_0)$ denote the open disk of radius ε centered at k_0 for a small $\varepsilon>0$ and $\mathscr{X}^\varepsilon=\mathscr{X}\cap D_\varepsilon(k_0)$. We can approximate $M^{(3)}$ in the neighborhood $D_\varepsilon(k_0)$ of k_0 by

$$M^{(k_0)}(x,t,k)=\eta^{-\widehat{\sigma}}M^X(x,t,z). \tag{3.64}$$

Lemma 3.4. *For each $\xi\in\mathscr{I}$ and $t>0$, the function $M^{(k_0)}(x,t,k)$ defined in (3.64) is an analytic function of $k\in D_\varepsilon(k_0)\setminus\mathscr{X}^\varepsilon$. Furthermore,*

$$|M^{(k_0)}(x,t,k)-I|\le C,\quad t>3,\ \xi\in\mathscr{I},\ k\in\overline{D_\varepsilon(k_0)}\setminus\mathscr{X}^\varepsilon. \tag{3.65}$$

Across $\mathscr{X}^\varepsilon$, $M^{(k_0)}$ satisfied the jump condition $M_+^{(k_0)}=M_-^{(k_0)}J^{(k_0)}$, where the jump matrix $J^{(k_0)}$ satisfies the following estimates for $1\le p\le\infty$:

$$\|J^{(3)}-J^{(k_0)}\|_{L^p(\mathscr{X}^\varepsilon)}\le Ct^{-\frac{1}{2}-\frac{1}{2p}}\ln t,\quad t>3,\ \xi\in\mathscr{I}, \tag{3.66}$$

where $C>0$ is a constant independent of t,ξ,z. Moreover, as $t\to\infty$,

$$\|(M^{(k_0)})^{-1}(x,t,k)-I\|_{L^\infty(\partial D_\varepsilon(k_0))}=O(t^{-1/2}), \tag{3.67}$$

and

$$-\frac{1}{2\pi i}\int_{\partial D_\varepsilon(k_0)}\big((M^{(k_0)})^{-1}(x,t,k)-I\big)dk=\frac{\eta^{-\widehat{\sigma}}M_1^X}{\sqrt{8t}}+O(t^{-1}), \tag{3.68}$$

where M_1^X is defined by

$$M_1^X=-i\begin{pmatrix}0 & \beta^X\\ (\beta^X)^\dagger & 0\end{pmatrix}. \tag{3.69}$$

Proof. The analyticity of $M^{(k_0)}$ is obvious. Since $|\eta|=1$, thus, the estimate (3.65) follows from the definition of $M^{(k_0)}$ in (3.64) and the estimate (3.63).

On the other hand, we have $J^{(3)}-J^{(k_0)}=J^{(3)}-\widetilde{J}+\widetilde{J}-J^{(k_0)}$. However, according to the Lemma 89 in [15], we conclude that

$$\|\widetilde{J}-J^{(k_0)}\|_{L^\infty(\mathscr{X}_1^\varepsilon)}\le C|e^{\frac{i\gamma}{2}z^2}|t^{-1/2}\ln t,\quad 0<\gamma<\frac{1}{2},\ t>3,\ \xi\in\mathscr{I},$$

for $k\in\mathscr{X}_1^\varepsilon$, that is, $z=\sqrt{8t}ue^{\frac{i\pi}{4}}$, $0\le u\le\varepsilon$. Thus, for ε small enough, it follows from (3.55) that

$$\|J^{(3)}-J^{(k_0)}\|_{L^\infty(\mathscr{X}_1^\varepsilon)}\le C|e^{\frac{i\gamma}{2}z^2}|t^{-1/2}\ln t. \tag{3.70}$$

Then we have

$$\|J^{(3)}-J^{(k_0)}\|_{L^1(\mathscr{X}_1^\varepsilon)}\le Ct^{-1}\ln t,\quad t>3,\ \xi\in\mathscr{I}. \tag{3.71}$$

By the general inequality $\|f\|_{L^p} \le \|f\|_{L^\infty}^{1-1/p}\|f\|_{L^1}^{1/p}$, we find

$$\|J^{(2)} - J^{(k_0)}\|_{L^p(\mathscr{X}_1^\varepsilon)} \le Ct^{-1/2-1/2p}\ln t, \quad t > 3, \ \xi \in \mathscr{I}. \tag{3.72}$$

The norms on $\mathscr{X}_j^\varepsilon$, $j = 2, 3, 4$, are estimated in a similar way. Therefore, (3.66) follows.

If $k \in \partial D_\varepsilon(k_0)$, the variable $z = \sqrt{8t}(k-k_0)$ tends to infinity as $t \to \infty$. It follows from (3.61) that

$$M^X(x,t,z) = I + \frac{M_1^X}{\sqrt{8t}(k-k_0)} + O\left(\frac{1}{t}\right), \quad t \to \infty, \ k \in \partial D_\varepsilon(k_0). \tag{3.73}$$

Since

$$M^{(k_0)}(x,t,k) = \eta^{-\widehat{\sigma}} M^X(x,t,z),$$

thus we have

$$(M^{(k_0)})^{-1}(x,t,k) - I = -\frac{\eta^{-\widehat{\sigma}} M_1^X}{\sqrt{8t}(k-k_0)} + O\left(\frac{1}{t}\right), \quad t \to \infty, \ k \in \partial D_\varepsilon(k_0). \tag{3.74}$$

The estimate (3.67) immediately follows from (3.74) and $|M_1^X| \le C$. By Cauchy's formula and (3.74), we derive (3.68). $\square$

3.3. *Derivation of the asymptotic formula*

Define the approximate solution $M^{(app)}(x,t,k)$ by

$$M^{(app)} = \begin{cases} M^{(k_0)}, & k \in D_\varepsilon(k_0), \\ I, & \text{elsewhere}. \end{cases} \tag{3.75}$$

Let $\widehat{M}(x,t,k)$ be

$$\widehat{M} = M^{(3)}(M^{(app)})^{-1}, \tag{3.76}$$

then $\widehat{M}(x,t,k)$ satisfies the following RH problem

$$\begin{cases} \widehat{M}_+(x,t,k) = \widehat{M}_-(x,t,k)\widehat{J}(x,t,k), & k \in \widehat{\Sigma}, \\ \widehat{M}(x,t,k) \to I, & k \to \infty. \end{cases} \tag{3.77}$$

where the jump contour $\widehat{\Sigma} = \Sigma^{(3)} \cup \partial D_\varepsilon(k_0)$ is depicted in Fig. 5, and the jump matrix $\widehat{J}(x,t,k)$ is given by

$$\widehat{J} = \begin{cases} M_-^{(k_0)} J^{(3)} (M_+^{(k_0)})^{-1}, & k \in \widehat{\Sigma} \cap D_\varepsilon(k_0), \\ (M^{(k_0)})^{-1}, & k \in \partial D_\varepsilon(k_0), \\ J^{(3)}, & k \in \widehat{\Sigma} \setminus \overline{D_\varepsilon(k_0)}. \end{cases} \tag{3.78}$$

Let $\widehat{W} = \widehat{J} - I$, and we rewrite $\widehat{\Sigma}$ as follows:

$$\widehat{\Sigma} = \partial D_\varepsilon(k_0) \cup \mathscr{X}^\varepsilon \cup \widehat{\Sigma}_1 \cup \widehat{\Sigma}_2,$$

where

$$\widehat{\Sigma}_1 = \bigcup_1^6 \Sigma_j^{(3)} \setminus D_\varepsilon(k_0), \quad \widehat{\Sigma}_2 = \bigcup_7^{10} \Sigma_j^{(3)},$$

and $\{\Sigma_j^{(3)}\}_1^{10}$ denoting the restriction of $\Sigma^{(3)}$ to the contour labeled by j in Fig. 3. Then the following inequalities are valid.

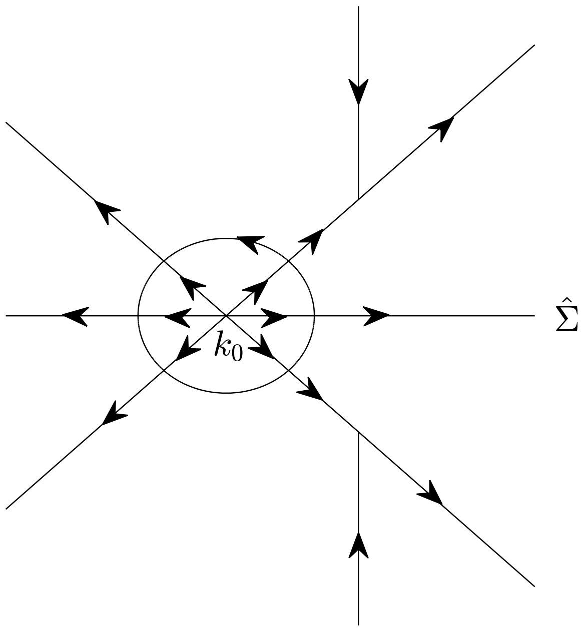

Fig. 5. The contour $\widehat{\Sigma}$.

Lemma 3.5. *For $1 \le p \le \infty$, the following estimates hold for $t > 3$ and $\xi \in \mathscr{I}$,*

$$\|\widehat{W}\|_{L^p(\partial D_\varepsilon(k_0))} \le Ct^{-\frac{1}{2}}, \tag{3.79}$$

$$\|\widehat{W}\|_{L^p(\mathscr{X}^\varepsilon)} \le Ct^{-\frac{1}{2}-\frac{1}{2p}} \ln t, \tag{3.80}$$

$$\|\widehat{W}\|_{L^p(\widehat{\Sigma}_1)} \le Ce^{-ct}, \tag{3.81}$$

$$\|\widehat{W}\|_{L^p(\widehat{\Sigma}_2)} \le Ct^{-\frac{3}{2}}. \tag{3.82}$$

Proof. The inequality (3.79) is a consequence of (3.67) and (3.78). For $k \in \mathscr{X}^\varepsilon$, we find

$$\widehat{W} = M_-^{(k_0)}(J^{(3)} - J^{(k_0)})(M_+^{(k_0)})^{-1}.$$

Therefore, it follows from (3.65) and (3.66) that the estimate (3.80) holds. For $k \in D_2 \cap (\mathscr{X}_1 \setminus \overline{D_\varepsilon(k_0)})$, $\widehat{W}$ only has a nonzero $\delta^{-1}(r_{1,a}+h)(\det\delta)^{-1}\mathrm{e}^{t\Phi}$ in (21) entry. Hence, for $t \ge 1$, by (3.20), we get

$$\begin{aligned}|\widehat{W}_{21}| &\le C|r_{1,a}+h_a|\mathrm{e}^{t\mathrm{Re}\Phi}\\ &\le \frac{C}{1+|k|^2}\mathrm{e}^{-3|k-k_0|^2 t} \le C\mathrm{e}^{-3\varepsilon^2 t}.\end{aligned}$$

In a similar way, the estimates on $\mathscr{X}_j \setminus \overline{D_\varepsilon(k_0)}$, $j = 2, 3, 4$ hold. This proves (3.81). Since the matrix $\widehat{W}$ on $\widehat{\Sigma}_2$ only involves the small remainders h_r, $r_{1,r}$ and $r_{2,r}$, thus, by Lemmas 3.1 and 3.2, the estimate (3.82) follows. □

The results in Lemma 3.5 imply that:

$$\begin{aligned}\|\widehat{W}\|_{L^\infty(\widehat{\Sigma})} &\le Ct^{-1/2}\ln t,\\ \|\widehat{W}\|_{L^1\cap L^2(\widehat{\Sigma})} &\le Ct^{-1/2},\end{aligned} \qquad t>3,\ \ \xi\in\mathscr{I}. \tag{3.83}$$

This uniformly vanishing bound on $\widehat{W}$ shows that the RH problem (3.77) is a small-norm RH problem, for which there is a well known existence and uniqueness theorem. In fact, we may write

$$\widehat{M}(x,t,k) = I + \frac{1}{2\pi \mathrm{i}}\int_{\widehat{\Sigma}} \frac{(\widehat{\mu}\widehat{W})(x,t,\zeta)}{\zeta-k}\mathrm{d}\zeta, \quad k\in\mathbb{C}\setminus\widehat{\Sigma}, \tag{3.84}$$

where the 3×3 matrix-valued function $\widehat{\mu}(x,t,k)$ is the unique solution of

$$\widehat{\mu} = I + \widehat{C}_{\widehat{W}}\widehat{\mu}, \tag{3.85}$$

and

$$\|\widehat{\mu}(x,t,\cdot)-I\|_{L^2(\widehat{\Sigma})} = O(t^{-1/2}), \quad t\to\infty,\ \ \xi\in\mathscr{I}. \tag{3.86}$$

The singular integral operator $\widehat{C}_{\widehat{W}}\colon L^2(\widehat{\Sigma})+L^\infty(\widehat{\Sigma})\to L^2(\widehat{\Sigma})$ is defined by

$$\begin{aligned}\widehat{C}_{\widehat{W}}f &= \widehat{C}_-(f\widehat{W}),\\ (\widehat{C}_-f)(k) &= \lim_{k\to\widehat{\Sigma}_-}\int_{\widehat{\Sigma}}\frac{f(\zeta)}{\zeta-k}\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}},\end{aligned}$$

where $\widehat{C}_-$ is the well-known Cauchy operator. Moreover, by (3.83), we find

$$\|\widehat{C}_{\widehat{W}}\|_{B(L^2(\widehat{\Sigma}))} \le C\|\widehat{W}\|_{L^\infty(\widehat{\Sigma})} \le Ct^{-1/2}\ln t, \tag{3.87}$$

where $B(L^2(\widehat{\Sigma}))$ denotes the Banach space of bounded linear operators $L^2(\widehat{\Sigma})\to L^2(\widehat{\Sigma})$. Therefore, the resolvent operator $(I-\widehat{C}_{\widehat{W}})^{-1}$ is existent and thus of both $\widehat{\mu}$ and $\widehat{M}$ for large t.

It follows from (3.84) that

$$\lim_{k\to\infty} k(\widehat{M}(x,t,k)-I) = -\frac{1}{2\pi i}\int_{\widehat{\Sigma}}(\widehat{\mu}\widehat{W})(x,t,k)dk. \tag{3.88}$$

Using (3.80) and (3.86), we have

$$\begin{aligned}\int_{\mathscr{X}^\varepsilon}(\widehat{\mu}\widehat{W})(x,t,k)dk &= \int_{\mathscr{X}^\varepsilon}\widehat{W}(x,t,k)dk + \int_{\mathscr{X}^\varepsilon}(\widehat{\mu}(x,t,k)-I)\widehat{W}(x,t,k)dk \\ &\le \|\widehat{W}\|_{L^1(\mathscr{X}^\varepsilon)} + \|\widehat{\mu}-I\|_{L^2(\mathscr{X}^\varepsilon)}\|\widehat{W}\|_{L^2(\mathscr{X}^\varepsilon)} \\ &\le Ct^{-1}\ln t, \quad t\to\infty.\end{aligned}$$

Similarly, by (3.81) and (3.86), the contribution from $\widehat{\Sigma}_1$ to the right-hand side of (3.88) is

$$O(\|\widehat{W}\|_{L^1(\widehat{\Sigma}_1)} + \|\widehat{\mu}-I\|_{L^2(\widehat{\Sigma}_1)}\|\widehat{W}\|_{L^2(\widehat{\Sigma}_1)}) = O(e^{-ct}), \quad t\to\infty.$$

By (3.82) and (3.86), the contribution from $\widehat{\Sigma}_2$ to the right-hand side of (3.88) is

$$O(\|\widehat{W}\|_{L^1(\widehat{\Sigma}_2)} + \|\widehat{\mu}-I\|_{L^2(\widehat{\Sigma}_2)}\|\widehat{W}\|_{L^2(\widehat{\Sigma}_2)}) = O(t^{-3/2}), \quad t\to\infty.$$

Finally, by (3.68), (3.79) and (3.86), we can get

$$\begin{aligned}&-\frac{1}{2\pi i}\int_{\partial D_\varepsilon(k_0)}(\widehat{\mu}\widehat{W})(x,t,k)dk \\ &= -\frac{1}{2\pi i}\int_{\partial D_\varepsilon(k_0)}\widehat{W}(x,t,k)dk - \frac{1}{2\pi i}\int_{\partial D_\varepsilon(k_0)}(\widehat{\mu}(x,t,k)-I)\widehat{W}(x,t,k)dk \\ &= -\frac{1}{2\pi i}\int_{\partial D_\varepsilon(k_0)}\Big((M^{(k_0)})^{-1}(x,t,k)-I\Big)dk + O\big(\|\widehat{\mu}-I\|_{L^2(\partial D_\varepsilon(k_0))}\|\widehat{W}\|_{L^2(\partial D_\varepsilon(k_0))}\big) \\ &= \frac{\eta^{-\widehat{\sigma}}M_1^X}{\sqrt{8t}} + O(t^{-1}), \quad t\to\infty.\end{aligned}$$

Thus, we obtain the following important relation

$$\lim_{k\to\infty} k(\widehat{M}(x,t,k)-I) = \frac{\eta^{-\widehat{\sigma}}M_1^X}{\sqrt{8t}} + O(t^{-1}\ln t), \quad t\to\infty. \tag{3.89}$$

Taking into account that (3.7), (3.10), (3.13), (3.42), (3.76) and (3.89), for sufficient large $k\in\mathbb{C}\setminus\widehat{\Sigma}$, we get

$$\begin{aligned}\big(u(x,t)\ \ v(x,t)\big) &= 2\lim_{k\to\infty}(kM(x,t,k))_{12} \\ &= 2\lim_{k\to\infty}k(\widehat{M}(x,t,k)-I)_{12} \\ &= \frac{\big(\eta^{-\widehat{\sigma}}M_1^X\big)_{12}}{\sqrt{2t}} + O\Big(\frac{\ln t}{t}\Big) \\ &= \frac{-i\beta^X\eta^2}{\sqrt{2t}} + O\Big(\frac{\ln t}{t}\Big), \quad t\to\infty.\end{aligned} \tag{3.90}$$

In view of (3.49) and (3.62), we obtain our main results stated as the Theorem 1.1.

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