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## A simple-looking relative of the Novikov, Hirota-Satsuma and Sawada-Kotera equations

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We study the simple-looking scalar integrable equation  $f_{xxt} - 3(f_x f_t - 1) = 0$ , which is related (in different ways) to the Novikov, Hirota-Satsuma and Sawada-Kotera equations. For this equation we present a Lax pair, a Bäcklund transformation, soliton and merging soliton solutions (some exhibiting instabilities), two infinite hierarchies of conservation laws, an infinite hierarchy of continuous symmetries, a Painlevé series, a scaling reduction to a third order ODE and its Painlevé series, and the Hirota form (giving further multisoliton solutions).

*Keywords:* Integrable equation; Novikov; Hirota-Satsuma; Sawada-Kotera; Bäcklund transformation

2000 Mathematics Subject Classification: 35Q58, 37K05, 37K10, 37K35, 37K40, 37K45, 70G65

### 1. Introduction

Early in the history of integrable systems it was found that there are remarkable transformations connecting different equations [1]. These “hidden connections” abound, and complicate classification attempts, see for example [2]. But, in general, discovery of such a connection is very useful, as typically most of the integrability properties of an equation can be mapped to other related equations. Thus, for example, more research focuses on the KdV equation than on the (defocusing) MKdV equation, and on the NLS equation rather than the many systems related to it by Miura maps (see Figure 1 in [3]). Naturally, the focus tends to fall on the equation in the class which is simplest to write down.

One of the simple, archetypal equations in integrable systems theory is the Camassa-Holm equation, the first integrable equation to be discovered with peakon solutions [4, 5]. This was followed by the Depasperi-Procesi equation [6, 7] and by the Novikov equation [8, 9]

$$m_t + 3uu_x m + u^2 m_x = 0, \quad (1.1)$$

$$m = u - u_{xx}. \quad (1.2)$$

In [10] we announced that the Novikov equation is related, by a chain of transformations, to the particularly simple equation

$$f_{xxt} - 3(f_x f_t - 1) = 0, \quad (1.3)$$

which we called the associated Novikov equation (aN). We believe that [10] is the first time (1.3) was written down explicitly as a single equation. However, it is trivial to derive from equations (2.6a) and (2.8) in [11] by introducing a potential (writing  $W = -F_\tau$ ,  $U = 3F_y$ ) followed by a rescaling (and changes of names of the variables). See also [12], equation (9). Matsuno derived a different scalar equation from equations (2.6a) and (2.8) in [11]. In our notation, note that we can solve (1.3) to find  $f_t$  in terms of  $g = f_x$ . Differentiating this with respect to  $x$  and rearranging gives

$$gg_{xxt} - g_x g_{xt} - 3g^2 g_t - 3g_x = 0, \tag{1.4}$$

cf. (2.11) in [11]. Note also that by rescaling  $t$  it is possible to replace the constant term in (1.3) by any non-zero constant. The equation with vanishing constant term, i.e.

$$f_{xxt} - 3f_x f_t = 0, \tag{1.5}$$

was studied in [13], and we will discuss this equation further at the end of this introduction.

Since the derivation of the various equations in [11] from the Novikov equation is clear, we do not give the full derivation of aN from Novikov here. Proceeding from (1.3), however, make the substitution

$$f(x,t) = h(x,t) + \beta x + \frac{t}{\beta},$$

where  $\beta \neq 0$  is a constant, to obtain

$$h_{xxt} - 3h_x h_t + \beta h_t + \frac{h_x}{\beta} = 0.$$

Differentiating with respect to  $x$  and writing  $h = r_x$ , so  $r = -\int_x^\infty h(y,t) dy$ , we obtain

$$r_{xxt} + 3r_x \int_x^\infty r_t(y,t) dy - 3rr_t + \beta r_t + \frac{r_x}{\beta} = 0. \tag{1.6}$$

Modulo rescalings, this is equation (2) in [14], known as the Hirota-Satsuma equation. In [11], Matsuno established the connection between his equation (1.4) and the Hirota-Satsuma equation.

Thus we see that aN is related to the Novikov equation (1.1)-(1.2), the Matsuno equation (1.4) and the Hirota-Satsuma equation (1.6). We note, though, that aN is rather more simple in form than any of these other equations, and thus is surely the natural first object for study. In fact there is a further equation to which aN is related, though this time it is not by a sequence of transformations. We recall that  $\eta$  is an infinitesimal symmetry for (1.3) if, as a consequence of (1.3),  $f + \varepsilon\eta$  is a solution of (1.3) up to linear order in  $\varepsilon$ . That is, if

$$\eta_{xxt} - 3f_x \eta_t - 3f_t \eta_x = 0.$$

A direct calculation shows that  $\eta = -\frac{1}{15}f_{xxxxx} + f_x f_{xxx} - f_x^3$  satisfies the necessary condition. It follows that the flow

$$f_s = \frac{1}{15}f_{xxxxx} - f_x f_{xxx} + f_x^3 \tag{1.7}$$

(now  $f$  is being considered as a function of three variables,  $x$ ,  $t$  and  $s$ ) is consistent with the flow defined by aN. This last equation is a potential form of the Sawada-Kotera equation (SK), viz. equation (44) in [15]. The link between aN and potential SK is not that there are transformations

from one to the other, but they define consistent flows, i.e. they belong to the same *hierarchy*. Although the link is less direct, integrable equations in the same hierarchy also share properties, such as conservation laws and soliton solutions. The fact that the Novikov equation is related to the SK hierarchy was already established in [9], see also [16]. Although in this case aN has a complication not in the potential SK flow, in that in equation (1.7) only a single  $t$  derivative term appears, aN remains superficially simpler, so we would argue that aN is the right equation to study first.

We have thus seen in this introduction how aN, equation (1.3), is related to numerous equations of interest, but is, at least superficially, the simplest of all. The aim of this paper is to present the basic properties of aN, without reference to any other equation, with the intention that ultimately, aN can be used as a base for study of the other equations. In section 2 we present the Lax pair and Bäcklund transformation (BT) for aN. The Lax pair, which we derived from the Lax pair for the Novikov equation given in [9], is identical to that given for (1.5) in [13]. In section 3 we present the basic soliton solutions of aN, and also solutions describing the “merger” of two solitons, obtained using the BT. However, the BT also gives rise to a solution showing that at least some of the basic soliton solutions are *unstable*. This limits the possible physical relevance of aN, but on the other hand provides a simple analytic example of the phenomenon of soliton instability, the possibility of which is often overlooked. In section 4 we discuss conservation laws, giving two infinite hierarchies. In section 5 we consider symmetries, but only succeed to give one infinite hierarchy. We believe a second hierarchy should exist (in parallel to what we have found for conservation laws), but leave this as an open problem. We show the existence of a second infinite hierarchy in the case of the Camassa-Holm equation to demonstrate that this is feasible. In section 6 we show the Painlevé property for aN. In section 7 we discuss a scaling reduction to a third order ODE and give its Painlevé series and the simplest solution. In section 8 we give the Hirota form of aN and a formula for multisoliton solutions. In section 9 we conclude.

Before leaving this introduction, we briefly discuss equation (1.5) that was studied in [13]. As mentioned above, by rescaling  $t$  it is possible to replace the constant in (1.3) by any nonzero constant, and thus it is possible to derive properties of the equation (1.5) from those of the equation (1.3) by a limit process. In this way, for example, it is possible to find (several families of) soliton solutions of the (1.5). For a reason that will become clear in the next section, there is a single Lax pair for the equation, irrespective of the value of the constant term. However, in general we are not aware of a process to derive properties of (1.3) from those of (1.5) and we believe the two equations are nonequivalent. Furthermore, it is (1.3) that arises in the context of the Novikov equation. Thus in this paper we study (1.3).

## 2. Lax Pair and Bäcklund transformation

The Lax pair for aN is

$$\phi_{xxx} = 3f_x\phi_x + \theta\phi, \tag{2.1}$$

$$\phi_t = \frac{1}{\theta}(f_t\phi_{xx} - f_{xt}\phi_x). \tag{2.2}$$

As mentioned above, this Lax pair can be derived, through the necessary transformations, from the Lax pair for the Novikov equation given in [9]. In fact the consistency condition for the Lax pair is not exactly equation (1.3), but rather its  $x$ -derivative. This explains why it also provides a Lax pair

for the equation (1.5), as studied in [13], and indeed it coincides with the Lax pair given there. The first equation of the Lax pair coincides with the first equation of the Lax pair for SK [17].

The Bäcklund transformation (BT) for aN is

$$f \rightarrow f - 2v, \tag{2.3}$$

where  $v$  satisfies the following system:

$$v_{xx} = -v^3 - 3vv_x + 3f_xv + \theta, \tag{2.4}$$

$$v_t = f_t - \frac{f_tv^3 + f_tv v_x - f_{tx}v^2 - 3v}{\theta}. \tag{2.5}$$

The second equation can be rewritten

$$v_t = \frac{1}{\theta}(f_tv_x - f_{xt}v + f_tv^2)_x. \tag{2.6}$$

The equation (2.4) has appeared before in [18, 19] as part of the BT for SK. The BT (2.4,2.5) is connected to the Lax pair (2.1,2.2) via the substitution

$$v = \frac{\phi_x}{\phi}. \tag{2.7}$$

By the relevant transformations, this BT is equivalent to the BT given by Yadong Shang [20] for the Hirota-Satsuma equation.

### 3. Soliton and merging soliton solutions

Using the standard procedure for finding travelling wave solutions shows that aN has soliton solutions

$$f = \beta x + \frac{t}{\beta} - 2B \tanh B(x - ct), \quad B^2 = \frac{3}{4} \left( \beta - \frac{1}{c\beta} \right).$$

We call this a soliton solution as the associated profile  $g = f_x$  of the Matsuno equation (1.4) has a  $\text{sech}^2$  profile, on the nonzero background  $\beta$  (in fact the profile is  $-\text{sech}^2$ , so it might more correctly be called an antisoliton). Note that here  $\beta \neq 0$  and  $c$ , the wave speed, are arbitrary, subject to the constraint

$$\beta > \frac{1}{c\beta}.$$

So if  $\beta$  is negative, then  $0 < c < \frac{1}{\beta^2}$ , and if  $\beta$  is positive then either  $c$  is negative or  $c > \frac{1}{\beta^2}$ . In the solution it is possible to replace  $\tanh$  by  $\coth$  to get a singular soliton solution.

Application of the BT (2.3), (2.4), (2.5) to the “trivial” solution

$$f = \beta x + \frac{t}{\beta}$$

gives the solution

$$f = \beta x + \frac{t}{\beta} - 2 \frac{C_1 \lambda_1 e^{\lambda_1 x + \frac{\lambda_1^2 t}{\beta \theta}} + C_2 \lambda_2 e^{\lambda_2 x + \frac{\lambda_2^2 t}{\beta \theta}} + C_3 \lambda_3 e^{\lambda_3 x + \frac{\lambda_3^2 t}{\beta \theta}}}{C_1 e^{\lambda_1 x + \frac{\lambda_1^2 t}{\beta \theta}} + C_2 e^{\lambda_2 x + \frac{\lambda_2^2 t}{\beta \theta}} + C_3 e^{\lambda_3 x + \frac{\lambda_3^2 t}{\beta \theta}}}. \tag{3.1}$$

where  $C_1, C_2, C_3$  are constants and  $\lambda_1, \lambda_2, \lambda_3$  are the roots of the cubic equation  $\lambda^3 = 3\beta\lambda + \theta$ . The analysis of these solutions proceeds in a similar manner as in [10,21]. We restrict the analysis to the case  $4\beta^3 > \theta^2$ , so the roots  $\lambda_1, \lambda_2, \lambda_3$  are real and distinct; note, in particular, that this implies  $\beta > 0$ . When one of the constants  $C_i$  vanishes, say  $C_3$ , we recover a travelling wave solution with speed  $c = \frac{\lambda_3}{\beta\theta}$ , which is a soliton if  $C_1, C_2$  have the same sign and a singular soliton if they have different signs. It is possible to check that for a suitable choice of  $\theta$  this gives all possible speeds  $c$  with  $c < 0$  or  $c > \frac{1}{\beta^2}$ .

Continuing now to the case that all the three constants  $C_1, C_2, C_3$  are nonzero, if all of them have the same sign then evidently the solution (3.1) will be nonsingular, but if there are differing signs then we expect a singularity. Without loss of generality assume  $\lambda_1 < \lambda_2 < \lambda_3$ . Then it can be checked that for  $\theta > 0$  the solution with  $C_1, C_2, C_3$  all of the same sign describes solitons of speeds  $\frac{\lambda_1}{\beta\theta}$  and  $\frac{\lambda_3}{\beta\theta}$  merging to give a soliton of speed  $\frac{\lambda_2}{\beta\theta}$ , while for  $\theta < 0$  the opposite happens, with the soliton of speed  $\frac{\lambda_2}{\beta\theta}$  splitting into two solitons of speeds  $\frac{\lambda_1}{\beta\theta}$  and  $\frac{\lambda_3}{\beta\theta}$ . See Figure 1 (for the case  $\theta > 0$ ). Note that  $\lambda_1 + \lambda_2 + \lambda_3 = 0$  (so in the mergers there is “conservation of speed”), and  $\lambda_1 \lambda_2 \lambda_3 = \theta$ . So in general  $\lambda_1 < 0$  and  $\lambda_3 > 0$ , while the sign of  $\lambda_2$  is opposite to that of  $\theta$ .

When  $C_1, C_2, C_3$  have differing signs, various scenarios emerge. Figure 2 shows a case of expulsion of a soliton by a singular soliton. However, the most interesting case is when  $\theta < 0$  and the sign of  $C_2$  differs from that of  $C_1$  and  $C_3$ . In this case the solution describes the splitting of a left moving soliton into a pair of singular solitons, see Figure 3. For large negative  $t$  this solution is a small perturbation of the soliton solution obtained by setting  $C_2 = 0$ . However, as  $t$  increases, the small perturbation grows, ultimately causing a divergence. Thus we deduce that the original soliton solution is unstable. A brief calculation shows this only happens for solitons with  $-\frac{1}{2\beta^2} < c < 0$ . The stability of solitons for other values of  $c$  (and for the case of negative  $\beta$ ) remains open.

#### 4. Conservation laws

To find the conservation laws of aN, it is just necessary to observe that (2.6) has the form of a conservation law. Thus  $v$ , which is the solution of (2.4)-(2.5) and depends on  $\theta$ , provides a generating function for (densities of) conservation laws. Similar to what was observed for the Camassa-Holm equation in [22], this can be expanded in various different ways to obtain series of conservation laws.

For large  $|\theta|$ , the solution  $v$  to (2.4) can be expanded in an asymptotic series of the form

$$v \sim \sum_{i=-1}^{\infty} \theta^{-i/3} v_i. \tag{4.1}$$

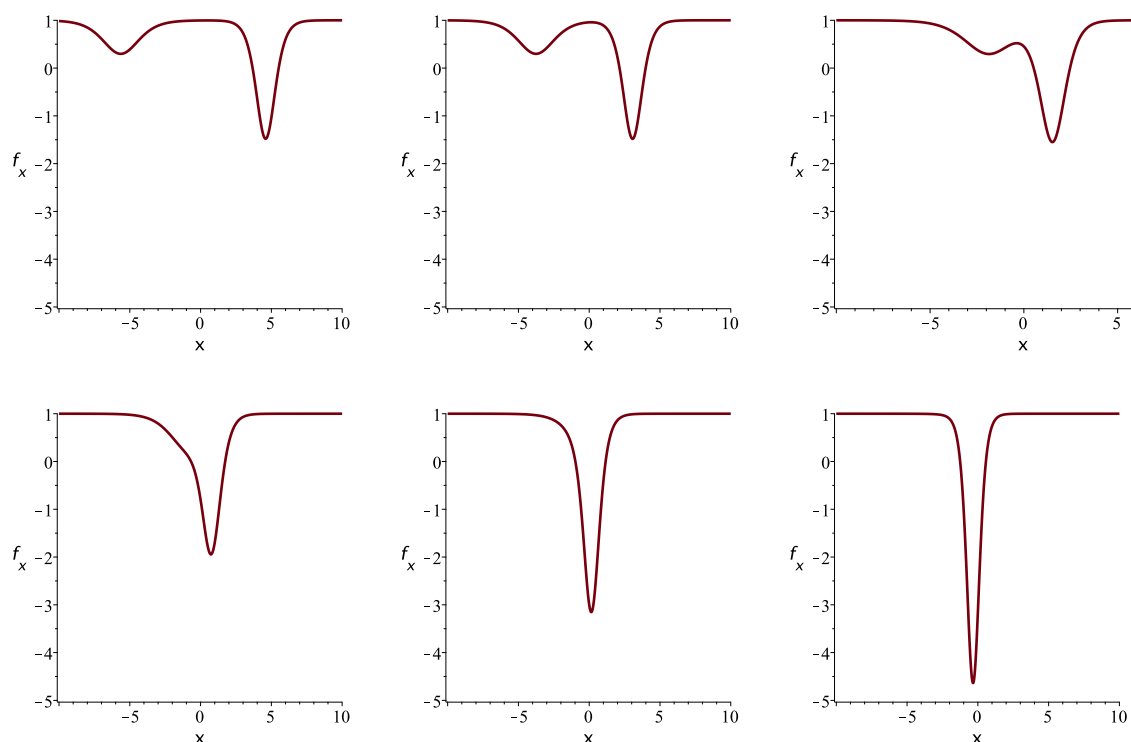


Fig. 1. The merging soliton. Parameter values are  $\beta = 1$  and  $\theta = 1$ , so  $\lambda_1, \lambda_2, \lambda_3 \approx -1.53, -0.35, 1.88$ . The constants  $C_1, C_2, C_3$  are all taken to be 1. Plots of  $f_x$ , with  $f$  given by (3.1), displayed for times  $t = -3, -2, -1, -0.5, 0, 1$ .

Each of the coefficients  $v_i$  is the density for a conservation law. The first few coefficients are given as follows:

$$v_{-1}^3 = 1, \quad v_0 = 0, \quad v_1 = \frac{f_x}{v_{-1}}, \quad v_2 = -\frac{f_{xx}}{v_{-1}^2}, \quad v_3 = \frac{2}{3}f_{xxx}, \quad v_4 = -\frac{f_{xxxx}}{3v_{-1}}.$$

Further terms can be computed using the recurrence relation

$$v_{k+2} = \frac{1}{3v_{-1}^2} \left( 3f_x v_k - \sum_{j=-1}^{k+1} \left( \sum_{i=\max(-1, -j-1)}^{\min(k+1, k-j+1)} v_{k-i-j} v_i v_j \right) - 3 \sum_{i=-1}^{k+1} v_i v_{(k-i)x} - v_{kxx} \right), \quad k \geq 1.$$

So, for example

$$\begin{aligned} v_5 &= \frac{f_{5x} + 3f_x f_{xxx} - 3f_x^3}{9v_{-1}^2}, \\ v_6 &= 2f_x^2 f_{xx} - \frac{2}{3}f_{xx} f_{xxx} - \frac{2}{3}f_x f_{xxxx}, \\ v_7 &= \frac{1}{27v_{-1}} (21f_{xxx}^2 - 72f_x^2 f_{xxx} - 117f_x f_{xx}^2 + 39f_{xx} f_{xxxx} + 21f_x f_{5x} + 9f_x^4 - f_{7x}). \end{aligned}$$

Here  $f_{5x}, f_{6x}, \dots$  denote 5th, 6th etc. derivatives of  $f$  with respect to  $x$ . For each  $i = 1, 2, \dots$   $v_i$  is the density  $F$  of a conservation law  $F_t + G_x = 0$ . For  $i = 1, 2, 3, 4, 6$  the conservation laws are evidently

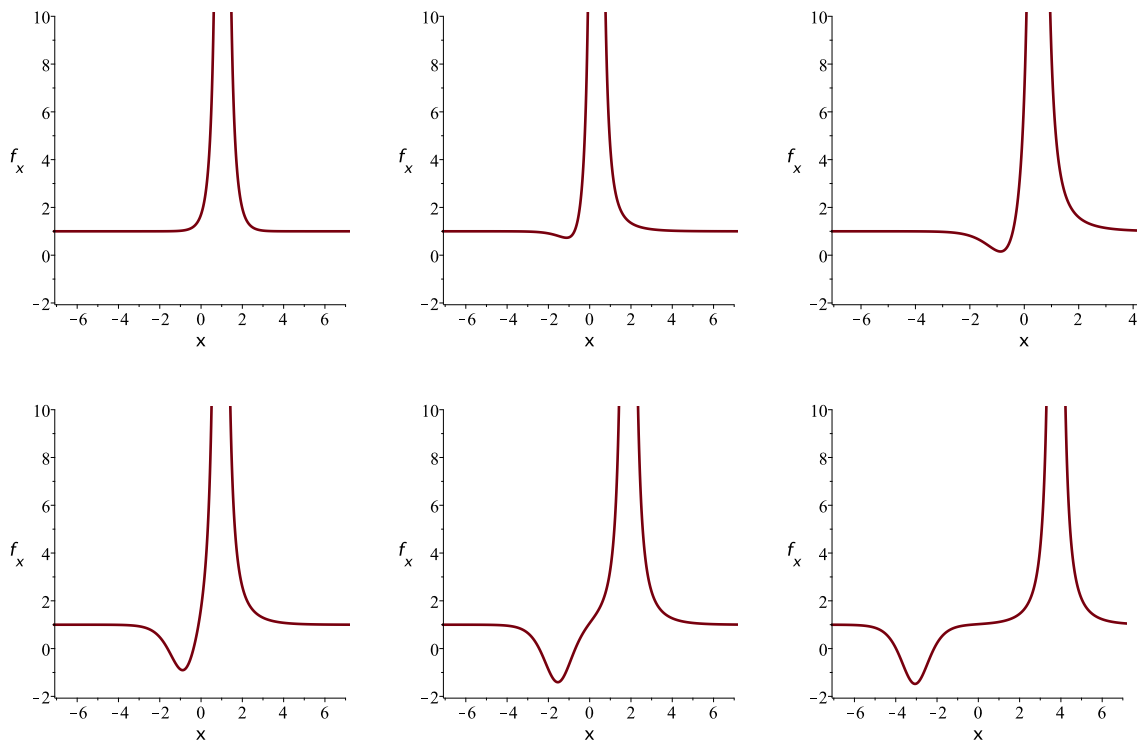


Fig. 2. The expulsion of soliton by a singular soliton. Parameter values are  $\beta = 1$  and  $\theta = -1$ , so  $\lambda_1, \lambda_2, \lambda_3 \approx -1.88, 0.35, 1.53$ , and  $C_1 = C_2 = 1, C_3 = -1$ . Plots of  $f_x$ , displayed for times  $t = -3, 0, 0.2, 0.5, 1, 2$ .

trivial ( $F = H_x, G = -H_t$  for some  $H$ ). For  $i = 5, 7$  we obtain the following conservation laws (after eliminating trivial parts):

$$\begin{aligned}
 F_5 &= f_x f_{xxx} - f_x^3, & F_7 &= 24f_x^2 f_{xxx} - 3f_x^4 + 39f_x f_{xx}^2 + f_{xx} f_{xxx}, \\
 G_5 &= -(3f_x + f_{xx} f_x), & G_7 &= 3(18f_x^2 - 20f_t f_x^3 - f_t f_x f_{xxx} - 15f_{tx} f_x f_{xx} + f_{xxx}).
 \end{aligned}$$

Note that there are 3 possible series for  $v$ , corresponding to the 3 possible choices of  $v_{-1}$ . The dependence of  $v_1, v_2, \dots$  on the choice of  $v_{-1}$  is clear, and can be verified to be consistent with the recursion relation. We denote the three associated asymptotic series by  $v^{(1)}, v^{(2)}, v^{(3)}$ . If we define  $\sigma = v^{(1)} + v^{(2)} + v^{(3)}$ , then  $\sigma$  has asymptotic series  $\sum_{i=1}^{\infty} 3v_{3i}\theta^{-i}$ . However, if we define

$$A = (v^{(2)} - v^{(3)})v_x^{(1)} + (v^{(3)} - v^{(1)})v_x^{(2)} + (v^{(1)} - v^{(2)})v_x^{(3)} - (v^{(1)} - v^{(2)})(v^{(2)} - v^{(3)})(v^{(3)} - v^{(1)}), \quad (4.2)$$

then it can be verified (using (2.4) for each of the functions  $v^{(1)}, v^{(2)}, v^{(3)}$ ) that

$$\sigma = v^{(1)} + v^{(2)} + v^{(3)} = -(\log A)_x. \quad (4.3)$$

It follows that  $v_{3i}$  is a total  $x$  derivative for all  $i$ , and the associated conservation laws are trivial. We also conjecture that  $v_{2i}$  is a total  $x$  derivative for all  $i$ , though do not yet have a direct proof of this.



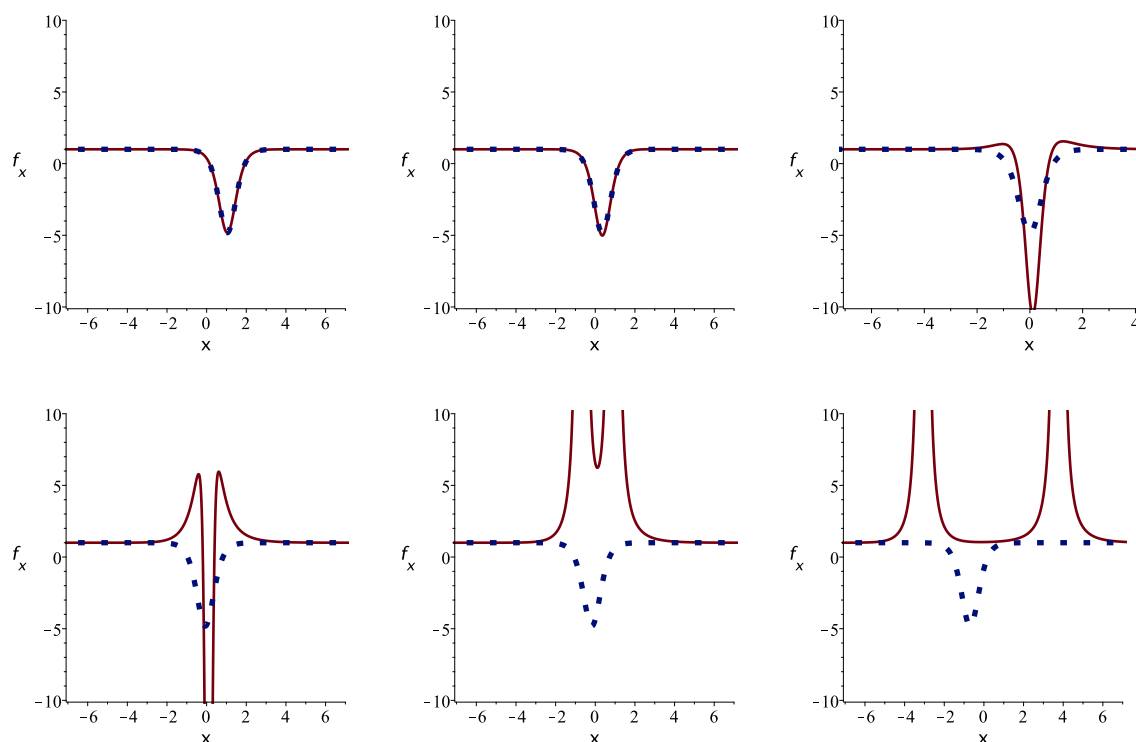


Fig. 3. The splitting of a perturbed soliton into two singular solitons. Parameter values  $\beta = 1$  and  $\theta = -1$ , so  $\lambda_1, \lambda_2, \lambda_3 \approx -1.88, 0.35, 1.53$ , and  $C_1 = C_3 = 1, C_2 = -1$ . Plots of  $f_x$ , displayed for times  $t = -3, -1, 0, 0.2, 0.5, 2$ . The blue, dotted plots show the unperturbed soliton solution, given by the same parameters except  $C_2 = 0$ .

Another possible expansion of  $v$ , this time for small  $|\theta|$ , is

$$v \sim \sum_{i=1}^{\infty} \theta^i w_i. \tag{4.4}$$

The coefficients of this series are obtained by plugging (4.4) into (2.5), and involve derivatives with the respect to  $t$ , which can not be eliminated. We call it the “expansion in the  $t$  direction”, as opposed to the (4.1) which is the “expansion in the  $x$  direction”. The first few coefficients of (4.4) are

$$w_1 = -\frac{1}{3}f_t, \quad w_2 = -\frac{1}{9}f_{tt}, \quad w_3 = \frac{1}{81}(f_t^2 f_{ttx} - f_t^4 - f_t f_{tt} f_{tx} - 3f_{ttt}).$$

For each  $i = 1, 2, \dots$ ,  $w_i$  is the density of a conservation law. The conservation law arising from  $w_2$  is trivial, but from  $w_1, w_3$ , after elimination of some trivial parts, we obtain the following conservation laws:

$$\begin{aligned} \bar{F}_1 &= -3f_t, & \bar{F}_3 &= -3f_t(f_t^3 + 3f_{tt}f_{tx}), \\ \bar{G}_1 &= f_t f_{ttx} - f_t^3 - f_{tx} f_{tt}, & \bar{G}_3 &= 15f_t f_{tt}^2 - 3f_{tt} f_{ttt} + 3f_{ttt} f_{ttx} - 2(f_t^3 - f_t f_{ttx} + f_{tt} f_{tx})^2. \end{aligned}$$

### 5. Symmetries

By the direct computation of the first order generalized symmetries [23] of aN we obtain

$$\begin{aligned} \eta^{(f)} &= 1, \\ \eta^{(x)} &= f_x, \\ \eta^{(t)} &= f_t, \\ \eta^{(s)} &= x f_x - 3 t f_t + f, \end{aligned}$$

corresponding to invariance of aN under translations of  $f, x, t$  and a scaling symmetry.

Following the ideas of [24], and with some inspiration from formulas that appeared in [10, 21], it is possible to find a generating function for symmetries of aN in terms of multiple solutions of the BT (2.4)-(2.5):

$$Q = \frac{1}{A} ((v^{(1)} v_x^{(2)} - v_x^{(1)} v^{(2)}) v^{(3)} - (v^{(1)} - v^{(2)}) (v^{(1)} v^{(2)} v^{(3)} + \theta)). \tag{5.1}$$

Here  $A$  is defined by (4.2). It is straightforward to check directly that  $\eta = Q$  satisfies the linearized aN equation

$$\eta_{xxt} - 3(f_x \eta_t + f_t \eta_x) = 0. \tag{5.2}$$

To generate local symmetries for aN from (5.1) we take  $v^{(1)}, v^{(2)}, v^{(3)}$  to be given by the 3 possible asymptotic expansions of  $v$  of form (4.1), as given in the previous section. Substitution of these asymptotic expansions into  $Q$  and expansion in inverse powers of  $\theta^{1/3}$  gives an infinite hierarchy of symmetries. The first two symmetries in this hierarchy are  $\eta^{(f)}, \eta^{(x)}$ . From the higher order terms we obtain

$$\begin{aligned} \eta^{(5)} &= \frac{1}{15} f_{5x} + f_x^3 - f_x f_{xxx}, \\ \eta^{(7)} &= \frac{1}{21} f_{7x} - 3 f_x^4 + 6 f_x^2 f_{xxx} + 3 f_x f_{xx}^2 - f_x f_{5x} - f_{xx} f_{xxx} - f_{xxx}^2. \end{aligned}$$

Observe that  $\eta^{(5)}$  defines the potential Sawada-Kotera flow (1.7).

We have not succeeded in using the second expansion (4.4) of  $v$  in (5.1) to generate a second hierarchy of symmetries, as the coefficients in this expansion are unique. However, we note that there are other equations for which such an approach is possible. Specifically, for the Camassa-Holm equation

$$m_t + 2u_x m + u m_x = 0, \quad m = u - u_{xx}, \tag{5.3}$$

the system of the BT for the Camassa-Holm equation is

$$s_x = -\frac{s^2}{2\alpha} + \frac{1}{2}(m + \alpha), \tag{5.4}$$

$$s_t = -s^2 \left(1 - \frac{u}{2\alpha}\right) - u_x s + \frac{1}{2}(2\alpha^2 + \alpha u - um). \tag{5.5}$$

One possible expansion of  $s$ , the “expansion in the  $x$  direction”, is in the form

$$s = \sum_{n=1}^{\infty} s_n \alpha^{n/2}. \tag{5.6}$$

There are two versions of this expansion (related by replacing  $\alpha^{1/2}$  by  $-\alpha^{1/2}$ ), and these were used in [25] to construct a hierarchy of symmetries. However there is also an “expansion in the  $t$  direction”:

$$s = \sum_{n=0}^{\infty} s_n \alpha^{1-n}. \tag{5.7}$$

The first few coefficients are

$$s_0^2 = 1, \quad s_1 = \frac{1}{2s_0}(u - s_0 u_x), \quad s_2 = \frac{1}{8s_0}(2u_{tx} - 2s_0 u u_x - u^2 + 2u u_{xx} - 2s_0 u_t + u_x^2).$$

Once again, there are two versions of this expansion, corresponding to the choices  $s_0 = \pm 1$ . A generating symmetry for the Camassa-Holm equation was found in [25] and its form is

$$Q_{ch} = \frac{s^{(1)} + s^{(2)}}{s^{(2)} - s^{(1)}}. \tag{5.8}$$

In [25] the two expansions of  $s$  of form (5.6) were used in this formula to produce a hierarchy of symmetries. A second hierarchy is found by using the two expansions of form (5.7). The first few symmetries we get are

$$\begin{aligned} \eta^{(1)} &= u_x, \\ \eta^{(2)} &= u_t, \\ \eta^{(3)} &= (2u u_{xt} - 2u_{tt} + u u_x^2 - 2u_t u_x - 5u^3 + 4u^2 u_{xx})_x, \\ \eta^{(4)} &= u_x u^3 - 2u_{ttt} - 2u_{xxx} u^3 - 6u_x u_{xx} u^2 - 3u_t (3u^2 - 2u u_{xx} - u_x^2). \end{aligned}$$

This hierarchy of symmetries is different from the one found in [25]. We note that this is a hierarchy of hyperbolic, not evolutionary, symmetries. These symmetries have already been found, for example, in [26]. The existence of hierarchies of such symmetries has been also discussed in [27]. In parallel to the fact that we have been able to find two hierarchies of conservation laws for aN, we expect there to be two hierarchies of symmetries. But finding the second hierarchy remains an open problem.

## 6. Painlevé Property

It is lengthy, but straightforward, to verify that aN has the WTC Painlevé property [28]. There are formal series solutions about the singularity manifold  $\phi(x, t) = 0$  of the form

$$f = \phi^{-1} \sum_{j=0}^{\infty} f_j \phi^j$$

with  $\phi$ ,  $f_1$ ,  $f_6$  arbitrary, except that  $\phi_x$  and  $\phi_t$  must not vanish. As usual, the truncated Painlevé series (see [29]), which has the simple form

$$f = f_1 - \frac{2\phi_x}{\phi},$$

gives the BT, cf. equations (2.3) and (2.7).

### 7. Scaling Reduction

aN has the obvious scaling invariance  $f \rightarrow \lambda f$ ,  $\partial_x \rightarrow \lambda \partial_x$ ,  $\partial_t \rightarrow \lambda^{-3} \partial_t$ . Thus we look for a reduction to an ordinary differential equation by taking

$$f(x, t) = \frac{F(z)}{3x}, \quad z = 3xt^{1/3}.$$

This gives the third order equation

$$F''' = \frac{zF'^2 - FF''}{z^2} - 1. \tag{7.1}$$

This equation has the Painlevé property, with a pole series

$$F = \sum_{j=0}^{\infty} F_j (z - z_0)^{j-1}$$

with  $z_0, F_1, F_6$  arbitrary, except that  $z_0 \neq 0$ . The equation has a first integral

$$E = zF''^2 - F'F''' - \frac{2}{3}F'^3 + \frac{FF''^2}{z} + 2zF' - 3F. \tag{7.2}$$

Note that if  $F$  satisfies the third order equation (7.1) then  $E$  is constant. But in the other direction,  $E$  being constant implies that either  $F$  satisfies the third order equation (7.1) or  $F = C_1 + C_2 z^{3/2}$  where  $C_1, C_2$  are constants. Remarkably, the functions

$$F = \frac{1}{4} \pm \frac{2}{\sqrt{3}} z^{3/2}$$

also satisfy the third order equation (7.1), and illustrate the fact that although (7.1) has the Painlevé property, solutions can have a non-pole singularity at  $z = 0$ . These give rise to the solutions  $f = \frac{1}{12x} \pm 2\sqrt{xt}$  of aN.

By the substitution

$$F(z) = \frac{1}{4} - 9G(w), \quad w = z^{3/2},$$

equation (7.2) is brought to the form

$$w^2 G''(w)^2 = -4(wG'(w) - G(w))G'(w)^2 + \frac{16}{243}(wG'(w) - G(w)) + \frac{4(4E + 3)}{6561}.$$

This is a special case of equation SD-I.b, equation (5.5) in [30], which can be solved, as explained in [30], in terms of either Painlevé III or Painlevé V transcendents.

## 8. Hirota form

Substituting

$$f = -2 \frac{\phi_x}{\phi}$$

in aN we obtain the Hirota bilinear form

$$(D_x^3 D_t - 3) \phi \cdot \phi = 0.$$

See, for example, [31] for notation. This has an unusual form, in that as far as we are aware, most studies of equations in Hirota form assume the form  $P(D_x, D_t, \dots) \phi \cdot \phi = 0$  with  $P(0, 0, \dots) = 0$  (see, for example, [31], equation (17)). However, by substituting

$$\phi = C(t) e^{-\beta x^2/4 - tx/2\beta} \psi,$$

where  $C(t)$  is an arbitrary function, we obtain

$$\left( D_x^3 D_t + \frac{6}{\beta} D_x^2 + 6\beta D_x D_t \right) \psi \cdot \psi = 0$$

Modulo rescalings, this is the Hirota form of the Hirota-Satsuma equation [31–33]. The basic soliton solution is

$$\psi = 1 + e^\eta, \quad \eta = a(x - ct) + b$$

where  $a, b, c$  are constants with

$$a^2 = 3 \left( \beta - \frac{1}{\beta c} \right).$$

As observed in our previous discussion of soliton solutions, the speed  $c$  is restricted by the requirement  $\frac{1}{\beta c} < \beta$ . The two soliton solution is

$$\psi = 1 + e^{\eta_1} + e^{\eta_2} + A e^{\eta_1 + \eta_2}$$

where

$$\begin{aligned} \eta_1 &= a_1(x - c_1 t) + b_1, & a_1^2 &= 3 \left( \beta - \frac{1}{\beta c_1} \right), \\ \eta_2 &= a_2(x - c_2 t) + b_2, & a_2^2 &= 3 \left( \beta - \frac{1}{\beta c_2} \right), \\ A &= \frac{(3c_1 c_2 (c_1 + c_2) \beta^2 - (c_1^2 + 4c_1 c_2 + c_2^2) - \beta c_1 c_2 a_1 a_2 (c_1 + c_2))^2}{(c_1 - c_2)^2 (3\beta^2 c_1 c_2 (c_1 + c_2) + c_1^2 + c_1 c_2 + c_2^2)}. \end{aligned}$$

Here we have written the phase shift factor  $A$  in a form in which it is clear that at least in the case that both  $c_1$  and  $c_2$  are positive,  $A$  is positive, thus guaranteeing a non-singular two soliton solution of aN. Note that for certain values of the parameters  $A$  can either vanish, or be ill-defined, as the

denominator vanishes. It can be checked that this happens if and only if

$$\frac{c_1^3}{1 + 3\beta^2 c_1} = \frac{c_2^3}{1 + 3\beta^2 c_2}$$

In this case the two soliton solution reduces to the “merging soliton” solutions described in section 3. The two soliton solution extends to multisoliton solutions in the usual way for integrable equations of KdV type [31, 32]. So, for example, the three soliton solution is, in the obvious notation,

$$\psi = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} + A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3}.$$

We have not yet succeeded in determining whether there exist nonsingular superpositions of merging solitons and regular solitons.

## 9. Conclusion

In this paper we have researched the aN equation, an integrable nonlinear equation, which is notable for taking a particularly simple form, with just a single, quadratic nonlinear term, and for having relationships with various other significant integrable equations. Among the interesting properties of this equation we would mention the existence of merging solitons, the fact that some of the soliton solutions are unstable, which can be demonstrated very effectively, and the existence of two infinite hierarchies of conserved quantities. Among the matters that we have not been able to resolve fully and which merit further research, we would mention the lack of a superposition principle for the Bäcklund transformation, the need to establish the stability of other solutions, and the need for a fuller exploration of the space of solutions, particularly to determine if there are nonsingular ways to superpose merging solitons. Also, we have only been able to give a single hierarchy of local symmetries so far. We note that the aN equation is not in evolutionary form, which limits its potential application; but its superficial simplicity and rich properties make it notable, nonetheless.

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