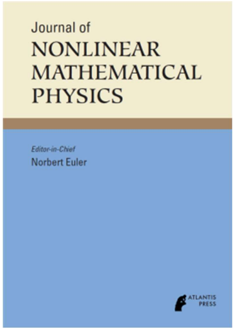

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Bilinear Identities and Squared Eigenfunction Symmetries of the BC_r-KP Hierarchy

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The BC_r-KP hierarchy is an important sub hierarchy of the KP hierarchy, which includes the BKP and CKP hierarchies as the special cases. Some properties of the BC_r-KP hierarchy and its constrained case are investigated in this paper, including bilinear identities and squared eigenfunction symmetries. We firstly discuss the bilinear identities of the BC_r-KP hierarchy, and then generalize them into the constrained case. Next, we investigate the squared eigenfunction symmetries for the BC_r-KP hierarchy and its constrained case, and also the connections with the additional symmetries. It is found that the constrained BC_r-KP hierarchy can be defined by identifying the time flow with the squared eigenfunction symmetries.

Keywords: the BC_r-KP hierarchy; the constrained BC_r-KP hierarchy; bilinear identities; squared eigenfunction symmetries.

2000 Mathematics Subject Classification: 35Q53, 37K10, 37K40

1. Introduction

In mathematical physics and integrable systems, the Kadomtsev-Petviashvili (KP) hierarchy [8, 11] is an important research object. For the KP hierarchy, there is a kind of important sub hierarchy called the BC_r-KP hierarchy [9, 12, 28], including the BKP and CKP hierarchies [3, 8, 9, 19, 21, 28] as the special cases. The BC_r-KP hierarchy is introduced in [9], then it is rewritten by Zuo et al in [28], and also is named by the BC_r-KP hierarchy for brevity in [28]. Zuo *et al.* [28] construct additional symmetries of the BC_r-KP hierarchy and its constrained case, which shows that all of them from a $w_\infty^{BC_r}$-algebra and a Witt algebra respectively. The gauge transformations of the BC_r-KP hierarchy and its constrained cases are constructed, and the relations with the additional symmetries are investigated in [12]. In this paper, we continue to study the bilinear identities and squared eigenfunction symmetries of the BC_r-KP hierarchy and its constrained case respectively.

Bilinear identity [7–9, 11, 20] is a bilinear residue identity for wave functions, which is an important equivalent form of the KP hierarchy. From the bilinear identity, one can know the whole information of the KP hierarchy. And the bilinear identity can provide a crucial role when discussing the existence of the tau functions [8, 11]. Also the Hirota's bilinear equations [8] can be derived easily from the bilinear identity. By now, there are many results on bilinear identity. For example, the constrained KP hierarchy [7, 20], the constrained BKP hierarchy [25, 26] and the extended KP and BKP hierarchies [15, 16]. In this paper, we will consider the bilinear identity of the BC_r-KP hierarchy. In fact, the BC_r-KP hierarchy is equivalent to the sub hierarchy in [9], where the corresponding bilinear identities are discussed. But there are some differences in the expression forms

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between the BC_r-KP hierarchy and the sub hierarchy in [9]. And also the bilinear identities in [9] are not discussed completely. Therefore, it will be important to find the bilinear identity of the BC_r-KP hierarchy for itself. And what's more, as far as we know, the bilinear identities for the constrained BC_r-KP hierarchy have not discussed in literature.

The squared eigenfunction symmetry [2, 22–24], sometimes called the "ghost" symmetry [2], plays an important symmetries in the integrable system. The squared eigenfunction symmetry can be traced back to [22], where Oevel studied the solutions of the constrained KP hierarchy in the first time. Then it is widely investigated in [4, 5, 13, 14]. The squared eigenfunction symmetry can be used to define the new integrable system, such as the extended integrable system [17, 18] and the symmetry constraint [6, 7, 19–21, 24–27] and the additional symmetry [1, 2, 10, 28]. Recently, many researches have been done in the squared eigenfunction symmetries, for instance, the Toda lattice hierarchy and its sub hierarchy of B and C type [4, 5], the discrete KP [14] and modified discrete KP [13] hierarchies are investigated recently. In this paper, we will consider some properties of the squared eigenfunction symmetries of the BC_r-KP hierarchy and its constrained case.

The structure of this paper is as follows. The backgrounds of the BC_r-KP hierarchy will be reviewed in Section 2. In Section 3, we will study the bilinear identities of the BC_r-KP and its constrained case. In Section 4, the squared eigenfunction symmetries associated with the BC_r-KP and constrained BC_r-KP hierarchy are constructed. In Section 5, some conclusions and discussions are given.

2. The BC_r-KP Hierarchy

The BC_r-KP hierarchy is the sub hierarchy of the KP hierarchy [9, 28], which is defined by the pseudo-differential operators. The algebra $\mathfrak{g}$ of the pseudo-differential operators [11] is given by

$$\mathfrak{g} = \left\{ \sum_{i \ll \infty} u_i \partial_x^i \right\}, \tag{2.1}$$

where $u_i = u_i(t_1 = x, t_2, t_3, \dots)$. The multiplication of ∂_x^i with f obeys the Leibnitz rule [11]

$$\partial_x^i f = \sum_{j \geq 0} \binom{i}{j} f^{(j)} \partial_x^{i-j}, \quad i \in \mathbb{Z}. \tag{2.2}$$

In this paper, for $A = \sum_i a_i \partial_x^i \in \mathfrak{g}$, we denote $\mathrm{Res}_{\partial_x} \mathrm{A} = a_{-1}$, $A_{\geq k} = \sum_{i \geq k} a_i \partial_x^i$ and $A_{<k} = \sum_{i<k} a_i \partial_x^i$, $A_+ = A_{\geq 0}$, and $A_- = A_{<0}$. For $A, B \in \mathfrak{g}$ and a function f, $*$ is the conjugate operation: $(AB)^* = B^* A^*$, $\partial^* = -\partial$, $f^* = f$, and Af or $A \cdot f$ indicates that the multiplication of A and f, while $A(f)$ denotes the action of A on f.

Lemma 2.1 ([22]). *For arbitrary operator $A \in \mathfrak{g}$*

$$Res_{\partial_x} A \partial_x^{-1} = A_0, \quad Res_{\partial_x} A = -Res_{\partial_x} A^*, \tag{2.3}$$

$$(A_{\geq 0} \partial_x^{-1})_{<0} = A_0 \partial_x^{-1}, \quad (\partial_x^{-1} A_{\geq 0})_{<0} = \partial_x^{-1} (A^*)_0. \tag{2.4}$$

where $(\quad)_0$ denotes the zeroth-order term.

The KP hierarchy [9, 28] is defined by

$$\partial_{t_n} L = [B_n, L], \quad B_n = (L^n)_{\geq 0}, \quad n = 1, 2, 3, \ldots. \tag{2.5}$$

Here the Lax operator $L \in \mathfrak{g}$ is given by

$$L = \partial_x + u_1\partial_x^{-1} + u_2\partial_x^{-2} + u_3\partial_x^{-3} + \cdots. \tag{2.6}$$

The Lax operator L for the KP hierarchy can be expressed by the dressing operator S,

$$L = S\,\partial_x S^{-1}, \tag{2.7}$$

where S is given by

$$S = 1 + s_1\partial_x^{-1} + s_2\partial_x^{-2} + s_3\partial_x^{-3} + \cdots. \tag{2.8}$$

Then the Lax equation Eq. (2.5) is equivalent to

$$\partial_{t_n} S = -(L^n)_{<0} S = -(S\,\partial_x^n S^{-1})_{<0} S. \tag{2.9}$$

The eigenfunction ϕ and the adjoint eigenfunction ψ of the KP hierarchy are defined by

$$\partial_{t_n}\phi = B_n(\phi), \quad \partial_{t_n}\psi = -B_n^*(\psi), \tag{2.10}$$

respectively. The wave and adjoint wave functions of the KP hierarchy are defined by the following way:

$$w(t,\lambda) = S(e^{\xi(t,\lambda)}) = (1 + s_1\lambda^{-1} + s_2\lambda^{-2} + \cdots)e^{\sum_{i=1}^{\infty} t_i\lambda^i}, \tag{2.11}$$

$$w^*(t,\lambda) = (S^{-1})^*(e^{-\xi(t,\lambda)}) = (1 + s_1^*\lambda^{-1} + s_2^*\lambda^{-2} + \cdots)e^{-\sum_{i=1}^{\infty} t_i\lambda^i}, \tag{2.12}$$

where λ is the spectral parameter. And w and w^* satisfy

$$L^n w(t,\lambda) = \lambda^n w(t,\lambda), \quad \partial_{t_n} w(t,\lambda) = B_n w(t,\lambda), \tag{2.13}$$

$$(L^n)^* w^*(t,\lambda) = \lambda^n w^*(t,\lambda), \quad \partial_{t_n} w^*(t,\lambda) = -B_n^* w^*(t,\lambda). \tag{2.14}$$

Next, we discuss the BC_r-KP hierarchy. Define

$$Q = (S^{-1})^*\partial^r S^{-1}, \quad r \in \mathbb{Z}_{\geq 0}, \tag{2.15}$$

where S is the dressing operator of the KP hierarchy. Obviously, Q is a r-order differential operator and has the properties:

$$Q^* = (-1)^r Q, \quad QL + L^*Q = 0, \tag{2.16}$$

$$Q_{t_n} = (1 + (-1)^n)(S^{-1})^*\partial_x^{r+n} S^{-1} - QB_n - B_n^* Q. \tag{2.17}$$

The BC_r-KP hierarchy is defined by the following constraints,

$$Q = Q_+. \tag{2.18}$$

Therefore for the BC_r-KP hierarchy, by taking the negative part of Eq. (2.17), one obtains $1 + (-1)^n = 0$, so there are only odd flows in the BC_r-KP hierarchy, and the Lax equation of the BC_r-KP hierarchy is given in the following way

$$\partial_{t_{2n+1}} L = [B_{2n+1}, L], \quad n \in \mathbb{Z}_{\geq 0}, \tag{2.19}$$

In particular,

- BC_0-KP is the CKP hierarchy: when $r=0$, $Q=1$, then $L^*=-L$,
- BC_1-KP is the BKP hierarchy: when $r=1$, $Q=\partial_x$, then $L^*=-\partial_x L\partial_x^{-1}$.

According to Eq. (2.10) and Eq. (2.17), the adjoint eigenfunction of the BC_r-KP hierarchy can be expressed by $Q(\phi)$, where ϕ is eigenfunction. The constrained BC_r-KP hierarchy is defined by imposing the below constraints on the Lax operator of the BC_r-KP hierarchy

$$L^k=(L^k)_{\geq 0}+\sum_{j=1}^{m}\Big(q_{1j}\partial_x^{-1}\cdot Q(q_{2j})+(-1)^r q_{2j}\partial_x^{-1}\cdot Q(q_{1j})\Big),\quad k=1,3,\ldots, \tag{2.20}$$

where q_{1j} and q_{2j} are independent eigenfunctions of the BC_r-KP hierarchy.

3. Bilinear Identities of the BC_r-KP Hierarchy and its Constrained Case

Bilinear identity formulations of the BC_r-KP hierarchy and its constrained case will be studied in this section. Before discussion, the following lemmas will be needed.

Lemma 3.1 ([11]). *For A, $B\in\mathfrak{g}$,*

$$Res_\lambda A(e^{x\lambda})\cdot B(e^{-x\lambda})=Res_{\partial_x}AB^*, \tag{3.1}$$

where B^* is the adjoint operator of B.

Lemma 3.2. *If we let $A(x)=\sum_i a_i(x)\partial_x^i$ and $B(x')=\sum_j b_j(x')\partial_{x'}^j$ be two operators, then*

$$A(x)B^*(x)\partial_x(\Delta^0)=Res_\lambda A(x)(e^{x\lambda})\cdot B(x')(e^{-x'\lambda}), \tag{3.2}$$

where $\Delta^0=(x-x')^0$ and

$$\partial_x^{-a}(\Delta^0)=\begin{cases}0, & a<0,\\ \dfrac{(x-x')^a}{a!}, & a\geq 0.\end{cases} \tag{3.3}$$

Proof. Firstly, by the formal expansion of $B(x')$ at $x'=x$ and according to Lemma 3.1

$$\begin{aligned}
&Res_\lambda A(x)(e^{x\lambda})\cdot B(x')(e^{-x'\lambda})\\
&=Res_\lambda A(x)(e^{x\lambda})\sum_{n=0}^{\infty}\frac{(x'-x)^n}{n!}\partial_x^n B(x)(e^{-x\lambda})\\
&=\sum_{n=0}^{\infty}\frac{(x'-x)^n}{n!}Res_{\partial_x}A(x)B^*(x)(-1)^n\partial_x^n\\
&=\sum_{n=0}^{\infty}\frac{(x-x')^n}{n!}Res_{\partial_x}A(x)B^*(x)\partial_x^n\\
&=\sum_{n=0}^{\infty}\partial_x^{-n}(\Delta^0)Res_{\partial_x}A(x)B^*(x)\partial_x^n.
\end{aligned} \tag{3.4}$$

Then if set $A(x)B^*(x)=\sum_{i\in\mathbb{Z}}c_i\partial_x^i$ and notice that

$$Res_{\partial_x}A(x)B^*(x)\partial_x^n=Res_{\partial_x}\sum_{i\in\mathbb{Z}}c_i\partial_x^{i+n}=c_{-n-1}, \tag{3.5}$$

one can obtain

$$
\begin{aligned}
& Res_\lambda A(x)(e^{x\lambda})B(x')(e^{-x'\lambda}) \\
&= \sum_{n=0}^{\infty} c_{-n-1}\partial_x^{-n}\Delta^0 \xlongequal{-n-1=i} \sum_{k\in\mathbb{Z}} c_i \partial_x^{i+1}\Delta^0 \\
&= A(x)B^*(x)\partial_x(\Delta^0) \qquad (3.6)
\end{aligned}
$$

□

Proposition 3.1. *The wave and adjoint wave functions of the BC_r-KP hierarchy satisfy the bilinear identities*

$$Res_\lambda \lambda^r w^*(t,-\lambda)w^*(t',\lambda) = 0. \qquad (3.7)$$

Proof. The wave and adjoint wave functions of the BC_r-KP hierarchy satisfy the following bilinear identities

$$\partial_{t_{2n+1}} w(x,\bar{t},\lambda) = (S\partial_x^{2n+1}S^{-1})_{\geq 0}(w(x,\bar{t},\lambda)), \quad \partial_{t_{2n+1}} w^*(x,\bar{t},\lambda) = -(S\partial_x^{2n+1}S^{-1})^*_{\geq 0}(w^*(x,\bar{t},\lambda)), \quad (3.8)$$

where $\bar{t} = (t_3, t_5, \ldots)$. Therefore $\partial_t^\alpha w^*(x',\bar{t},\alpha)$ can be written in the following way

$$\partial_t^\alpha w^*(x',\bar{t},\lambda) = P_\alpha(x',\bar{t})(w^*(x',\bar{t},\lambda)), \qquad (3.9)$$

where $\partial_t^\alpha = \prod_{n=1}^\infty \partial_{t_{2n+1}}^{\alpha_{2n+1}}$ and $P_\alpha(x',\bar{t}) = \sum_{i\geq 0} a_{\alpha,i}(x',\bar{t})\partial_{x'}^i$ is a differential operator. Next by considering the formal expansion of w^* with respect to $\bar{t}'$ at $\bar{t}$

$$w^*(x',\bar{t}',\lambda) = \sum_{\alpha=(\alpha_3,\alpha_5,\ldots)\geq 0} \frac{(\bar{t}'-\bar{t})^\alpha}{\alpha!}\partial_t^\alpha w^*(x',\bar{t},\lambda), \qquad (3.10)$$

with

$$(\bar{t}'-\bar{t})^\alpha = \prod_{n=1}^\infty (t'_{2n+1}-t_{2n+1})^{\alpha_{2n+1}}, \quad \alpha! = \prod_{n=1}^\infty \alpha_{2n+1}!, \ \ \alpha \geq 0 \ \Leftrightarrow \ \alpha_{2n+1}\geq 0, \ \ n = 1,2,\ldots, \quad (3.11)$$

one can obtain

$$w^*(x',\bar{t}',\lambda) = \sum_{\alpha\geq 0}\frac{(\bar{t}'-\bar{t})^\alpha}{\alpha!}P_\alpha(x',\bar{t})S^{-1}(x',\bar{t})^* e^{-\sum_{n=1}^\infty (-1)^{2n+1}\bar{t}_{2n+1}\partial_{x'}^{2n+1}}(e^{-x'\lambda}). \qquad (3.12)$$

Thus according to Lemma 3.2

$$
\begin{aligned}
& Res_\lambda \lambda^r w^*(x,\bar{t},-\lambda)w^*(x',\bar{t}',\lambda) \\
&= Res_\lambda (S^{-1}(x,\bar{t}))^*\partial_x^r(e^{x\lambda})\sum_{\alpha\geq 0}\frac{(\bar{t}'-\bar{t})^\alpha}{\alpha!}P_\alpha(x',\bar{t})(S^{-1}(x',\bar{t}))^*(e^{-x'\lambda}) \\
&= \sum_{\alpha\geq 0}\frac{(\bar{t}'-\bar{t})^\alpha}{\alpha!}(S^{-1}(x,\bar{t}))^*\partial_x^r S^{-1}(x,\bar{t})P_\alpha^*(x,\bar{t})\partial_x(\Delta^0) \\
&= \sum_{\alpha\geq 0}\frac{(\bar{t}'-\bar{t})^\alpha}{\alpha!}Q(t)P_\alpha^*(x,\bar{t})\partial_x(\Delta^0) = 0, \qquad (3.13)
\end{aligned}
$$

where we have used the fact $Q = Q_+$ and P_α are differential operators. □

Proposition 3.2. *Let $w(t,\lambda)$ and $w^*(t,\lambda)$ be expressed by $w = S(e^{\xi(t,\lambda)})$ and $w^* = (S^{-1})^*(e^{-\xi(t,\lambda)})$ respectively, where $S = 1+\sum_{n=1}^{\infty} s_n\partial^{-n}$ and $\xi(t,\lambda) = \sum_{n=0}^{\infty} t_{2n+1}\lambda^{2n+1}$. And define $Q = (S^{-1})^*\partial^r S^{-1}$.*

Then if w and w^* satisfy Eq. (3.7), one can obtain $Q = Q_+$ and $\partial_{t_{2n+1}}S = -(S\partial_x^{2n+1}S^{-1})_{<0}S$, which means that $w(t,\lambda)$ and $w^*(t,\lambda)$ are the wave and adjoint wave functions of the BC_r-KP hierarchy.

Proof. Starting from the bilinear identity Eq. (3.7) and using Lemma 3.2

$$\begin{aligned}
0 &= Res_\lambda \lambda^r w^*(x,\bar{t},-\lambda)w^*(x',\bar{t}',\lambda)\\
&= Res_\lambda\Big(S^{-1}(x,\bar{t})^* e^{\sum_{n\geq1} t_{2n+1}\partial_x^{2n+1}}\partial_x^r\Big)(e^{x\lambda})\cdot\Big(S^{-1}(x',\bar{t}')^* e^{-\sum_{n\geq1} t'_{2n+1}(-\partial_{x'})^{2n+1}}\Big)(e^{-x'\lambda})\\
&= S^{-1}(x,\bar{t})^*\partial_x^r e^{\sum_{n\geq1}(t_{2n+1}-t'_{2n+1})\partial_x^{2n+1}} S^{-1}(x,\bar{t}')\partial_x(\Delta^0).
\end{aligned} \tag{3.14}$$

Let $\bar{t} = \bar{t}'$, then

$$(S^{-1})^*\partial_x^r S^{-1}\partial_x(\Delta^0) = Q(t)\partial_x(\Delta^0) = 0, \tag{3.15}$$

which implies Q is a differential operator.

On the other hand

$$w^*(t,\lambda) = (S^{-1})^*(e^{-\xi(t,\lambda)}) = Q(t)S\partial_x^{-r}(e^{-\xi(t,\lambda)}) = (-1)^r\lambda^{-r}Q(t)(w(t,-\lambda)). \tag{3.16}$$

So

$$0 = Res_\lambda \lambda^r w^*(x,\bar{t},-\lambda)w^*(x',\bar{t}',\lambda) = Res_\lambda Q(t)\big(w(x,\bar{t},\lambda)\big)w^*(x',\bar{t}',\lambda). \tag{3.17}$$

Apply $Q^{-1}(t)$ on both sides of Eq. (3.17) and let $Q^{-1}(t)(0) = \alpha(x,\bar{t})$,

$$\begin{aligned}
\alpha(x,\bar{t}) &= Res_\lambda w(x,\bar{t},\lambda)w^*(x',\bar{t}',\lambda)\\
&= Res_\lambda\Big(S(x,\bar{t})e^{\sum_{n\geq1} t_{2n+1}\partial_x^{2n+1}}\Big)(e^{x\lambda})\Big(\big(S^{-1}(x',\bar{t}')\big)^* e^{-\sum_{n\geq1} t'_{2n+1}(-\partial_{x'})^{2n+1}}\Big)(e^{-x'\lambda})\\
&= S(x,\bar{t})e^{\sum_{n\geq1}(t_{2n+1}-t'_{2n+1})\partial_x^{2n+1}} S^{-1}(x,\bar{t}')\partial_x(\Delta^0).
\end{aligned} \tag{3.18}$$

If $\bar{t} = \bar{t}'$, we have

$$\alpha(x,\bar{t}) = S(t)S^{-1}(t)\partial_x(\Delta^0) = 0, \tag{3.19}$$

Thus $Res_\lambda w(t,\lambda)w^*(t',\lambda) = 0$, which satisfies the bilinear identity of KP hierarchy, it is obvious that

$$\partial_{t_{2n+1}}S = -(S\partial_x^{2n+1}S^{-1})_{<0}S. \tag{3.20}$$

□

Corollary 3.1. *If $Q^{-1}(t) = \sum_{i=0}^{\infty} a_i(t)\partial_x^{-r-i}$, where $a_0 = 1$, $a_1 = 0$, the bilinear identity Eq. (3.7) can be rewritten as*

$$Res_\lambda \lambda^{-r} w(x,\bar{t},\lambda)w(x',\bar{t}',-\lambda) = \begin{cases} 0, & r=0, \quad (CKP), \\ 1, & r=1, \quad (BKP), \\ \dfrac{(x-x')^{r-1}}{(r-1)!} + \displaystyle\sum_{i>1}^{\infty} a_i(t')\dfrac{(x-x')^{r+i-1}}{(r+i-1)!}, & r>1. \end{cases} \tag{3.21}$$

Proof. From the above proof in Proposition 3.2 and Eq. (3.16),

$$Res_\lambda \lambda^{-r} w(x,\bar{t},\lambda)Q(t')w(x',\bar{t},-\lambda) = 0. \tag{3.22}$$

By applying $Q^{-1}(t')$, and letting $\beta(x',\bar{t}') = Q^{-1}(t')(0)$ and $\bar{t}' = \bar{t}$,

$$\begin{aligned} \beta(x',\bar{t}) &= S(x,\bar{t})\partial_x^{-r}S^*(x,\bar{t})\partial_x(\Delta^0) \\ &= Q^{-1}(t)\partial_x(\Delta^0) \\ &= \begin{cases} 0, & r=0, \\ 1, & r=1, \\ \dfrac{(x-x')^{r-1}}{(r-1)!} + \displaystyle\sum_{i>1}^{\infty} a_i(x',\bar{t})\dfrac{(x-x')^{r+i-1}}{(r+i-1)!}, & r>1. \end{cases} \end{aligned} \tag{3.23}$$

□

Next, we discuss the bilinear identity formulation of the constrained BC_r-KP hierarchy. For convenience, we will introduce squared eigenfunction potential $\Omega(\psi,\phi)$ [22, 23], which is determined by the following conditions

$$\Omega(\psi,\phi)_x = \psi\phi, \quad \Omega(\psi,\phi)_{t_{2n+1}} = Res\partial_x^{-1}\psi(L^{2n+1})_{\geq 0}\phi\partial_x^{-1}. \tag{3.24}$$

Proposition 3.3. *For the constrained BC_r-KP hierarchy Eq.* (2.20)*,*

$$\sum_{j=1}^{m}\Big(Q(q_{1j})(t)\cdot Q(q_{2j})(t') + (-1)^r Q(q_{2j})(t)\cdot Q(q_{1j})(t')\Big) = Res_\lambda \lambda^{k+r} w^*(t,-\lambda)w^*(t',\lambda), \tag{3.25}$$

$$Q(q_{lj})(t) = -Res_\lambda\Big(\lambda^r w^*(t,\lambda)\Omega\Big(q_{lj}(t'), w^*(t',\lambda)\Big)\Big). \tag{3.26}$$

where $l = 1, 2$.

Proof. Firstly, consider the residue of $L^k\partial_x^p$ for an arbitrary integer $p \geq 0$, and notice that $Res_{\partial_x}(L^k)_-\partial_x^p = Res_{\partial_x}L^k\partial_x^p$. Then according to Lemma 3.1 and $S = Q^{-1}(S^{-1})^*\partial_x^r$,

$$\begin{aligned} &\sum_{j=1}^{m}\Big(q_{1j}(t)(-1)^p\partial_x^p\Big(Q(q_{2j})(t)\Big) + (-1)^r q_{2j}(t)(-1)^p\partial_x^p\Big(Q(q_{1j})(t)\Big)\Big) \\ &= Res_{\partial_x}L^k\partial_x^p = Res_{\partial_x}S\partial_x^k S^{-1}\partial_x^p \\ &= (-1)^p Res_\lambda S\partial_x^k(e^{\xi(t,\lambda)})\partial_x^p(S^{-1})^*(e^{-\xi(t,\lambda)}) \\ &= (-1)^p Res_\lambda Q^{-1}(S^{-1})^*\partial_x^{k+r}(e^{\xi(t,\lambda)})\partial_x^p(S^{-1})^*(e^{-\xi(t,\lambda)}) \\ &= (-1)^p Res_\lambda \lambda^{k+r}Q^{-1}(t)w^*(t,-\lambda)\partial_x^p w^*(t,\lambda). \end{aligned} \tag{3.27}$$

Further by using Eq. (2.10) and the formal expansion at $t' = t$,

$$\sum_{j=1}^{m}\Big(q_{1j}(t)Q(q_{2j})(t') + (-1)^r q_{2j}(t)Q(q_{1j})(t')\Big) = Res_\lambda \lambda^{k+r} Q^{-1}(t) w^*(t,-\lambda) w^*(t',\lambda). \tag{3.28}$$

Lastly, by applying $Q(t)$ on both sides of Eq. (3.28), one can obtain Eq. (3.25).

According to Eq. (2.14) and Eq. (3.7),

$$\begin{aligned}
&\sum_{j=1}^{m}\Big(Q(q_{1j})(t)Q(q_{2j})(t') + (-1)^r Q(q_{2j})(t)Q(q_{1j})(t')\Big)\\
&= Res_\lambda \lambda^r w^*(t,-\lambda)(L^k)^*(t') w^*(t',\lambda)\\
&= Res_\lambda \lambda^r w^*(t,-\lambda)\Big(B_k^*(t') - \sum_{j=1}^{m}(Q(q_{2j})(t')\partial_{x'}^{-1}\cdot q_{1j}(t')\\
&\quad +(-1)^r Q(q_{1j})(t')\partial_{x'}^{-1}\cdot q_{2j}(t'))\Big) w^*(t',\lambda)\\
&= -\sum_{j=1}^{m} Res_\lambda \lambda^r w^*(t,-\lambda)\Big(Q(q_{2j})(t')\Omega\Big(q_{1j}(t'), w^*(t',\lambda)\Big)\\
&\quad +(-1)^r Q(q_{1j})(t')\Omega\Big(q_{2j}(t'), w^*(t',\lambda)\Big)\Big).
\end{aligned} \tag{3.29}$$

According to $Q(q_{1j})(t')$ and $Q(q_{2j})(t')$ are independent, and from the comparison of both sides

$$Q(q_{lj})(t) = -Res_\lambda \lambda^r w^*(t,-\lambda)\Omega\Big(q_{lj}(t'), w^*(t',\lambda)\Big), \quad l = 1, 2. \tag{3.30}$$

□

Proposition 3.4. *Let $w(t,\lambda)$ and $w^*(t,\lambda)$ be expressed by $w = S(e^{\xi(t,\lambda)})$ and $w^* = (S^{-1})^*(e^{-\xi(t,\lambda)})$ respectively, where $w = 1 + \sum_{n=1}^{\infty} s_n \partial^{-n}$ and $\xi(t,\lambda) = \sum_{n=0}^{\infty} t_{2n+1}\lambda^{2n+1}$. And define $Q = (S^{-1})^*\partial^r S^{-1}$.*

If w and w^* satisfy Eq. (3.25) and Eq. (3.26), then $Q = Q_+$, $\partial_{t_{2n+1}} S = -(S\partial_x^{2n+1} S^{-1})_{<0} S$ and $(L^k)_{<0} = \sum_{j=1}^{m}\Big(q_{1j}\partial_x^{-1}\cdot Q(q_{2j}) + (-1)^r q_{2j}\partial_x^{-1}\cdot Q(q_{1j})\Big)$. Therefore, Eq. (3.25) and Eq. (3.26) satisfy the characteristics of the constrained BC_r-KP hierarchy.

Proof. Firstly, we prove Eq. (3.26) implies the bilinear identity Eq. (3.7). By differentiating both sides of Eq. (3.26) with respect to x',

$$\partial_{x'}(Q(q_{lj})(t)) = 0 = -Res_\lambda \lambda^r w^*(t,-\lambda) q_{lj}(t') w^*(t',\lambda), \quad l = 1,2. \tag{3.31}$$

It is obvious that

$$Res_\lambda \lambda^r w^*(t,-\lambda) w^*(t',\lambda) = 0. \tag{3.32}$$

On the other hand, by differentiating both sides of Eq. (3.25) with respect to t_{2n+1},

$$\begin{aligned}
&\sum_{j=1}^{m}\Big(\partial_{t_{2n+1}} Q(q_{1j})(t)Q(q_{2j})(t') + (-1)^r \partial_{t_{2n+1}} Q(q_{2j})(t)Q(q_{1j})(t')\Big)\\
&= -Res_\lambda \lambda^{k+r} B_{2n+1}^*(t) w^*(t,-\lambda) w^*(t',\lambda)\\
&= -\sum_{j=1}^{m}\Big(B_{2n+1}^*(Q(q_{1j})(t))Q(q_{2j})(t') + (-1)^r B_{2n+1}^*(Q(q_{2j})(t))Q(q_{1j})(t')\Big),
\end{aligned} \tag{3.33}$$

which means that $Q(q_{lj})(t)$ is the adjoint eigenfunction, since $Q(q_{1j})(t)$ and $Q(q_{2j})(t)$ are independent. By differentiating both sides of $q_{lj}(t) = Q^{-1}\Big(Q(q_{lj})(t)\Big)$ with respect to t_{2n+1}, according to Eq. (2.10) and $Q(q_{lj})(t)$ is the adjoint eigenfunction

$$\partial_{t_{2n+1}} q_{lj}(t) = -Q^{-1}\partial_{t_{2n+1}} Q \cdot Q^{-1}\Big(Q(q_{lj})(t)\Big) + Q^{-1}\cdot \partial_{t_{2n+1}}\Big(Q(q_{lj})(t)\Big) = B_{2n+1}(q_{lj}(t)). \tag{3.34}$$

So $q_{lj}(t)$ is the eigenfunction.

Finally, by applying $Q^{-1}(t)$ on Eq. (3.25), formally expanding at $t' = t$ and according to Eq. (2.13) and Lemma 3.1

$$\begin{aligned}
&\sum_{j=1}^{m}\Big(q_{1j}(t)\partial_x^p Q(q_{2j})(t) + (-1)^r q_{2j}(t)\partial_x^p Q(q_{1j})(t)\Big)\\
&= Res_\lambda \lambda^{k+r} Q^{-1}(t) w^*(t,-\lambda)\partial_x^p w^*(t,\lambda)\\
&= Res_\lambda L^k S\, e^{x\lambda}\partial_x^p (S^{-1})^* e^{-x\lambda}\\
&= (-1)^p Res_{\partial_x} L^k \partial_x^p.
\end{aligned} \tag{3.35}$$

This implies that the negative part of the Lax operator L^k has the form Eq. (2.20). It is proved that Eq. (3.25) and Eq. (3.26) satisfy the characteristics of the constrained BC_r-KP hierarchy. □

4. Squared Eigenfunction Symmetries of the BC_r-KP Hierarchy and its Constrained Case

We construct the squared eigenfunction symmetries associated with the BC_r-KP hierarchy and its constrained case in this section.

Denoting the corresponding group parameter as α, we define the squared eigenfunction flow by

$$\partial_\alpha S = \sum_{i=1}^{m}\Big(\phi_{1i}\partial_x^{-1}\cdot Q(\phi_{2i}) + (-1)^r \phi_{2i}\partial_x^{-1}\cdot Q(\phi_{1i})\Big)S, \tag{4.1}$$

or

$$\partial_\alpha L = \Big[\sum_{i=1}^{m}\Big(\phi_{1i}\partial_x^{-1}\cdot Q(\phi_{2i}) + (-1)^r \phi_{2i}\partial_x^{-1}\cdot Q(\phi_{1i})\Big), L\Big], \tag{4.2}$$

It is essential that the definition of ∂_α must keep the BC_r-constraint. The next proposition will explain the definition of ∂_α is reasonable.

Proposition 4.1. *For the BC_r-KP hierarchy,*

$$\partial_\alpha Q = (\partial_\alpha Q)_+. \tag{4.3}$$

Proof. According to Eq. (2.15)

$$\begin{aligned}
\partial_\alpha Q &= -(S^{-1})^*\partial_\alpha S^*\cdot (S^{-1})^*\partial_x^r S^{-1} - (S^{-1})^*\partial_x^r S^{-1}\partial_\alpha S\cdot S^{-1}\\
&= -(\partial_\alpha S\cdot S^{-1})^* Q - Q\partial_\alpha S\cdot S^{-1}.
\end{aligned} \tag{4.4}$$

According to Eq. (2.4), Eq. (4.1) and $Q^* = (-1)^r Q$, we can obtain that

$$\begin{aligned}
(\partial_\alpha Q)_- &= -\Big((\partial_\alpha S\cdot S^{-1})^* Q + Q\partial_\alpha S\cdot S^{-1}\Big)_- \\
&= -\sum_{i=1}^m \Big((\phi_{1i}\partial_x^{-1}\cdot Q(\phi_{2i}) + (-1)^r\phi_{2i}\partial_x^{-1}\cdot Q(\phi_{1i}))^* Q \\
&\quad + Q(\phi_{1i}\partial_x^{-1}\cdot Q(\phi_{2i}) + (-1)^r\phi_{2i}\partial_x^{-1}\cdot Q(\phi_{1i}))\Big)_- \\
&= \sum_{i=1}^m \Big(Q(\phi_{2i})\partial_x^{-1}\cdot Q^*(\phi_{1i}) + (-1)^r Q(\phi_{1i})\partial_x^{-1}\cdot Q^*(\phi_{2i}) \\
&\quad - Q(\phi_{1i})\partial_x^{-1}\cdot Q(\phi_{2i}) - (-1)^r Q(\phi_{2i})\partial_x^{-1}\cdot Q(\phi_{1i})\Big) \\
&= 0. \qquad (4.5)
\end{aligned}$$

□

Next, we need to check $[\partial_\alpha, \partial_{t_{2n+1}}] = 0$, which means that ∂_α is the symmetry of the BC_r-KP hierarchy.

Proposition 4.2. *For the BC_r-KP hierarchy,*

$$[\partial_\alpha, \partial_{t_{2n+1}}] = 0. \qquad (4.6)$$

Proof. For convenience, assume $M' = \sum_{i=1}^m \Big(\phi_{1i}\partial_x^{-1}\cdot Q(\phi_{2i}) + (-1)^r\phi_{2i}\partial_x^{-1}\cdot Q(\phi_{1i})\Big)$. Then

$$\partial_\alpha L = [M', L] \quad\Longrightarrow\quad \partial_\alpha L^{2n+1} = [M', L^{2n+1}], \qquad (4.7)$$

whence the projection to differential orders ≥ 0 yields to $\partial_\alpha B_{2n+1} = [M', L^{2n+1}]_{\geq 0}$. Because $M' \in \mathfrak{g}_{<0}$ and $[\mathfrak{g}_{<0}, \mathfrak{g}_{<0}] \subset \mathfrak{g}_{<0}$

$$\partial_\alpha B_{2n+1} = [M', (L^{2n+1})_{\geq 0} + (L^{2n+1})_{<0}]_{\geq 0} = [M', B_{2n+1}]_{\geq 0}. \qquad (4.8)$$

On the other hand, according to Eq. (2.4)

$$\begin{aligned}
[B_{2n+1}, M']_{<0} &= \sum_{i=1}^m \Big(B_{2n+1}\Big(\phi_{1i}\partial_x^{-1}\cdot Q(\phi_{2i}) + (-1)^r\phi_{2i}\partial_x^{-1}\cdot Q(\phi_{1i})\Big)\Big)_{<0} \\
&\quad - \sum_{i=1}^m \Big(\Big(\phi_{1i}\partial_x^{-1}\cdot Q(\phi_{2i}) + (-1)^r\phi_{2i}\partial_x^{-1}\cdot Q(\phi_{1i})\Big)B_{2n+1}\Big)_{<0} \\
&= \sum_{i=1}^m \Big(B_{2n+1}(\phi_{1i})\partial_x^{-1}\cdot Q(\phi_{2i}) + (-1)^r B_{2n+1}(\phi_{2i})\partial_x^{-1}\cdot Q(\phi_{1i})\Big) \\
&\quad - \sum_{i=1}^m \Big(\phi_{1i}\partial_x^{-1}\cdot B_{2n+1}^*(Q(\phi_{2i})) + (-1)^r\phi_{2i}\partial_x^{-1}\cdot B_{2n+1}^*(Q(\phi_{1i})B_n)\Big), \qquad (4.9)
\end{aligned}$$

one concludes

$$[B_{2n+1}, M']_{<0} = \partial_{t_{2n+1}} M'. \qquad (4.10)$$

The pseudo-differential zero curvature equation will be obtained by Eq. (4.8) and Eq. (4.10),

$$\partial_\alpha B_{2n+1} - \partial_{t_{2n+1}} M' = [M', B_{2n+1}]. \qquad (4.11)$$

This establishes the commutativity of the squared eigenfunction flow and the Lax hierarchy:

$$
\begin{aligned}
\partial_{t_{2n+1}} L_\alpha - \partial_\alpha L_{t_{2n+1}} &= [M', L]_{t_{2n+1}} - [B_{2n+1}, L]_\alpha \\
&= [M'_{t_{2n+1}} - \partial_\alpha B_{2n+1}, L] + [M', L_{t_{2n+1}}] - [B_{2n+1}, L_\alpha] \\
&= \Big[\partial_{t_{2n+1}} M' - \partial_\alpha B_{2n+1} + [M', B_{2n+1}], L\Big] = 0. \qquad (4.12)
\end{aligned}
$$

□

Remark 4.1. If identifying the squared eigefunction symmetry flow ∂_α with the time flow $-\partial_{t_{2k-1}}$ in the BC_r-KP hierarchy, one can obtain the following constraint on the Lax operator

$$
(L^k)_{<0} = \sum_{i=1}^{m} \Big(\phi_{1i} \cdot \partial_x^{-1} \cdot Q(\phi_{2i}) + (-1)^r \phi_{2i} \cdot \partial_x^{-1} \cdot Q(\phi_{1i})\Big), \qquad (4.13)
$$

which is just the definition of the constrained BC_r-KP hierarchy and k is odd.

Then we will consider the actions of ∂_α on eigenfunctions.

Proposition 4.3. *The squared eigenfunction symmetry of Proposition 4.2 is the compatibility condition of the linear problems:*

$$
\phi_{t_{2n+1}} = (L^{2n+1})_{\geq 0}(\phi), \quad \phi_\alpha = \sum_{i=1}^{m} \Big(\phi_{1i} \Omega(Q(\phi_{2i}), \phi) + (-1)^r \phi_{2i} \Omega(Q(\phi_{1i}), \phi)\Big). \qquad (4.14)
$$

Proof. The aim is to prove $(\phi_{t_{2n+1}})_\alpha = (\phi_\alpha)_{t_{2n+1}}$ with the help of Eq. (4.1) or Eq. (4.2). At first,

$$
\begin{aligned}
\partial_{t_{2n+1}} \phi_\alpha &= \sum_{i=1}^{m} \Big(B_{2n+1}(\phi_{1i}) \Omega(Q(\phi_{2i}), \phi) + (-1)^r B_{2n+1}(\phi_{2i}) \Omega(Q(\phi_{1i}), \phi)\Big) \\
&\quad + \sum_{i=1}^{m} Res_\partial \Big(\phi_{1i} \partial_x^{-1} \cdot Q(\phi_{2i}) B_{2n+1} \phi \cdot \partial_x^{-1} + (-1)^r \phi_{2i} \partial_x^{-1} \cdot Q(\phi_{1i}) B_{2n+1} \phi \cdot \partial_x^{-1}\Big) \\
&= \sum_{i=1}^{m} \Big(B_{2n+1}(\phi_{1i}) \Omega(Q(\phi_{2i}), \phi) + (-1)^r B_{2n+1}(\phi_{2i}) \Omega(Q(\phi_{1i}), \phi)\Big) \\
&\quad + Res_\partial (M' B_{2n+1} \phi \partial_x^{-1}). \qquad (4.15)
\end{aligned}
$$

According to Eq. (4.11), we obtain that

$$
\partial_\alpha B_{2n+1} - \partial_{t_{2n+1}} M' = M' B_{2n+1} - B_{2n+1} M'. \qquad (4.16)
$$

Therefore

$$
\begin{aligned}
\partial_{t_{2n+1}} \phi_\alpha &= \sum_{i=1}^{m} \Big(B_{2n+1}(\phi_{1i}) \Omega(Q(\phi_{2i}), \phi) + (-1)^r B_{2n+1}(\phi_{2i}) \Omega(Q(\phi_{1i}), \phi)\Big) \\
&\quad + Res_{\partial_x} (\partial_\alpha B_{2n+1} \cdot \phi \partial_x^{-1}) - Res_\partial (M'_{t_{2n+1}} \phi \partial_x^{-1}) + Res_\partial (B_{2n+1} M' \phi \partial_x^{-1}). \qquad (4.17)
\end{aligned}
$$

According to Eq. (2.3), the second term yields

$$
Res_{\partial_x} (\partial_\alpha B_{2n+1} \cdot \phi \partial_x^{-1}) = \partial_\alpha B_{2n+1}(\phi). \qquad (4.18)
$$

Because $M'_{t_{2n+1}}\partial_x^{-1} \in \mathfrak{g}_{<0}$, the third term yields

$$Res_{\partial_x}(M'_{t_{2n+1}}\phi\partial_x^{-1}) = 0. \tag{4.19}$$

According to $f_x = \partial_x f - f\partial_x$, the fourth term yields

$$\begin{aligned}
Res_{\partial_x}(B_{2n+1}M'\phi\partial_x^{-1}) &= \sum_{i=1}^{m} Res_{\partial_x}\Big(B_{2n+1}\phi_{1i}\partial_x^{-1}\cdot Q(\phi_{2i})\phi\partial_x^{-1} + (-1)^r B_{2n+1}\phi_{2i}\partial_x^{-1}\cdot Q(\phi_{1i})\phi\partial_x^{-1}\Big)\\
&= \sum_{i=1}^{m} Res_{\partial_x}\Big(B_{2n+1}\phi_{1i}\partial_x^{-1}\cdot(\partial_x\Omega(Q(\phi_{2i}),\phi) - \Omega(Q(\phi_{2i}),\phi)\partial_x)\cdot\partial_x^{-1}\Big)\\
&\quad + \sum_{i=1}^{m} Res_{\partial_x}\Big((-1)^r B_{2n+1}\phi_{2i}\partial_x^{-1}\cdot(\partial_x\Omega(Q(\phi_{1i}),\phi) - \Omega(Q(\phi_{1i}),\phi)\partial_x)\cdot\partial_x^{-1}\Big)\\
&= B_{2n+1}\Big(\sum_{i=1}^{m}(\phi_{1i}\Omega(Q(\phi_{2i}),\phi) + (-1)^r\phi_{2i}\Omega(Q(\phi_{1i}),\phi))\Big)\\
&\quad - \sum_{i=1}^{m}\Big(B_{2n+1}(\phi_{1i})\Omega(Q(\phi_{2i}),\phi) + (-1)^r B_{2n+1}(\phi_{2i})\Omega(Q(\phi_{1i}),\phi)\Big),
\end{aligned} \tag{4.20}$$

whence

$$\begin{aligned}
\partial_{t_{2n+1}}\phi_\alpha &= \sum_{i=1}^{m}\Big(B_{2n+1}(\phi_{1i})\Omega(Q(\phi_{2i}),\phi) + (-1)^r B_{2n+1}(\phi_{2i})\Omega(Q(\phi_{1i}),\phi)\Big)\\
&\quad + \partial_\alpha B_{2n+1}(\phi) + B_{2n+1}\Big(\sum_{i=1}^{m}(\phi_{1i}\Omega(Q(\phi_{2i}),\phi) + (-1)^r\phi_{2i}\Omega(Q(\phi_{1i}),\phi))\Big)\\
&\quad - \sum_{i=1}^{m}\Big(B_{2n+1}(\phi_{1i})\Omega(Q(\phi_{2i}),\phi) + (-1)^r B_{2n+1}(\phi_{2i})\Omega(Q(\phi_{1i}),\phi)\Big)\\
&= \partial_\alpha B_{2n+1}(\phi) + B_{2n+1}(\phi_\alpha) = \partial_\alpha\phi_{t_{2n+1}}.
\end{aligned} \tag{4.21}$$

□

Now we study the commutativity of two squared eigenfunction symmetries generated by eigenfunctions ϕ_{li} and q_{lj} ($l = 1, 2,\ i = 1, ,\ldots, m,\ j = 1, 2, \ldots, m$), separately.

Proposition 4.4. *Let ϕ_{li} and q_{lj} ($l = 1, 2,\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, m$) satisfy*

$$\partial_{t_{2n+1}}(\phi_{li}) = (L^{2n+1})_{\geq 0}(\phi_{li}), \quad \partial_{t_{2n+1}}(q_{lj}) = (L^{2n+1})_{\geq 0}(q_{lj}), \tag{4.22}$$

and

$$\partial_{\alpha_2}(\phi_{li}) = \sum_{j=1}^{m}\Big(q_{1j}\Omega(Q(q_{2j}),\phi_{li}) + (-1)^r q_{2j}\Omega(Q(q_{1j}),\phi_{li})\Big), \tag{4.23}$$

$$\partial_{\alpha_1}(q_{lj}) = \sum_{i=1}^{m}\Big(\phi_{1i}\Omega(Q(\phi_{2i}),q_{lj}) + (-1)^r\phi_{2i}\Omega(Q(\phi_{1i}),q_{lj})\Big). \tag{4.24}$$

Then

$$M' = \sum_{i=1}^{m}\Big(\phi_{1i}\partial_x^{-1}\cdot Q(\phi_{2i}) + (-1)^r\phi_{2i}\partial_x^{-1}\cdot Q(\phi_{1i})\Big), \tag{4.25}$$

$$M'' = \sum_{j=1}^{m}\Big(q_{1j}\partial_x^{-1}\cdot Q(q_{2j}) + (-1)^r q_{2j}\partial_x^{-1}\cdot Q(q_{1j})\Big), \tag{4.26}$$

satisfy the zero curvature equation $M'_{\alpha_2} - M''_{\alpha_1} = [M'', M']$, further $[\partial_{\alpha_1}, \partial_{\alpha_2}] = 0$.

Proof. At first, we consider M'_{α_2} and M''_{α_1}

$$\begin{aligned} M'_{\alpha_2} = &\sum_{i=1}^{m}\Big((\phi_{1i})_{\alpha_2}\partial_x^{-1}\cdot Q(\phi_{2i}) + (-1)^r(\phi_{2i})_{\alpha_2}\partial_x^{-1}\cdot Q(\phi_{1i})\Big) \\ &+ \sum_{i=1}^{m}\Big(\phi_{1i}\partial_x^{-1}\cdot (Q(\phi_{2i}))_{\alpha_2} + (-1)^r\phi_{2i}\partial_x^{-1}\cdot (Q(\phi_{1i}))_{\alpha_2}\Big), \end{aligned} \tag{4.27}$$

$$\begin{aligned} M''_{\alpha_1} = &\sum_{j=1}^{m}\Big((q_{1j})_{\alpha_1}\partial_x^{-1}\cdot Q(q_{2j}) + (-1)^r(q_{2j})_{\alpha_1}\partial_x^{-1}\cdot Q(q_{1j})\Big) \\ &+ \sum_{j=1}^{m}\Big(q_{1j}\partial_x^{-1}\cdot (Q(q_{2j}))_{\alpha_1} + (-1)^r q_{2j}\partial_x^{-1}\cdot (Q(q_{1j}))_{\alpha_1}\Big). \end{aligned} \tag{4.28}$$

According to Eq. (4.25), Eq. (4.26) and $\partial_x^{-1} f_x \partial_x^{-1} = f\partial_x^{-1} - \partial_x^{-1} f$, we calculate $[M'', M'] = M''M' - M'M''$

$$\begin{aligned} M''M' = &\sum_{j=1}^{m}\sum_{i=1}^{m}\Big(q_{1j}\partial_x^{-1}\cdot Q(q_{2j})\phi_{1i}\partial_x^{-1}Q(\phi_{2i}) + (-1)^r q_{1j}\partial_x^{-1}\cdot Q(q_{2j})\phi_{2i}\partial_x^{-1}Q(\phi_{1i}) \\ &+(-1)^r q_{2j}\partial_x^{-1}\cdot Q(q_{1j})\phi_{1i}\partial_x^{-1}Q(\phi_{2i}) + q_{2j}\partial_x^{-1}\cdot Q(q_{1j})\phi_{2i}\partial_x^{-1}Q(\phi_{1i})\Big) \\ = &\sum_{j=1}^{m}\sum_{i=1}^{m}\Big(q_{1j}\Omega(Q(q_{2j}),\phi_{1i})\partial_x^{-1}\cdot Q(\phi_{2i}) - q_{1j}\partial_x^{-1}\cdot\Omega(Q(q_{2j}),\phi_{1i})Q(\phi_{2i}) \\ &+(-1)^r q_{1j}\Omega(Q(q_{2j}),\phi_{2i})\partial_x^{-1}\cdot Q(\phi_{1i}) - (-1)^r q_{1j}\partial_x^{-1}\cdot\Omega(Q(q_{2j}),\phi_{2i})Q(\phi_{1i}) \\ &+(-1)^r q_{2j}\Omega(Q(q_{1j}),\phi_{1i})\partial_x^{-1}\cdot Q(\phi_{2i}) - (-1)^r q_{2j}\partial_x^{-1}\cdot\Omega(Q(q_{1j}),\phi_{1i})Q(\phi_{2i}) \\ &+q_{2j}\Omega(Q(q_{1j}),\phi_{2i})\partial_x^{-1}\cdot Q(\phi_{1i}) - q_{2j}\partial_x^{-1}\cdot\Omega(Q(q_{1j}),\phi_{2i})Q(\phi_{1i})\Big) \end{aligned} \tag{4.29}$$

We can get $M'M''$ by exchanging ϕ_{lj} with q_{lj} in Eq. (4.29), where $l = 1, 2$. So

$$M'_{\alpha_2} - M''_{\alpha_1} - [M'', M'] = M'_{\alpha_2} - M''_{\alpha_1} - M''M' + M'M'' = 0. \tag{4.30}$$

Hence

$$\begin{aligned} [\partial_{\alpha_1}, \partial_{\alpha_2}]L &= \partial_{\alpha_1}[M'', L] - \partial_{\alpha_2}[M', L] \\ &= [\partial_{\alpha_1}M'' - \partial_{\alpha_2}M', L] + \Big[M'', [M', L]\Big] - \Big[M', [M'', L]\Big] \\ &= \Big[M'_{\alpha_2} - M''_{\alpha_1} - [M'', M'], L\Big] = 0. \end{aligned} \tag{4.31}$$

□

At last, we consider the squared eigenfunction symmetries associated with the constrained BC_r-KP hierarchy.

Proposition 4.5. *The constrained BC_r-KP hierarchy*

$$L^k = (L^k)_{\geq 0} + \sum_{j=1}^{m}\Big(q_{1j}\partial_x^{-1}\cdot Q(q_{2j}) + (-1)^r q_{2j}\partial_x^{-1}\cdot Q(q_{1j})\Big)$$

is invariant under the squared eigenfunction flow

$$\partial_\alpha L = \Big[\sum_{i=1}^{m}\Big(\phi_{1i}\partial_x^{-1}\cdot Q(\phi_{2i}) + (-1)^r\phi_{2i}\partial_x^{-1}\cdot Q(\phi_{1i})\Big), L\Big], \tag{4.32}$$

$$\partial_\alpha q_{lj} = \sum_{i=1}^{m}\Big(\phi_{1i}\Omega\big(Q(\phi_{2i}), q_{lj}\big) + (-1)^r\phi_{2i}\Omega\big(Q(\phi_{1i}), q_{lj}\big)\Big), \tag{4.33}$$

if $\phi_{l1},\dots,\phi_{lm}$ and $Q(\phi_{l1}),\dots,Q(\phi_{lm})$ satisfy

$$B_k(\phi_{1i}) + \sum_{j=1}^{m}\Big(q_{1j}\Omega\big(Q(q_{2j}),\phi_{1i}\big) + (-1)^r q_{2j}\Omega\big(Q(q_{1j}),\phi_{1i}\big)\Big) = \lambda_i\phi_{1i}, \tag{4.34}$$

$$B_k(\phi_{2i}) + \sum_{j=1}^{m}\Big(q_{1j}\Omega\big(Q(q_{2j}),\phi_{2i}\big) + (-1)^r q_{2j}\Omega\big(Q(q_{1j}),\phi_{2i}\big)\Big) = (-1)^r\lambda_i\phi_{2i}. \tag{4.35}$$

with arbitrary spectral parameters, $l = 1,2$, $\lambda_i \in C$ and k is odd.

Proof. M' and M'' are defined by Eq. (4.25) and Eq. (4.26) respectively,

$$\begin{aligned}
&\Big((L^k)_{<0} - \sum_{j=1}^{m}\Big(q_{1j}\partial_x^{-1}\cdot Q(q_{2j}) + (-1)^r q_{2j}\partial_x^{-1}\cdot Q(q_{1j})\Big)\Big)_\alpha \\
&= \partial_\alpha(L^k)_{<0} - M'_\alpha = [M', B_k + M'']_{<0} - M'_\alpha \\
&= \underbrace{[M', B_k]_{<0}}_{(a)} + \underbrace{[M', M'']_{<0} - M'_\alpha}_{(b)}.
\end{aligned} \tag{4.36}$$

According to Eq. (4.9),

$$\begin{aligned}
(a) = &-\sum_{i=1}^{m}\Big(B_k(\phi_{1i})\partial_x^{-1}\cdot Q(\phi_{2i}) + (-1)^r B_k(\phi_{2i})\partial_x^{-1}\cdot Q(\phi_{1i})\Big) \\
&+\sum_{i=1}^{m}\Big(\phi_{1i}\partial_x^{-1}\cdot B_k^*(Q(\phi_{2i})) + (-1)^r\phi_{2i}\partial_x^{-1}\cdot B_k^*(Q(\phi_{1i}))\Big).
\end{aligned} \tag{4.37}$$

So according to Eq. (2.4), Eq. (4.29) and Eq. (4.33),

$$\begin{aligned}
(b) = -\sum_{j=1}^{m}\sum_{i=1}^{m}\Big(&q_{1j}\Omega\big(Q(q_{2j}),\phi_{1i}\big)\partial_x^{-1}\cdot Q(\phi_{2i}) - q_{1j}\partial_x^{-1}\cdot\Omega\big(Q(q_{2j}),\phi_{1i}\big)Q(\phi_{2i}) \\
&+(-1)^r q_{1j}\Omega\big(Q(q_{2j}),\phi_{2i}\big)\partial_x^{-1}\cdot Q(\phi_{1i}) - (-1)^r q_{1j}\partial_x^{-1}\cdot\Omega\big(Q(q_{2j}),\phi_{2i}\big)Q(\phi_{1i}) \\
&+(-1)^r q_{2j}\Omega\big(Q(q_{1j}),\phi_{1i}\big)\partial_x^{-1}\cdot Q(\phi_{2i}) - (-1)^r q_{2j}\partial_x^{-1}\cdot\Omega\big(Q(q_{1j}),\phi_{1i}\big)Q(\phi_{2i}) \\
&+q_{2j}\Omega\big(Q(q_{1j}),\phi_{2i}\big)\partial_x^{-1}\cdot Q(\phi_{1i}) - q_{2j}\partial_x^{-1}\cdot\Omega\big(Q(q_{1j}),\phi_{2i}\big)Q(\phi_{1i})\Big).
\end{aligned} \tag{4.38}$$

By Eq. (4.34) and Eq. (4.35), one can notice that $(a)+(b)=0$. □

5. Conclusions and Discussions

Firstly, the bilinear identities of the BC_r-KP hierarchy are investigated in Proposition 3.1 and Proposition 3.2, and the bilinear identities of the constraint case are studied in Proposition 3.3 and Proposition 3.4. Next, the squared eigenfunction symmetries are constructed in Eq. (4.1) and Eq. (4.2). And the corresponding definition is showed to be reasonable in Proposition 4.1 and Proposition 4.2. What's more, the actions of the squared eigenfunctions symmetry on the eigenfunction are given in Proposition 4.3. Also the commutativity of two different squared eigenfunction symmetries are discussed in Proposition 4.4. Lastly the squared eigenfunction symmetries associated with the constrained BC_r-KP hierarchy are studied in Proposition 4.5.

The bilinear identities of the BC_r-KP hierarchy can help us to discuss the existence of the tau functions. Though the tau functions of the BKP and CKP hierarchies are investigated in [3, 8], it is still worth studying the tau functions of the BC_r-KP hierarchies for $r \geq 2$. Just as we have said in Section 1, the squared eigenfunction symmetry can used to define the new integrable systems and study the additional symmetry. In fact, the constrained BC_r-KP hierarchy can be obtained by identify ∂_α with $-\partial_{t_{2n+1}}$. The generating operator $Y^{BC_r}(\lambda,\mu)$ [28] of the additional symmetries of the BC_r-KP hierarchy can be used to define the squared symmetry ∂_α, and therefore,

$$\partial_\alpha = -\sum_{m=0}^{\infty} \frac{(\mu-\lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} \partial^*_{m,m+l}. \tag{5.1}$$

So the squared eigenfunction symmetry defined by the (adjoint) wave functions $w(t,\mu)$ and $w^*(t,\lambda)$ can be viewed as the generator of the additional symmetries.

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