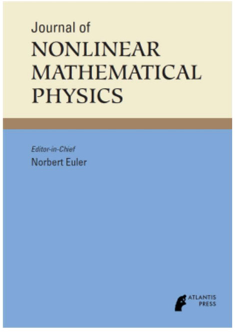

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On Slant Magnetic Curves in S-manifolds

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We consider slant normal magnetic curves in $(2n+1)$-dimensional S-manifolds. We prove that γ is a slant normal magnetic curve in an S-manifold $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ if and only if it belongs to a list of slant φ-curves satisfying some special curvature equations. This list consists of some specific geodesics, slant circles, Legendre and slant helices of order 3. We construct slant normal magnetic curves in $\mathbb{R}^{2n+s}(-3s)$ and give the parametric equations of these curves.

Keywords: Magnetic curve; slant curve; S-manifold.

2010 Mathematics Subject Classification: 53C25, 53C40, 53A04

1. Introduction

Let (M,g) be a Riemannian manifold, F a closed 2-form and let us denote the Lorentz force on M by Φ, which is a $(1,1)$-type tensor field. If F is associated by the relation

$$g(\Phi X, Y) = F(X,Y), \ \ \forall X, Y \in \chi(M), \tag{1.1}$$

then it is called a *magnetic field* ([1], [4] and [9]). Let ∇ be the Riemannian connection associated to the Riemannian metric g and $\gamma : I \to M$ a smooth curve. If γ satisfies the Lorentz equation

$$\nabla_{\gamma'(t)}\gamma'(t) = \Phi(\gamma'(t)), \tag{1.2}$$

then it is called a *magnetic curve* or a *trajectory* for the magnetic field F. The Lorentz equation is a generalization of the equation for geodesics. A curve which satisfies the Lorentz equation is called *magnetic trajectory*. Magnetic trajectories have constant speed. If the speed of the magnetic curve γ is equal to 1, then it is called a *normal magnetic curve* [10]. For extensive information about almost contact metric manifolds and Sasakian manifolds, we refer to Blair's book [2].

In [1], Adachi studied curvature bound and trajectories for magnetic fields on a Hadamard surface. He showed that every normal trajectory is unbounded in both directions in a 2-dimensional complete simply connected Riemannian manifold satisfying some special curvature conditions. In [5], Baikoussis and Blair considered Legendre curves in contact 3-manifolds and they proved that the torsion of a Legendre curve in a 3-dimensional Sasakian manifold is equal to 1. Moreover,

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in [8], Cho, Inoguchi and Lee proved that a non-geodesic curve in a Sasakian 3-manifold is a slant curve if and only if the ratio of $(\tau \pm 1)$ and κ is constant, where τ is the geodesic torsion and κ is the geodesic curvature. Cabrerizo, Fernandez and Gomez gave a nice geometric construction of an almost contact metric structure compatible with an assigned metric on a 3-dimensional oriented Riemannian manifold in [6]. In the paper [10], Druţă-Romaniuc, Inoguchi, Munteanu and Nistor studied the magnetic trajectories of the contact magnetic field $F_q = q\Omega$ on a Sasakian $(2n+1)$-manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$, where Ω is the fundamental 2-form. The main objective of [11] is the study of trajectories for particles moving under the influence of a contact magnetic curve in a cosymplectic manifold. The paper [14] is concerned with closed magnetic trajectories on 3-dimensional Berger spheres. In [15], the authors studied magnetic trajectories in an almost contact metric manifold. They proved that normal magnetic curves are helices of maximum order 5. Moreover, in [16], Jleli and Munteanu worked in the context of a para-Kaehler manifold, showing that spacelike and timelike normal magnetic curves corresponding to the para-Kaehler 2-forms are circles. In [17], the authors gave a complete classification of Killing magnetic curves with unit speed. Furthermore, in [18], the same authors proved that a normal magnetic curve on the Sasakian sphere S^{2n+1} lies on a totally geodesic sphere S^3. They also considered two particular magnetic fields on three-dimensional torus obtained from two different contact forms on the Euclidean space E^3 and studied their closed normal magnetic trajectories in their recent paper [19]. In [23], the authors investigated some special curves in 3-dimensional semi-Riemannian manifolds, such as T-magnetic curves, N-magnetic curves and B-magnetic curves, that are defined by means of their Frenet elements. Calvaruso, Munteanu and Perrone provided a complete classification of the magnetic trajectories of a Killing characteristic vector field on an arbitrary normal paracontact metric manifold of dimension 3 in [7]. The present authors considered biharmonic Legendre curves of S-space forms in [21]. The second author studied magnetic curves in the 3-dimensional Heisenberg group in [22]. In [20], Nakagawa introduced the notion of framed f-structures, which is a generalization of almost contact structures. On the other hand, Hasegawa, Okuyama and Abe defined a pth Sasakian manifold and gave some typical examples in [13].

Motivated by the above studies, in the present paper, we consider slant normal magnetic curves in $(2n+s)$-dimensional S-manifolds. S-manifolds are interesting generalizations of Sasakian manifolds, which have a tensor field of type $(1,1)$ called a φ-structure, satisfying $\varphi^3 + \varphi = 0$. So, in Section 2, we give the definitions and brief information on S-manifolds and magnetic curves. In Section 3, we prove that γ is a slant normal magnetic curve in an S-manifold $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ if and only if it belongs to a list of slant φ-curves. This list consists of some specific geodesics, slant circles, Legendre and slant helices of order 3. Finally, in Section 4, after the definition of $\mathbb{R}^{2n+s}(-3s)$ and its structures, we construct slant normal magnetic curves in $\mathbb{R}^{2n+s}(-3s)$ and give the parametric equations of these curves in two cases.

2. Preliminaries

In this section, we give brief information on S-manifolds and magnetic curves. Let $\left(M^{2n+s}, g\right)$ be a differentiable manifold, φ a $(1,1)$-type tensor field, η^α 1-forms, ξ_α vector fields for $\alpha = 1, \ldots, s$, satisfying

$$\varphi^2 = -I + \sum_{\alpha=1}^{s} \eta^\alpha \otimes \xi_\alpha, \tag{2.1}$$

$$\begin{gathered}
\eta^{\alpha}\left(\xi_{\beta}\right)=\delta_{\beta}^{\alpha}, \quad \varphi \xi_{\alpha}=0, \quad \eta^{\alpha}(\varphi X)=0, \quad \eta^{\alpha}(X)=g\left(X, \xi_{\alpha}\right), \\
g(\varphi X, \varphi Y)=g(X, Y)-\sum_{\alpha=1}^{s} \eta^{\alpha}(X) \eta^{\alpha}(Y), \\
d \eta^{\alpha}(X, Y)=-d \eta^{\alpha}(Y, X)=g(X, \varphi Y),
\end{gathered} \tag{2.2}$$

where $X, Y \in TM$. Then $(\varphi, \xi_{\alpha}, \eta^{\alpha}, g)$ is called *framed φ-structure* and $(M^{2n+s}, \varphi, \xi_{\alpha}, \eta^{\alpha}, g)$ is called *framed φ-manifold* [20]. $(M^{2n+s}, \varphi, \xi_{\alpha}, \eta^{\alpha}, g)$ is also called *framed metric manifold* [25] or *almost r-contact metric manifold* [24]. If the Nijenhuis tensor of φ is equal to $-2d\eta^{\alpha} \otimes \xi_{\alpha}$, then $(M^{2n+s}, \varphi, \xi_{\alpha}, \eta^{\alpha}, g)$ is called an *S-manifold* [3]. For $s=1$, an S-structure becomes a Sasakian structure. For an S-structure, the following properties are satisfied [3]:

$$(\nabla_X \varphi) Y=\sum_{\alpha=1}^{s}\left\{g(\varphi X, \varphi Y) \xi_{\alpha}+\eta^{\alpha}(Y) \varphi^2 X\right\}, \tag{2.3}$$

$$\nabla \xi_{\alpha}=-\varphi, \quad \alpha \in\{1, \ldots, s\}. \tag{2.4}$$

Let $M^{2n+s}=(M^{2n+s}, \varphi, \xi_{\alpha}, \eta^{\alpha}, g)$ be an S-manifold and Ω *the fundamental* 2*-form* of M^{2n+s} defined by

$$\Omega(X, Y)=g(X, \varphi Y), \tag{2.5}$$

(see [20] and [24]). From the definition of framed φ-structure, we have $\Omega = d\eta^{\alpha}$. Hence, the fundamental 2-form Ω on M^{2n+s} is closed. The *magnetic field F_q* on M^{2n+s} can be defined by

$$F_q(X, Y)=q \Omega(X, Y),$$

where X and Y are vector fields on M^{2n+s} and q is a real constant. F_q is called the *contact magnetic field with strength q* [15]. If $q=0$ then the magnetic curves are geodesics of M^{2n+s}. Because of this reason we shall consider $q \neq 0$ (see [6] and [10]).

From (1.1) and (2.5), the Lorentz force Φ associated to the contact magnetic field F_q can be written as

$$\Phi_q=-q \varphi.$$

So the Lorentz equation (1.2) can be written as

$$\nabla_T T=-q \varphi T, \tag{2.6}$$

where $\gamma : I \subseteq R \rightarrow M^{2n+s}$ is a smooth unit-speed curve and $T=\gamma'$ (see [10] and [15]).

3. Slant magnetic curves in S-manifolds

Let (M^n, g) be a Riemannian manifold. A unit-speed curve $\gamma : I \rightarrow M$ is said to be *a Frenet curve of osculating order r*, if there exists positive functions $\kappa_1, \ldots, \kappa_{r-1}$ on I satisfying

$$\begin{aligned}
T &= v_1=\gamma', \\
\nabla_T T &= k_1 v_2, \\
\nabla_T v_2 &= -k_1 T+k_2 v_3, \\
&\cdots \\
\nabla_T v_r &= -k_{r-1} v_{r-1},
\end{aligned} \tag{3.1}$$

where $1 \leq r \leq n$ and $T, v_2, \ldots, v_r$ are a g-orthonormal vector fields along the curve. The positive functions $\kappa_1, \ldots, \kappa_{r-1}$ are called *curvature functions* and $\{T, v_2, \ldots, v_r\}$ is called the *Frenet frame field*. A *geodesic* is a Frenet curve of osculating order $r = 1$. A *circle* is a Frenet curve of osculating order $r = 2$ with a constant curvature function κ_1. A *helix of order r* is a Frenet curve of osculating order r with constant curvature functions $\kappa_1, \ldots, \kappa_{r-1}$. A helix of order 3 is simply called a *helix*.

Let $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an S-manifold. For a unit-speed curve $\gamma : I \to M$, if

$$\eta^\alpha(T) = 0,$$

for all $\alpha = 1, \ldots, s$, then γ is called a *Legendre curve of M* [21]. More generally, if there exists a constant angle θ such that

$$\eta^\alpha(T) = \cos\theta,$$

for all $\alpha = 1, \ldots, s$, then γ is called a *slant curve* and θ is called the *contact angle of* γ, where $|\cos\theta| \leq 1/\sqrt{s}$ [12].

Let $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an S-manifold. A Frenet curve of osculating order $r \geq 3$ is called a *φ-curve in M* if its Frenet vector fields $T, v_2, \ldots, v_r$ span a φ-invariant space. A φ-curve of osculating order r with constant curvature functions $\kappa_1, \ldots, \kappa_{r-1}$ is called a *φ-helix of order r*. A curve of osculating order 2 is called a *φ-curve* if

$$sp\left\{T, v_2, \sum_{\alpha=1}^{s} \xi_\alpha\right\}$$

is a φ-invariant space.

Throughout the paper, when we state "slant magnetic curve", we mean "slant curves which satisfy equation (2.6)". For magnetic curves, $\eta^\alpha(T) = \cos\theta_\alpha$ does not have to be equal for all $\alpha = 1, \ldots, s$. By taking the curve as slant, we only study the equality case of the slant angles θ_α in the present paper. The complete classification of magnetic curves in S-manifolds is still an open problem.

Firstly, we state the following theorem:

Theorem 3.1. *Let $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an S-manifold and consider the contact magnetic field F_q for $q \neq 0$. Then γ is a slant normal magnetic curve associated to F_q in M^{2m+s} if and only if γ belongs to the following list:*

a) *geodesics obtained as integral curves of* $\left(\pm \frac{1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_\alpha\right)$;

b) *non-geodesic slant circles with the curvature $\kappa_1 = \sqrt{q^2 - s}$, having the contact angle $\theta = \arccos\left(\frac{1}{q}\right)$ and the Frenet frame field*

$$\left\{T, \frac{-\operatorname{sgn}(q)\varphi T}{\sqrt{1 - s\cos^2\theta}}\right\},$$

where $|q| > \sqrt{s}$;

c) *Legendre helices with curvatures $\kappa_1 = |q|$ and $\kappa_2 = \sqrt{s}$, having the Frenet frame field*

$$\left\{T, -\operatorname{sgn}(q)\varphi T, \frac{-\operatorname{sgn}(q)}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_\alpha\right\};$$

i.e., a class of 1*-dimensional integral submanifolds of the contact distribution;*

d) *slant helices with curvatures* $\kappa_1 = |q|\sqrt{1 - s\cos^2\theta}$ *and* $\kappa_2 = \sqrt{s}|1 - q\cos\theta|$, *having the Frenet frame field*

$$\left\{T, \frac{-\operatorname{sgn}(q)\varphi T}{\sqrt{1 - s\cos^2\theta}}, \frac{-\varepsilon \operatorname{sgn}(q)}{\sqrt{s}\sqrt{1 - s\cos^2\theta}}\left(-s\cos\theta T + \sum_{\alpha=1}^{s}\xi_\alpha\right)\right\},$$

where $\theta \neq \frac{\pi}{2}$ *is the contact angle satisfying* $|\cos\theta| < \frac{1}{\sqrt{s}}$ *and* $\varepsilon = \operatorname{sgn}(1 - q\cos\theta)$.

Proof. Let γ be a normal magnetic curve. If the magnetic curve is a geodesic, then

$$\nabla_T T = 0 = -q\varphi T$$

gives us

$$T \in sp\{\xi_1, \ldots, \xi_s\}.$$

If γ is slant, then we can write

$$T = \cos\theta \sum_{\alpha=1}^{s}\xi_\alpha.$$

Since γ is unit speed, we have $\cos\theta = \pm\frac{1}{\sqrt{s}}$. So the proof of a) is complete.

From now on, we suppose that γ is a non-geodesic Frenet curve of osculating order $r > 1$. Let us choose an $\alpha \in \{1, \ldots, s\}$. Applying ξ_α to $\nabla_T T = -q\varphi T$, we obtain

$$0 = g(-q\varphi T, \xi_\alpha) = g(\nabla_T T, \xi_\alpha) = \frac{d}{dt}g(T, \xi_\alpha) - g(T, \nabla_T \xi_\alpha). \tag{3.2}$$

From (2.4), we also have

$$\nabla_T \xi_\alpha = -\varphi T. \tag{3.3}$$

Using equations (3.2) and (3.3), we find

$$\frac{d}{dt}g(T, \xi_\alpha) = 0,$$

that is,

$$\eta^\alpha(T) = \cos\theta_\alpha = \text{constant}.$$

Let us assume $\theta_\alpha = \theta$ for all $\alpha = 1, \ldots, s$, i.e., γ is slant. So, we have

$$\eta^\alpha(T) = \cos\theta. \tag{3.4}$$

Equations (2.6) and (3.1) give us

$$\nabla_T T = \kappa_1 v_2 = -q\varphi T. \tag{3.5}$$

Then we get

$$\kappa_1 = |q|\,\|\varphi T\| = |q|\sqrt{1 - s\cos^2\theta}. \tag{3.6}$$

If we write (3.6) in (3.5), we find

$$-q\varphi T = \kappa_1 v_2 = |q|\sqrt{1-s\cos^2\theta}v_2,$$

which gives us

$$\varphi T = -\frac{|q|}{q}\sqrt{1-s\cos^2\theta}v_2 = -\operatorname{sgn}(q)\sqrt{1-s\cos^2\theta}v_2. \tag{3.7}$$

If $\kappa_2 = 0$, then the magnetic curve is a Frenet curve of osculating order $r = 2$. Since κ_1 is a constant, γ is a circle. From (3.7), we have

$$\eta^\alpha(\varphi T) = 0 = -\operatorname{sgn}(q)\sqrt{1-s\cos^2\theta}\eta^\alpha(v_2),$$

that is,

$$\eta^\alpha(v_2) = 0.$$

If we differentiate the last equation along the curve γ, we obtain

$$\nabla_T\eta^\alpha(v_2) = 0 = g(\nabla_T v_2, \xi_\alpha) + g(v_2, \nabla_T \xi_\alpha).$$

So, we calculate

$$g(-\kappa_1 T, \xi_\alpha) + g\left(v_2, \operatorname{sgn}(q)\sqrt{1-s\cos^2\theta}v_2\right) = 0.$$

Since $r = 2$, we find

$$-\kappa_1\cos\theta + \operatorname{sgn}(q)\sqrt{1-s\cos^2\theta} = 0.$$

Using equation (3.6) in the last equation, it is easy to see that

$$|q|\sqrt{1-s\cos^2\theta}\left(-\cos\theta + \frac{1}{q}\right) = 0.$$

Since γ is non-geodesic, we have

$$\cos\theta = \frac{1}{q}.$$

Then equation (3.6) becomes

$$\kappa_1 = |q|\sqrt{1-s\cos^2\theta} = \sqrt{q^2-s},$$

where $|q| > \sqrt{s}$. So the proof of b) is complete.

Let $\kappa_2 \neq 0$. From (2.1) and (3.4), we find

$$\varphi^2 T = -T + \cos\theta \sum_{\alpha=1}^{s} \xi_\alpha. \tag{3.8}$$

Using (2.3) and (3.4), we have

$$(\nabla_T \varphi)T = -s\cos\theta T + \sum_{\alpha=1}^{s} \xi_\alpha, \tag{3.9}$$

which gives us

$$\begin{aligned} \nabla_T \varphi T &= (\nabla_T \varphi)T + \varphi \nabla_T T \\ &= -s\cos\theta T + \sum_{\alpha=1}^{s} \xi_\alpha + \varphi(-q\varphi T) \\ &= -s\cos\theta T + \sum_{\alpha=1}^{s} \xi_\alpha - q\left(-T + \cos\theta \sum_{\alpha=1}^{s} \xi_\alpha\right). \end{aligned} \tag{3.10}$$

Differentiating (3.7), we also find

$$\nabla_T \varphi T = -sqn(q)\sqrt{1 - s\cos^2\theta}\,(-\kappa_1 T + \kappa_2 v_3). \tag{3.11}$$

By the use of (3.6), (3.10) and (3.11), after some calculations, we obtain

$$(1 - q\cos\theta)\left(-s\cos\theta T + \sum_{\alpha=1}^{s} \xi_\alpha\right) = -\operatorname{sgn}(q)\sqrt{1 - s\cos^2\theta}\,\kappa_2 v_3. \tag{3.12}$$

If we find the norm of both sides in (3.12), we get

$$\kappa_2 = \sqrt{s}\,|1 - q\cos\theta|. \tag{3.13}$$

Let us denote $\varepsilon = \operatorname{sgn}(1 - q\cos\theta)$. If we write (3.13) in (3.12), we obtain

$$\sum_{\alpha=1}^{s} \xi_\alpha = s\cos\theta T - \varepsilon \operatorname{sgn}(q)\sqrt{s}\sqrt{1 - s\cos^2\theta}\, v_3. \tag{3.14}$$

Applying φ to (3.14), we find

$$\varphi v_3 = -\varepsilon\sqrt{s}\cos\theta v_2.$$

If we apply φ to (3.7) and then use equations (3.4) and (3.14) together, we have

$$\varphi v_2 = \operatorname{sgn}(q)\sqrt{1 - s\cos^2\theta}\, T + \varepsilon\cos\theta\sqrt{s} v_3. \tag{3.15}$$

Let us choose a $\beta \in \{1, \ldots, s\}$. From (3.15), we calculate

$$\eta^\beta(v_3) = -\varepsilon \operatorname{sgn}(q)\frac{\sqrt{1 - s\cos^2\theta}}{\sqrt{s}}.$$

If we differentiate (3.14) along the curve γ, we get

$$\sum_{\alpha=1}^{s} \nabla_T \xi_\alpha = s\cos\theta \nabla_T T - \varepsilon \operatorname{sgn}(q)\sqrt{s}\sqrt{1 - s\cos^2\theta}\, \nabla_T v_3,$$

which gives us

$$-s(1-q\cos\theta)\varphi T = -\varepsilon\,\mathrm{sgn}(q)\sqrt{s}\sqrt{1-s\cos^2\theta}\,(-\kappa_2 v_2 + \kappa_3 v_4)\,.$$

Since $\varphi T \parallel v_2$, we find $\kappa_3 = 0$. This proves d) of the theorem.

Let us examine Legendre case separately, that is, $\theta = \frac{\pi}{2}$. Then we have $\varepsilon = 1$, $\kappa_1 = |q|$, $\kappa_2 = \sqrt{s}$, $\kappa_3 = 0$ and equation (3.14) gives us

$$v_3 = \frac{-\,\mathrm{sgn}(q)}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_\alpha.$$

This completes the proof of c).

Conversely, let γ satisfy one of a), b), c) or d). Using the Frenet frame field and Frenet equations, it is straightforward to show that $\nabla_T T = -q\varphi T$, i.e., γ is a slant normal magnetic curve.

□

The above theorem is a generalization of Theorem 3.1 of [10] (by Simona Luiza Druta-Romaniuc et al.) for S-manifolds. If we choose $s = 1$, since an S-manifold becomes a Sasakian manifold, we find their results.

Remark 3.1. The order of a slant magnetic curve in an S-manifold is still $r \leq 3$, as in the case of a magnetic curve of a Sasakian manifold, which was considered in [10].

Now, let us remove the slant condition from the hypothesis and show that the osculating order is still $r \leq 3$.

Theorem 3.2. *Let $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an S-manifold and consider the contact magnetic field F_q for $q \neq 0$. If γ is a normal magnetic curve associated to F_q in M^{2m+s}, then the osculating order $r \leq 3$.*

Proof. Let γ be a normal magnetic curve. Then, the Lorentz equation (2.6) gives us

$$\eta^\alpha(T) = \cos\theta_\alpha, \alpha = 1,\ldots,s.$$

If we differentiate this equation along the curve, we have

$$\eta^\alpha(E_2) = 0$$

for all $\alpha = 1,\ldots,s$. From the Frenet equations (3.1), we obtain

$$-q\varphi T = \kappa_1 v_2.$$

From the definition of framed φ-structure, we calculate

$$g(\varphi T, \varphi T) = 1 - A,$$

where we denote

$$A = \sum_{\alpha=1}^{s} \cos^2\theta_\alpha.$$

Then, we have

$$\|\varphi T\| = \sqrt{1-A}$$

and

$$\kappa_1 = |q|\sqrt{1-A}.$$

Thus, φT can be rewritten as

$$\varphi T = -\operatorname{sgn}(q)\sqrt{1-A}v_2. \tag{3.16}$$

Again, from the definition of framed φ-structure, we have

$$\varphi^2 T = -T + V,$$

where we denote

$$V = \sum_{\alpha=1}^{s} \cos\theta_\alpha \xi_\alpha.$$

After some calculations, we get

$$\begin{aligned}\nabla_T \varphi T &= (q-B)T + (1-A)\sum_{\alpha=1}^{s} \xi_\alpha \\ &\quad +(-q+B)V,\end{aligned}$$

which corresponds to equation (3.10). Here, we denote

$$B = \sum_{\alpha=1}^{s} \cos\theta_\alpha.$$

From equation (3.16), we also find

$$\nabla_T \varphi T = -\operatorname{sqn}(q)\sqrt{1-A}(-\kappa_1 T + \kappa_2 v_3),$$

which corresponds to equation (3.11). In this last equation, we can replace $\kappa_1 = |q|\sqrt{1-A}$. Finally, we have

$$\begin{aligned}-\operatorname{sqn}(q)\sqrt{1-A}\kappa_2 v_3 &= (1-A)\sum_{\alpha=1}^{s} \xi_\alpha + (-q+B)V \\ &\quad + (qA-B)T.\end{aligned} \tag{3.17}$$

So, if we denote the norm of the right hand side of equation (3.17) by C, we find

$$C = \sqrt{(1-A)(Aq^2 - As + B^2 - 2Bq + s)},$$

which is a constant. Hence, we obtain

$$\kappa_2 = \frac{C}{\sqrt{1-A}} = \sqrt{Aq^2 - As + B^2 - 2Bq + s} = \text{constant}.$$

From equation (3.17), we also have $v_3 \in \operatorname{span}\{T, \xi_1, \ldots, \xi_s\}$. The angles between v_3 and T, $\xi_1, \ldots, \xi_s$ are all constants since all the coefficients in equation (3.17) are constants. Then, we can write

$$v_3 = c_0 T + c_1 \xi_1 + \cdots + c_s \xi_s \tag{3.18}$$

for some constants $c_0, \dots, c_s$. If we differentiate equation (3.18), we get

$$-\kappa_2 v_2 + \kappa_3 v_4 = c_0 \kappa_1 v_2 - c_1 \varphi T - \cdots - c_s \varphi T.$$

Since φT is parallel to v_2, if we take the inner product of the last equation with v_4, we find $\kappa_3 = 0$. This proves the theorem. □

In particular, if γ is slant, i.e. $\theta_\alpha = \theta$ for all $\alpha = 1, \dots, s$, then we obtain the following corollary:

Corollary 3.1. *If $\theta_\alpha = \theta$, for all $\alpha = 1, \dots, s$, then*

$$A = s\cos^2\theta, \;\; B = s\cos\theta, \;\; V = \cos\theta \sum_{\alpha=1}^{s} \xi_\alpha,$$
$$C = \sqrt{(1 - s\cos^2\theta)s(1 - q\cos\theta)^2}$$

and $\kappa_2 = \sqrt{s}|1 - q\cos\theta|$.

Now, let us state the following proposition:

Proposition 3.1. *Let γ be a slant φ-helix of order 3 in an S-manifold $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ with contact angle θ. Then*

$$\sum_{\alpha=1}^{s} \xi_\alpha = s\cos\theta T + \rho v_3, \tag{3.19}$$

where $\rho = g\left(v_3, \sum_{\alpha=1}^{s} \xi_\alpha\right) = s\eta^\alpha(v_3)$ is a real constant such that $\rho^2 = s - s^2\cos^2\theta$. Hence, γ has the Frenet frame field

$$\left\{ T, \frac{\pm\varphi T}{\sqrt{1 - s\cos^2\theta}}, \frac{\pm 1}{\sqrt{s}\sqrt{1 - s\cos^2\theta}} \left(-s\cos\theta T + \sum_{\alpha=1}^{s} \xi_\alpha \right) \right\}.$$

Proof. From the assumption, the Frenet frame field $\{T, v_2, v_3\}$ is φ-invariant and

$$\eta^\alpha(T) = \cos\theta.$$

Differentiating the last equation along the curve, it is easy to see that

$$\eta^\alpha(v_2) = 0. \tag{3.20}$$

If we differentiate once again, we have

$$g(\varphi T, v_2) = -\kappa_1 \cos\theta + \kappa_2 \eta^\alpha(v_3), \tag{3.21}$$

which means the value of $\eta^\alpha(v_3)$ does not depend on α.

Firstly, let us assume that $\theta \neq \frac{\pi}{2}$. Since the space spanned by the Frenet frame field is φ-invariant, then $\varphi^2 T$ is in the set. Using (3.8) and (3.20), we can write

$$\sum_{\alpha=1}^{s} \xi_\alpha \in sp\{T, v_3\},$$

that is,

$$\sum_{\alpha=1}^{s} \xi_\alpha = s\cos\theta T + \rho v_3. \tag{3.22}$$

If we take the norm of both sides, we find $\rho^2 = s - s^2\cos^2\theta$. Since the value of $\eta^\alpha(v_3)$ does not depend on α, we obtain

$$\rho = g\left(v_3, \sum_{\alpha=1}^{s} \xi_\alpha\right) = s\eta^\alpha(v_3).$$

If we apply φT to (3.22), we get $g(\varphi T, v_3) = 0$. Since $\varphi T \perp T$, $\varphi T \perp v_3$ and $sp\{T, v_2, v_3\}$ is φ-invariant, we have $\varphi T \parallel v_2$. As a result, we find

$$v_2 = \frac{\pm\varphi T}{\|\varphi T\|} = \frac{\pm\varphi T}{\sqrt{1 - s\cos^2\theta}}.$$

Now let us consider the Legendre case, i.e., $\theta = \frac{\pi}{2}$. From (3.21), we find

$$\nabla_T g(\varphi T, v_2) = -\kappa_2 g(\varphi T, v_3). \tag{3.23}$$

Using (2.1) and (2.3), we calculate

$$\begin{aligned} \nabla_T \varphi T &= (\nabla_T \varphi)T + \varphi\nabla_T T \\ &= \sum_{\alpha=1}^{s} \xi_\alpha + \kappa_1 \varphi v_2. \end{aligned} \tag{3.24}$$

Using the last equation, we obtain

$$\begin{aligned} \nabla_T g(\varphi T, v_2) &= g(\nabla_T \varphi T, v_2) + g(\varphi T, \nabla_T v_2) \\ &= \kappa_2 g(\varphi T, v_3). \end{aligned} \tag{3.25}$$

Equations (3.23) and (3.25) give us $g(\varphi T, v_3) = 0$, that is, $\varphi T \parallel v_2$. Thus, we have $\varphi T = \pm v_2$. Consequently, the Frenet frame field becomes $\{T, \pm\varphi T, v_3\}$. Now, we must show that v_3 is parallel to $\sum_{\alpha=1}^{s} \xi_\alpha$. Since the space spanned by the Frenet frame field is φ-invariant, from orthonormal expansion, we can write

$$\varphi v_3 = g(\varphi v_3, T)T \pm g(\varphi v_3, \pm\varphi T)\,\varphi T + g(\varphi v_3, v_3)\,v_3,$$

which reduces to

$$\varphi v_3 = g(\varphi v_3, T)T. \tag{3.26}$$

If we apply φ to equation (3.26) and use (2.1), we find

$$-v_3 + \sum_{\alpha=1}^{s} \eta^\alpha(v_3)\xi_\alpha = g(\varphi v_3, T)\varphi T. \tag{3.27}$$

Applying φT to (3.27) and using the Frenet frame field, we have $g(\varphi v_3, T) = 0$. As a result, we get $\varphi v_3 = 0$ and equation (3.27) becomes

$$-v_3 + \sum_{\alpha=1}^{s} \eta^\alpha(v_3)\xi_\alpha = 0.$$

We have already shown that the value of $\eta^\alpha(v_3)$ does not depend on α; so, we can write

$$v_3 = \eta^\alpha(v_3)\sum_{\alpha=1}^{s} \xi_\alpha. \tag{3.28}$$

Since v_3 and ξ_α are unit for all $\alpha = 1,\ldots,s$, we find $\eta^\alpha(v_3) = \frac{\pm 1}{\sqrt{s}}$. Finally, for $\theta = \frac{\pi}{2}$, we have $\rho^2 = s$ and $\sum_{\alpha=1}^{s} \xi_\alpha = \rho v_3$, which completes the proof. □

Corollary 3.2. *Let γ be a Legendre φ-helix of order 3 in an S-manifold $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$. Then $\kappa_2 = \sqrt{s}$, $v_2 = \pm\varphi T$ and $v_3 = \pm\frac{1}{\sqrt{s}}\sum_{\alpha=1}^{s} \xi_\alpha$.*

Proof. From equation (3.28), we already have

$$v_3 = \pm\frac{1}{\sqrt{s}}\sum_{\alpha=1}^{s} \xi_\alpha.$$

If we differentiate this equation and use (3.1), we obtain

$$-\kappa_2 v_2 = \pm\frac{1}{\sqrt{s}}\sum_{\alpha=1}^{s} \nabla_T \xi_\alpha. \tag{3.29}$$

Using equations (2.4) and (3.29), we find that $\kappa_2 = \sqrt{s}$ and $v_2 = \pm\varphi T$. □

Finally, we can give the following theorem:

Theorem 3.3. *Let γ be a slant φ-helix of order $r \leq 3$ on an S-manifold $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$. Let θ denote the contact angle of γ. Then we have*

i. *If $\cos\theta = \pm\frac{1}{\sqrt{s}}$, then γ is an integral curve of $\pm\frac{1}{\sqrt{s}}\sum_{\alpha=1}^{s} \xi_\alpha$, hence it is a normal magnetic curve for F_q with an arbitrary q.*

ii. *If* $\cos\theta = 0$ *and* $\kappa_1 \neq 0$ *(i.e.* γ *is a non-geodesic Legendre curve), then* γ *is a magnetic curve for* $F_{\pm\kappa_1}$.

iii. *If* $\cos\theta = \frac{\varepsilon}{\sqrt{\kappa_1^2+s}}$, *then* γ *is a magnetic curve for* $F_{\varepsilon\sqrt{\kappa_1^2+s}}$, *where* $\varepsilon = -\operatorname{sgn}(g(\varphi T, v_2))$. *In this case,* γ *is a slant* φ*-circle.*

iv. *If* $\cos\theta = \frac{\varepsilon\sqrt{s}\pm\kappa_2}{\sqrt{s}\sqrt{\kappa_1^2+(\varepsilon\sqrt{s}\pm\kappa_2)^2}}$, *then* γ *is a magnetic curve for* $F_{\varepsilon\sqrt{\kappa_1^2+(\varepsilon\sqrt{s}\pm\kappa_2)^2}}$, *where* $\varepsilon = -\operatorname{sgn}(g(\varphi T, v_2))$ *and the sign* $\pm$ *corresponds to the sign of* $\eta^\alpha(v_3)$.

v. *Except the above cases,* γ *is not a magnetic curve for any* F_q.

Proof. Let $\cos\theta = \pm\frac{1}{\sqrt{s}}$. Then we have

$$T = \pm\frac{1}{\sqrt{s}}\sum_{\alpha=1}^{s}\xi_\alpha,$$

which gives us $\nabla_T T = 0$. We also have $\varphi T = 0$. So γ satisfies $\nabla_T T = -q\varphi T$ for any q, which proves *i*.

Now let $\cos\theta = 0$ and $\kappa_1 \neq 0$. Using Corollary 3.2, we have

$$\nabla_T T = \kappa_1(\pm\varphi T) = -q\varphi T,$$

which gives us $q = \pm\kappa_1$. This completes the proof of *ii*.

From Proposition 3.1, we have the Frenet frame field

$$\left\{T, \frac{\pm\varphi T}{\sqrt{1-s\cos^2\theta}}, \frac{\pm 1}{\sqrt{s}\sqrt{1-s\cos^2\theta}}\left(-s\cos\theta T + \sum_{\alpha=1}^{s}\xi_\alpha\right)\right\}$$

when $r = 3$ and

$$\left\{T, \frac{\pm\varphi T}{\sqrt{1-s\cos^2\theta}}\right\}$$

when $r = 2$. If we differentiate v_2 along the curve, after some calculations, in both cases, we find

$$\left(1 \pm \frac{\kappa_1\cos\theta}{\sqrt{1-s\cos^2\theta}}\right)\left(-s\cos\theta T + \sum_{\alpha=1}^{s}\xi_\alpha\right) = \pm\kappa_2\sqrt{1-s\cos^2\theta}\,v_3, \tag{3.30}$$

(taking $\kappa_2 = 0$, when $r = 2$).

Next, let us assume $\cos\theta = \frac{\varepsilon}{\sqrt{\kappa_1^2+s}}$, where we denote $\varepsilon = -\operatorname{sgn}(g(\varphi T, v_2))$. Then the left side of equation (3.30) vanishes. Thus we get $\kappa_2 = 0$. From the assumption, we also have $\kappa_1 =$ constant, that is, γ is a slant φ-circle. Using the Frenet frame field, we find $\nabla_T T = -q\varphi T = \kappa_1 v_2$, where $q = \varepsilon\sqrt{\kappa_1^2+s}$. So, we have just completed the proof of *iii*.

Finally, let us assume $\cos\theta = \frac{\varepsilon\sqrt{s}\pm\kappa_2}{\sqrt{s}\sqrt{\kappa_1^2+(\varepsilon\sqrt{s}\pm\kappa_2)^2}}$, where $\varepsilon = -\operatorname{sgn}(g(\varphi T, v_2))$ and the sign $\pm$ corresponds to the sign of $\eta^\alpha(v_3)$. In this case, let us take $\kappa_2 \neq 0$, since we have already investigated order $r = 2$. Using the Frenet frame field, after some calculations, we obtain $\nabla_T T = -q\varphi T = \kappa_1 v_2$, where $q = \varepsilon\sqrt{\kappa_1^2+(\varepsilon\sqrt{s}\pm\kappa_2)^2}$. Hence, the proof of *iv* is complete.

Since we have considered all cases, we can state that there exist no other slant magnetic φ-helices in M. □

From the proof of Theorem 3.1, we can give the following proposition:

Proposition 3.2. *Let* $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ *be an S-manifold. There exist no non-geodesic slant φ-circles as magnetic curves corresponding to F_q for* $0 < |q| \leq \sqrt{s}$.

Theorem 3.3 and Proposition 3.2 generalize Theorem 3.2 and Proposition 3.2 in [10] to S-manifolds, respectively. Under the condition $s = 1$, we obtain their results.

4. Construction of slant normal magnetic curves in $\mathbb{R}^{2n+s}(-3s)$

In this section, we find parametric equations of slant normal magnetic curves in $\mathbb{R}^{2n+s}(-3s)$. As a start, we recall structures defined on this S-manifold. Let us take $M = \mathbb{R}^{2n+s}$ with coordinate functions $\{x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_s\}$ and define

$$\xi_\alpha = 2\frac{\partial}{\partial z_\alpha}, \quad \alpha = 1, \ldots, s,$$
$$\eta^\alpha = \frac{1}{2}\left(dz_\alpha - \sum_{i=1}^{n} y_i dx_i\right), \quad \alpha = 1, \ldots, s,$$
$$\varphi X = \sum_{i=1}^{n} Y_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} X_i \frac{\partial}{\partial y_i} + \left(\sum_{i=1}^{n} Y_i y_i\right)\left(\sum_{\alpha=1}^{s} \frac{\partial}{\partial z_\alpha}\right),$$
$$g = \sum_{\alpha=1}^{s} \eta^\alpha \otimes \eta^\alpha + \frac{1}{4}\sum_{i=1}^{n}(dx_i \otimes dx_i + dy_i \otimes dy_i),$$

where

$$X = \sum_{i=1}^{n}\left(X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}\right) + \sum_{\alpha=1}^{s}\left(Z_\alpha \frac{\partial}{\partial z_\alpha}\right) \in \chi(M).$$

It is well-known that $\left(\mathbb{R}^{2n+s}, \varphi, \xi_\alpha, \eta^\alpha, g\right)$ is an S-space form with constant φ-sectional curvature $-3s$. Hence it is denoted by $\mathbb{R}^{2n+s}(-3s)$ [13]. The following vector fields

$$X_i = 2\frac{\partial}{\partial y_i}, \quad X_{n+i} = \varphi X_i = 2\left(\frac{\partial}{\partial x_i} + y_i \sum_{\alpha=1}^{s} \frac{\partial}{\partial z_\alpha}\right), \quad \xi_\alpha = 2\frac{\partial}{\partial z_\alpha}$$

form a g-orthonormal basis and the Levi-Civita connection is

$$\nabla_{X_i} X_j = \nabla_{X_{n+i}} X_{n+j} = 0, \quad \nabla_{X_i} X_{n+j} = \delta_{ij} \sum_{\alpha=1}^{s} \xi_\alpha, \quad \nabla_{X_{n+i}} X_j = -\delta_{ij} \sum_{\alpha=1}^{s} \xi_\alpha,$$

$$\nabla_{X_i} \xi_\alpha = \nabla_{\xi_\alpha} X_i = -X_{n+i}, \quad \nabla_{X_{n+i}} \xi_\alpha = \nabla_{\xi_\alpha} X_{n+i} = X_i.$$

(see [13]). Let $\gamma : I \to \mathbb{R}^{2n+s}(-3s)$ be a unit-speed slant curve with contact angle θ. Let us denote

$$\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t), \gamma_{n+1}(t), \ldots, \gamma_{2n}(t), \gamma_{2n+1}(t), \ldots, \gamma_{2n+s}(t)),$$

where t is the arc-length parameter. Then γ has the tangent vector field

$$T = \gamma'_1 \frac{\partial}{\partial x_1} + \cdots + \gamma'_n \frac{\partial}{\partial x_n} + \gamma'_{n+1} \frac{\partial}{\partial y_1} + \cdots + \gamma'_{2n} \frac{\partial}{\partial y_n} + \gamma'_{2n+1} \frac{\partial}{\partial z_1} + \cdots + \gamma'_{2n+s} \frac{\partial}{\partial z_s},$$

which can be written as

$$T=\frac{1}{2}\left[\gamma'_{n+1}X_1+\cdots+\gamma'_{2n}X_n+\gamma'_1X_{n+1}+\cdots+\gamma'_nX_{2n}\right.$$
$$+\left(\gamma'_{2n+1}-\gamma'_1\gamma_{n+1}-\cdots-\gamma'_n\gamma_{2n}\right)\xi_1+\cdots$$
$$\left.+\left(\gamma'_{2n+s}-\gamma'_1\gamma_{n+1}-\cdots-\gamma'_n\gamma_{2n}\right)\xi_s\right].$$

Since γ is slant curve, we have

$$\eta^{\alpha}(T)=\frac{1}{2}\left(\gamma'_{2n+\alpha}-\gamma'_1\gamma_{n+1}-\cdots-\gamma'_n\gamma_{2n}\right)=\cos\theta$$

for all $\alpha=1,\ldots,s$. So, we obtain

$$\gamma'_{2n+1}=\cdots=\gamma'_{2n+s}=2\cos\theta+\gamma'_1\gamma_{n+1}+\cdots+\gamma'_n\gamma_{2n}. \tag{4.1}$$

Since γ is a unit-speed, we can write

$$\left(\gamma'_1\right)^2+\cdots+\left(\gamma'_{2n}\right)^2=4\left(1-s\cos^2\theta\right). \tag{4.2}$$

These equations were obtained in our paper [12].

Now, our aim is to find parametric equations for slant normal magnetic curves. So, let us assume that $\gamma: I\to\mathbb{R}^{2n+s}(-3s)$ is a normal magnetic curve. From the Lorentz equation, we have

$$\nabla_T T=-q\varphi T, \tag{4.3}$$

where $q\neq 0$ is a constant. Using the Levi-Civita connection, we calculate

$$\nabla_T T=\frac{1}{2}\left\{\left(\gamma''_{n+1}+2s\cos\theta\gamma'_1\right)X_1+\cdots+\left(\gamma''_{2n}+2s\cos\theta\gamma'_n\right)X_n\right. \tag{4.4}$$
$$\left.+\left(\gamma''_1-2s\cos\theta\gamma'_{n+1}\right)X_{n+1}+\cdots+\left(\gamma''_n-2s\cos\theta\gamma'_{2n}\right)X_{2n}\right\}$$

and

$$\varphi T=\frac{1}{2}\left\{-\gamma'_1X_1-\cdots-\gamma'_nX_n+\gamma'_{n+1}X_{n+1}+\cdots+\gamma'_{2n}X_{2n}\right\}. \tag{4.5}$$

From equations (4.3), (4.4) and (4.5), we have

$$\frac{\gamma''_{n+1}+2s\cos\theta\gamma'_1}{-\gamma'_1}=\cdots=\frac{\gamma''_{2n}+2s\cos\theta\gamma'_n}{-\gamma'_n}$$
$$=\frac{\gamma''_1-2s\cos\theta\gamma'_{n+1}}{\gamma'_{n+1}}=\cdots=\frac{\gamma''_n-2s\cos\theta\gamma'_{2n}}{\gamma'_{2n}}=-q,$$

which is equivalent to

$$\frac{\gamma''_{n+1}}{-\gamma'_1}=\cdots=\frac{\gamma''_{2n}}{-\gamma'_n}=\frac{\gamma''_1}{\gamma'_{n+1}}=\cdots=\frac{\gamma''_n}{\gamma'_{2n}}=\lambda, \tag{4.6}$$

where we denote $\lambda=-q+2s\cos\theta$. Firstly, let us assume $\lambda\neq 0$. From equation (4.6), if we select pairs

$$\frac{\gamma''_{n+1}}{-\gamma'_1}=\frac{\gamma''_1}{\gamma'_{n+1}},\ldots,\frac{\gamma''_{2n}}{-\gamma'_n}=\frac{\gamma''_n}{\gamma'_{2n}},$$

solving ODEs, we have

$$\left(\gamma'_1\right)^2+\left(\gamma'_{n+1}\right)^2=c_1^2,\ldots,\left(\gamma'_n\right)^2+\left(\gamma'_{2n}\right)^2=c_n^2,$$

where $c_1,\ldots,c_n$ are arbitrary constants. Thus, we can write

$$\begin{aligned}\gamma'_1&=c_1\cos f_1,\ldots,\gamma'_n=c_n\cos f_n,\\ \gamma'_{n+1}&=c_1\sin f_1,\ldots,\gamma'_{2n}=c_n\sin f_n,\end{aligned}\tag{4.7}$$

where $f_1,\ldots,f_n$ are differentiable functions on I. From (4.6) and (4.7), we find

$$f'_1=\cdots=f'_n=-\lambda,$$

which gives us

$$f_i=-\lambda t+a_i,\ \ i=1,2,\ldots,n$$

where $a_1,\ldots,a_n$ are arbitrary constants. Now, if we integrate (4.7), we have

$$\gamma_1=\frac{c_1}{-\lambda}\sin f_1+b_1,\ldots,\gamma_n=\frac{c_n}{-\lambda}\sin f_n+b_n,$$

$$\gamma_{n+1}=\frac{c_1}{\lambda}\cos f_1+d_1,\ldots,\gamma_{2n}=\frac{c_n}{\lambda}\cos f_n+d_n,$$

where b_i and d_i are arbitrary constants ($i=1,\ldots,n$). Thus, we get

$$\gamma'_1\gamma_{n+1}+\cdots+\gamma'_n\gamma_{2n}=\sum_{i=1}^{n}\left(\frac{c_i^2}{\lambda}\cos^2 f_i+c_id_i\cos f_i\right).$$

Using the last equation with (4.1), we obtain

$$\gamma'_{2n+\alpha}=2\cos\theta+\sum_{i=1}^{n}\left(\frac{c_i^2}{\lambda}\cos^2 f_i+c_id_i\cos f_i\right),$$

where $\alpha=1,\ldots,s$. If we integrate this last equation, we find

$$\gamma_{2n+\alpha}=2t\cos\theta-\sum_{i=1}^{n}\left\{\frac{c_i^2}{4\lambda^2}\left[\sin\left(2f_i\right)+2f_i\right]+\frac{c_id_i}{\lambda}\sin f_i\right\}+h_\alpha,$$

for $\alpha=1,\ldots,s$ and $h_1,\ldots,h_s$ are arbitrary constants. Moreover, from (4.2) and (4.7), we have

$$c_1^2+\cdots+c_n^2=4\left(1-s\cos^2\theta\right).\tag{4.8}$$

Thus, we have just finished the case $\lambda\neq 0$.

Secondly, let $\lambda = 0$. In this case, we have

$$\gamma_1'' = \gamma_2'' = \cdots = \gamma_{2n}'' = 0,$$

which gives us

$$\gamma_i = c_i t + d_i,$$

for $i = 1, \ldots, 2n$, where c_i and d_i are arbitrary constants. Using the last equation, we calculate

$$\gamma_1' \gamma_{n+1} + \cdots + \gamma_n' \gamma_{2n} = \sum_{i=1}^{n} c_i \left(c_{n+i} t + d_{n+i} \right).$$

So, equation (4.1) becomes

$$\gamma_{2n+\alpha}' = 2\cos\theta + \sum_{i=1}^{n} c_i \left(c_{n+i} t + d_{n+i} \right),$$

which gives us

$$\gamma_{2n+\alpha} = 2t\cos\theta + \sum_{i=1}^{n} c_i \left(\frac{c_{n+i}}{2} t^2 + d_{n+i} t \right) + h_\alpha,$$

where h_α are arbitrary constants for $\alpha = 1, \ldots, s$. Since γ is unit-speed, from (4.2), we have

$$c_1^2 + \cdots + c_{2n}^2 = 4\left(1 - s\cos^2\theta\right).$$

To sum up, we give the following Theorem:

Theorem 4.1. *The slant normal magnetic curves on $\mathbb{R}^{2n+s}(-3s)$ satisfying the Lorentz equation $\nabla_T T = -q\varphi T$ have the parametric equations*

a)

$$\gamma_i(t) = \frac{c_i}{-\lambda} \sin f_i(t) + b_i,$$
$$\gamma_{n+i}(t) = \frac{c_i}{\lambda} \cos f_i(t) + d_i,$$

$$\gamma_{2n+\alpha}(t) = 2t\cos\theta - \sum_{i=1}^{n} \left\{ \frac{c_i^2}{4\lambda^2} \left[\sin\left(2f_i(t)\right) + 2f_i(t) \right] + \frac{c_i d_i}{\lambda} \sin f_i(t) \right\} + h_\alpha,$$
$$f_i(t) = -\lambda t + a_i,$$
$$\alpha = 1, \ldots, s, \;\; i = 1, 2, \ldots, n,$$
$$\lambda = -q + 2s\cos\theta \neq 0$$

where a_i, b_i, c_i, d_i and h_α are arbitrary constants such that c_i satisfies

$$c_1^2 + \cdots + c_n^2 = 4\left(1 - s\cos^2\theta\right);$$

or

b)

$$\gamma_i(t) = c_i t + d_i,$$
$$\gamma_{2n+\alpha}(t) = 2t\cos\theta + \sum_{i=1}^{n} c_i\left(\frac{c_{n+i}}{2}t^2 + d_{n+i}t\right) + h_\alpha,$$
$$\alpha = 1,\ldots,s,\ \ i = 1,\ldots,2n,$$

where c_i, d_i and h_α are arbitrary constants such that c_i satisfies

$$c_1^2 + \cdots + c_{2n}^2 = 4\left(1 - s\cos^2\theta\right).$$

In both cases, $q \neq 0$ is a constant and θ denotes the constant contact angle satisfying $|\cos\theta| \leq \frac{1}{\sqrt{s}}$.

In particular, if $s = 1$, we obtain Theorem 3.5 in [10].

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