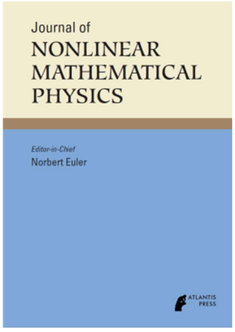

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LETTER TO THE EDITOR

A Note on the Equivalence of Methods to finding Nonclassical Determining Equations

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In this note we prove that the method of Bîlã and Niesen to determine nonclassical determining equations is equivalent to that of Nucci's method with heir-equations and thus in general is equivalent to using an appropriate form of generalised conditional symmetry.

Keywords: nonclassical symmetries, generalised conditional symmetries.

2000 Mathematics Subject Classification: 35K10, 58D19

1. Introduction

The focus here is on showing the equivalence of different approaches to finding the nonclassical determining equations for the partial differential equation (PDE),

$$u_t = K(x,t,u,u_x,u_{xx}). \tag{1.1}$$

In particular we address the problem posed by Hashemi and Nucci [4] who considered equations of the form (1.1): "We hope that an independent researcher will take up the task of comparing the two methods [of Bîlã and Niesen and Nucci] since we conjecture that Bîlã and Niesen's method, and its extension, as given in [2], are equivalent to Nucci's method."

We consider the symmetry generator

$$\Gamma = X(x,t,u)\frac{\partial}{\partial x} + T(x,t,u)\frac{\partial}{\partial t} + U(x,t,u)\frac{\partial}{\partial u} \tag{1.2}$$

and start by considering the case when the infinitesimal $T(x,t,u) \neq 0$..

2. $T \neq 0$

With $T \neq 0$, the corresponding invariant surface condition (ISC) is given by

$$Xu_x + Tu_t = U. \tag{2.1}$$

In the traditional approach, nonclassical symmetries of (1.1) are defined by

$$\Gamma^{(2)}[u_t - K(x,t,u,u_x,u_{xx})]|_{u_t=K\cap\{Xu_x+Tu_t=U\}} \tag{2.2}$$

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where $\Gamma^{(2)}$ is the second prolongation of Γ, namely,

$$\Gamma^{(2)} = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + U_{[x]}\frac{\partial}{\partial u_x} + U_{[t]}\frac{\partial}{\partial u_t} + U_{[xx]}\frac{\partial}{\partial u_{xx}}, \tag{2.3}$$

where

$$\begin{aligned} U_{[x]} &= D_x U - (D_x X)u_x - (D_x T)u_t,\ \ U_{[t]} = D_t U - (D_t X)u_x - (D_t T)u_t, \\ U_{[xx]} &= D_x(U_{[x]}) - D_x(X)u_{xx} - D_x(T)u_{xt}. \end{aligned} \tag{2.4}$$

Hence for nonclassical symmetries, we seek the invariance of the governing PDE subject to the PDE itself and the ISC (and its differential consequences). We note however that if $f(x,t,u)$ is an arbitrary function, then the prolongation formula implies $[f(x,t,u)\Gamma]^{(n)}|_{\{Xu_x+Tu_t=U\}} = f(x,t,u)\Gamma^{(n)}|_{\{Xu_x+Tu_t=U\}}$. That is, by imposing the ISC we have $[f(x,t,u)\Gamma]^{(n)}|_{\{Xu_x+Tu_t=U\}} = f(x,t,u)\Gamma^{(n)}$. Hence if Γ is a nonclassical symmetry then $f(x,t,u)\Gamma$ is also a nonclassical symmetry yielding the same invariant surface condition. This allows us to normalise any one of the nonzero coefficients of the vector field by setting it equal to one when finding nonclassical symmetries. Hence in the following, WLOG we set $T = 1$.

Applying (2.2) with $T = 1$ we get the condition

$$U_{[t]} - XK_x - K_t - UK_u - U_{[x]}K_{u_x} - U_{[xx]}K_{u_{xx}} = 0, \tag{2.5}$$

subject to $u_t = K \cap \{Xu_x + u_t = U\}$.

We can further expand this as

$$U_t + U_u(U - Xu_x) - X_t u_x - X_u(U - Xu_x)u_x - XK_x - K_t - UK_u - U_{[x]}K_{u_x} - U_{[xx]}K_{u_{xx}} = 0 \tag{2.6}$$

subject to $U - Xu_x = K$, where we have used $u_t = U - Xu_x$ and the definition of $U_{[t]}$.

Bîlã and Niesen [1] use the approach

$$\Gamma^{(2)}[\phi(x,t,u)/T'(x,t,u) - \xi(x,t,u)/T'(x,t,u)u_x - K(x,t,u,u_x,u_{xx})]|_{\phi/T'-\xi/T'u_x=K} \tag{2.7}$$

where the ISC $u_t = \phi/T' - \xi/T'u_x$ has been substituted into the governing equation before taking the second prolongation. Treating ϕ and ξ as arbitrary functions of x,t,u means that (2.7) is equivalent to finding the determining equations for classical symmetries of the ordinary differential equation $\phi/T' - \xi/T'u_x - K(x,t,u,u_x,u_{xx}) = 0$. Then substituting $\phi = U, \xi = X, T' = 1$ leads to the determining equations for nonclassical symmetries of (1.1).

Hence Bîlã and Niesen essentially use the approach

$$\Gamma^{(2)}[U - Xu_x - K(x,t,u,u_x,u_{xx})]|_{U-Xu_x=K}. \tag{2.8}$$

This gives

$$X[U_x - X_x u_x - K_x] + [U_t - X_t u_x - K_t] + U[U_u - X_u u_x - K_u] - XU_{[x]} - K_{u_x}U_{[x]} - K_{u_{xx}}U_{[xx]} = 0, \tag{2.9}$$

subject to $U - Xu_x = K$, or

$$\begin{aligned} &X[U_x - X_x u_x - K_x] + [U_t - X_t u_x - K_t] + U[U_u - X_u u_x - K_u] - X[U_x + U_u u_x - X_x u_x - X_u u_x^2] \\ &-K_{u_x}U_{[x]} - K_{u_{xx}}U_{[xx]} = 0, \end{aligned} \tag{2.10}$$

subject to $U - Xu_x = K$. The condition simplifies to

$$-XK_x + [U_t - X_t u_x - K_t] + U[U_u - X_u u_x - K_u] - X[U_u u_x - X_u u_x^2] - K_{u_x} U_{[x]} - K_{u_{xx}} U_{[xx]} = 0, \quad (2.11)$$

subject to $U - Xu_x = K$. As (2.6) and (2.11) are the same conditions they lead to the same nonclassical determining equations.

Comparison with Nucci's method

It has been shown in [3] that Nucci's method of heir-equations is essentially the same as the generalised conditional symmetries (GCS) method. Hence the method of finding the determining equations for nonclassical symmetries as described in [4] and [5] can be written as

$$\Gamma(\sigma)[u_t - K(x,t,u,u_x,u_{xx})]|_{\sigma=0 \cap u_t=K} \quad (2.12)$$

where $\sigma = K(x,t,u,u_x,u_{xx}) - U(x,t,u) + X(x,t,u)u_x$ and

$$\Gamma(\sigma) = \sigma\frac{\partial}{\partial u} + (D_t\sigma)\frac{\partial}{\partial u_t} + (D_x\sigma)\frac{\partial}{\partial u_x} + \cdots \quad (2.13)$$

This condition is equivalent to

$$\sigma_t + \sigma_u K + \sigma_{u_x}(D_x K) + \sigma_{u_{xx}}(D_{xx}K) = 0, \quad (2.14)$$

subject to $\sigma = 0$ (i.e. $U - Xu_x = K$) and its differential consequences with respect to x.
From (2.14) we get that the condition can be expressed as

$$\begin{aligned}
0 &= K_t - U_t + X_t u_x + (K_u - U_u + X_u u_x)K + (K_{u_x} + X)D_x K + K_{u_{xx}} D_{xx} K \\
&= K_t - U_t + X_t u_x + (K_u - U_u + X_u u_x)(U - Xu_x) \\
&\quad + (K_{u_x} + X)[U_x + U_u u_x - (X_x + X_u u_x)u_x - Xu_{xx}] + K_{u_{xx}} D_{xx} K \\
&= K_t - U_t + X_t u_x + (K_u - U_u + X_u u_x)(U - Xu_x) + (K_{u_x} + X)[U_{[x]} - Xu_{xx}] \\
&\quad + K_{u_{xx}}[D_x(U_{[x]} - Xu_{xx})] \\
&= K_t - U_t + X_t u_x + (K_u - U_u + X_u u_x)(U - Xu_x) + (K_{u_x} + X)[U_{[x]} - Xu_{xx}] \\
&\quad + K_{u_{xx}}[U_{[xx]} - Xu_{xxx}]
\end{aligned}$$

subject to $U - Xu_x = K$ as $D_x(U_{[x]} - Xu_{xx}) = U_{[xx]} + D_x(X)u_{xx} - D_x(X)u_{xx} - Xu_{xxx}$.

This can be further rewritten as

$$\begin{aligned}
&K_t - U_t + X_t u_x + (K_u - U_u + X_u u_x)U + (K_{u_x} + X)U_{[x]} + K_{u_{xx}} U_{[xx]} \\
&- Xu_x(K_u - U_u + X_u u_x) - Xu_{xx}(K_{u_x} + X) - Xu_{xxx}K_{u_{xx}} = 0,
\end{aligned} \quad (2.15)$$

subject to $U - Xu_x = K$.

Now consider

$$
\begin{aligned}
&-Xu_x(K_u - U_u + X_u u_x) - Xu_{xx}(K_{u_x} + X) - Xu_{xxx}K_{u_{xx}} \\
&= -X\{[K_x + u_x K_u + u_{xx}K_{u_x} + u_{xxx}K_{u_{xx}}] - [(U_x - X_x u_x) + (U_u u_x - X_u u_x^2) - Xu_{xx}]\} \\
&\quad +X(K_x - U_x + X_x u_x) \\
&= X(K_x - U_x + X_x u_x)
\end{aligned}
$$

Hence from (2.15), the condition is

$$
K_t - U_t + X_t u_x + (K_u - U_u + X_u u_x)U + (K_{u_x} + X)U_{[x]} + K_{u_{xx}}U_{[xx]} + X(K_x - U_x + X_x u_x) = 0, \quad (2.16)
$$

subject to $U - Xu_x = K$. Comparing (2.16) with (2.9) we see they are equivalent.

3. $T = 0$

When the infinitesimal symmetry $T = 0$ in (1.2), then as explained in the previous section, WLOG we can set $X = 1$. In the traditional approach we find the nonclassical determining equations using

$$
\Gamma_0^{(2)}[u_t - K(x,t,u,u_x,u_{xx})]|_{u_t=K\cap u_x=U} \quad (3.1)
$$

where $\Gamma_0^{(2)}$ is the second prolongation of $\Gamma_0 = \frac{\partial}{\partial x} + U(x,t,u)\frac{\partial}{\partial u}$, namely,

$$
\Gamma_0^{(2)} = \frac{\partial}{\partial x} + U\frac{\partial}{\partial u} + U_{[x]}\frac{\partial}{\partial u_x} + U_{[t]}\frac{\partial}{\partial u_t} + U_{[xx]}\frac{\partial}{\partial u_{xx}}. \quad (3.2)
$$

With $T = 0, X = 1$ we have $U_{[x]} = D_x U,\ U_{[t]} = D_t U,\ U_{[xx]} = D_x(U_{[x]})$.

Hence applying (3.1) we get the condition $U_{[t]} - K_x - UK_u - U_{[x]}K_{u_x} - U_{[xx]}K_{u_{xx}} = 0$, subject to $u_t = K \cap u_x = U$.

We can further expand this as

$$
U_{[t]} - K_x - UK_u - K_{u_x}[U_x + U_u U] - K_{u_{xx}}[U_{xx} + 2U_{xu}U + U_{uu}U^2 + U_u(U_x + U_u U)] = 0, \quad (3.3)
$$

subject to $u_t = K$, where we have used $u_x = U$ and the definition of $U_{[x]}$ and $U_{[xx]}$.

In [2], Bruzón and Gandarias extend the method of Bîlã and Niesen to the case $T = 0$. They use the approach

$$
\Gamma_0^{(2)}[u_t - K(x,t,u,U'/X',D_x(U'/X'))]|_{u_t=K(x,t,u,U'/X',D_x(U'/X'))} \quad (3.4)
$$

where the ISC $u_x = U'(x,t,u)/X'(x,t,u)$ has been substituted into the governing equation before taking the second prolongation. Treating U' and X' as arbitrary functions of x,t,u means that (3.4) is equivalent to finding the determining equations for classical symmetries of $u_t = K(x,t,u,U'/X',D_x(U'/X'))$. Then substituting $U' = U, X' = 1$, leads to the determining equations for nonclassical symmetries of (1.1).

Hence Bruzón and Gandarias essentially use the approach

$$\Gamma_0^{(2)}[u_t - K(x,t,u,U,U_x + U_u U]|_{u_t = K(x,t,u,U,U_x+U_uU)}. \tag{3.5}$$

Letting $z = U_x + U_u u_x (= u_{xx})$, this gives

$$U_{[t]} - [K_x + K_U U_x + K_z(U_{xx} + U_{ux}U)] - U[K_u + K_U U_u + K_z(U_{xu} + U_{uu}U)] - U_{[x]} K_z U_u = 0 \tag{3.6}$$

subject to $u_t = K$, or

$$\begin{aligned} &U_{[t]} - [K_x + K_U U_x + K_z(U_{xx} + U_{ux}U)] - U[K_u + K_U U_u + K_z(U_{xu} + U_{uu}U)] \\ &-K_z U_u(U_x + U_u U) = 0, \end{aligned} \tag{3.7}$$

subject to $u_t = K$. As (3.3) and (3.7) are the same conditions they lead to the same nonclassical determining equations.

Comparison with Nucci's method

With the infinitesimals $T = 0,\ X = 1$, Nucci's method can be expressed as

$$\Gamma(\sigma)[u_t - K(x,t,u,u_x,u_{xx})]|_{\sigma=0 \cap u_t = K} \tag{3.8}$$

where $\sigma = u_x - U(x,t,u)$ and

$$\Gamma(\sigma) = \sigma\frac{\partial}{\partial u} + (D_t\sigma)\frac{\partial}{\partial u_t} + (D_x\sigma)\frac{\partial}{\partial u_x} + \cdots \tag{3.9}$$

This is equivalent to

$$\sigma_t + \sigma_u K + \sigma_{u_x}(D_x K) + \sigma_{u_{xx}}(D_{xx}K) = 0, \tag{3.10}$$

subject to $u_t = K$, $\sigma = 0$ and its differential consequences with respect to x.

This leads to $-U_t - U_u K + D_x K = 0$, or with $z = u_{xx}$,

$$\begin{aligned} 0 &= -U_t - U_u K + K_x + K_u U + K_U(U_x + U_u U) + K_z[U_{xx} + 2U_{xu}U + U_{uu}U^2 + U_u(U_x + U_u U)] \\ &= -U_{[t]} + K_x + K_u U + K_U(U_x + U_u U) + K_z[U_{xx} + 2U_{xu}U + U_{uu}U^2 + U_u(U_x + U_u U)] \end{aligned} \tag{3.11}$$

subject to $u_t = K$. Comparing (3.11) with (3.3) and (3.7) we see they all lead to the same determining equations for nonclassical symmetries.

In conclusion, we find that the method of Bîlã and Niesen when the infinitesimal $T \neq 0$ and the method of Bruzón and Gandarias when $T = 0$ are equivalent to that of Nucci's method for finding nonclassical symmetries of the diffusion equation (1.1) in that they lead to the same determining equations.

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