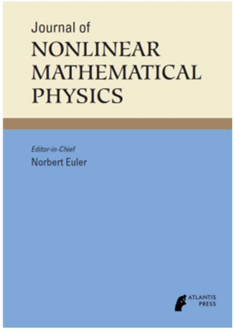

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Two Peculiar Classes of Solvable Systems Featuring 2 Dependent Variables Evolving in Discrete-Time via 2 Nonlinearly-Coupled First-Order Recursion Relations

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In this paper we identify certain peculiar systems of 2 *discrete-time* evolution equations,

$$\tilde{x}_n = F^{(n)}(x_1, x_2), \quad n = 1, 2,$$

which are *algebraically solvable*. Here ℓ is the "discrete-time" *independent* variable taking *integer* values ($\ell =$ 0, 1, 2, ...), $x_n \equiv x_n(\ell)$ are 2 dependent variables, and $\tilde{x}_n \equiv x_n(\ell+1)$ are the corresponding 2 *updated* variables. In a previous paper the 2 functions $F^{(n)}(x_1, x_2)$, $n = 1, 2$, were defined as follows: $F^{(n)}(x_1, x_2) = P_2(x_n, x_{n+1})$, $n = 1, 2 \bmod[2]$, with $P_2(x_1, x_2)$ a specific *second-degree homogeneous polynomial* in the 2 (*indistinguishable*!) dependent variables $x_1(\ell)$ and $x_2(\ell)$. In the present paper we further clarify some aspects of that model and we present its extension to the case when $F^{(n)}(x_1, x_2) = Q_k^{(n)}(x_1, x_2)$, $n = 1, 2$, with $Q_k^{(n)}(x_1, x_2)$ a specific *homogeneous function of arbitrary* (*integer*) *degree* k (hence a *polynomial of degree* k when $k > 0$) in the 2 dependent variables $x_1(\ell)$ and $x_2(\ell)$.

1. Introduction and main results

The results reported in this paper are a nontrivial extension of those reported in [1], to which the interested reader is referred: (i) for a terse overview of an old (see [2,3])—and recently substantially improved (see [4–10])—technique to identify *solvable* dynamical systems in *continuous-time* t; (ii) for an introduction to the extension of that approach to the case of *discrete-time* ℓ (see [11–13]); (iii) for a very terse review of previous results on *analogous solvable discrete-time* models (see [5]). To make the relevance of the present paper immediately clear we report already in this introductory section what we consider its *main* findings.

Notation 1.1. Hereafter $\ell = 0, 1, 2, \ldots$ denotes the *discrete-time independent* variable; the *dependent* variables are $x_n \equiv x_n(\ell)$ (with $n = 1, 2$), and the notation $\tilde{x}_n \equiv x_n(\ell+1)$ indicates the once-updated values of these variables. We shall also use other dependent variables, for instance $y_m \equiv y_m(\ell)$ (with $m = 1, 2$) and then of course likewise $\tilde{y}_m \equiv y_m(\ell+1)$. Variables such as x_n and y_m are generally assumed to be *complex* numbers (this does not exclude that they might in some cases take only *real* values); note that while these quantities generally depend on the *discrete-time* variable ℓ, only occasionally this is *explicitly* indicated. Parameters such as a, b, α, β, γ, B_n, C_j

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(with $n = 1, 2;\ j = 1, 2, 3$) are generally time-independent *complex* numbers, while k, q, r are *real integers*; and *indices* such as n, m, j are of course *positive integers* (the values they may take shall be explicitly indicated or be quite clear from the context). The quantity S denotes an *arbitrarily assigned* sign, $S = \pm$: note that generally the assignment of the sign S shall depend on the *discrete-time* ℓ, $S \equiv S(\ell)$ (and of course it has the same assigned value $S(\ell)$ for each value of ℓ). Finally: the convention is hereafter adopted according to which $\sum_{s=s_-}^{s_+} f(s) = 0$ and $\prod_{s=s_-}^{s_+} f(s) = 1$ whenever $s_+ < s_-$. ■

Remark 1.1. In this paper the term *solvable* generally characterizes systems of *discrete-time* evolution equations the initial-values problem of which is *explicitly solvable by algebraic operations*. ■

The main result of [1] is to provide the *explicit* solution of the initial-values problem for the system of 2 nonlinearly-coupled *discrete-time* evolution equations

$$\tilde{x}_n = -(x_1 + x_2)\left[a(x_1 + x_2) + Sb(x_n - x_{n+1})\right], \quad n = 1, 2 \bmod[2]. \tag{1.1}$$

(Note here the notational changes with respect to [1]: implying $\alpha = 2a$, $\beta = 2b$ and the *explicit* introduction of the *arbitrary* ℓ-dependent sign $S \equiv S(\ell)$, which is indeed implicit in the formulas (2.11b) and (1.2a) of [1].)

Remark 1.2. The characteristic of the *discrete-time* evolution of this model is that, if the sign $S(\ell)$ is *positive*, $S(\ell) = +$, then

$$\tilde{x}_n \equiv x_n(\ell+1) = -\left[x_1(\ell) + x_2(\ell)\right]\left\{a\left[x_1(\ell) + x_2(\ell)\right] + b\left[x_n(\ell) - x_{n+1}(\ell)\right]\right\}, \quad n = 1, 2 \bmod[2]; \tag{1.2}$$

while if $S(\ell) = -$, then

$$\tilde{x}_n \equiv x_n(\ell+1) = -\left[x_1(\ell) + x_2(\ell)\right]\left\{a\left[x_1(\ell) + x_2(\ell)\right] - b\left[x_n(\ell) - x_{n+1}(\ell)\right]\right\}, \quad n = 1, 2 \bmod[2]. \tag{1.3}$$

So it might appear that we are dealing here with a large plurality of *distinct* dynamical systems, as yielded by all the possible (ℓ-dependent!) assignments of the values—positive or negative sign—of $S(\ell)$. But this is not really the case, because it is evident that the different outcomes of these two systems (1.2) and (1.3) is merely to exchange the roles of the variables $x_1(\ell)$ and $x_2(\ell)$; hence the system (1.1) yields a well-defined, *unique* evolution if we consider the two variables $x_1(\ell)$ and $x_2(\ell)$ to identify 2 *indistinguishable* entities, such as the 2 different *zeros* of a generic second-degree polynomial. And it was indeed shown in [1] that the solution of the initial-values problem for the *discrete-time* evolution (1.1)—with the sign $S(\ell)$ being *arbitrarily* assigned for every value of ℓ—is provided by the 2 zeros $x_1(\ell)$ and $x_2(\ell)$ of a specific second-degree polynomial,

$$p_2(z;\ell) = z^2 + y_1(\ell)z + y_2(\ell) = \left[z - x_1(\ell)\right]\left[z - x_2(\ell)\right], \tag{1.4}$$

the 2 coefficients of which, $y_1(\ell)$ and $y_2(\ell)$, are *unambiguously* determined and indeed *explicitly* known in terms of the initial values $x_1(0)$ and $x_2(0)$.

This remark is reported here as an introduction to the somewhat more peculiar phenomenon associated with the more general models considered in the present paper, see below. ■

The *first main* result of the present paper is to provide (in the following **Sections 2** and **3**) the *explicit* solution of the initial-values problem for the following, more general, system of 2 nonlinearly-coupled *discrete-time* evolution equations

$$\tilde{x}_n = d\,(B_1x_1+B_2x_2)^k\,[\alpha g_n\,(B_1x_1+B_2x_2)+\beta(-1)^n S B_{n+1}\,(g_2x_1-g_1x_2)]\,,\;\; n=1,2,\;\text{mod}[2], \quad (1.5)$$

with (above and hereafter)

$$d=(1/2)\left[(B_1)^2C_2+(B_2)^2C_1-B_1B_2C_3\right]^{-1}, \quad (1.6)$$

$$g_1=2B_1C_2-B_2C_3, \quad g_2=2B_2C_1-B_1C_3, \quad (1.7)$$

and with k an *arbitrary* (*time-independent*, *integer*) parameter (of course k must be chosen to be a *positive integer* if one prefers that the right-hand side of these equations be homogeneous *polynomials* of degree $k+1$). The 7 parameters α, β and B_n, C_j (with $n=1,2;\ j=1,2,3$) are *arbitrary* (see **Notation 1.1**).

Remark 1.3. The restriction to *integer* values of the parameter k—and of the analogous exponents q and r, see below and **Notation 1.1**—is to make sure that the right-hand sides of the *main discrete-time* evolution equations we introduce and discuss in this paper are *analytic*, hence *unambiguously* defined, functions. ■

For $k=1$ the system (1.5)–(1.7) can of course be re-written as follows:

$$\tilde{x}_n = a_{n1}(x_1)^2+a_{n2}(x_2)^2+a_{n3}x_1x_2,\;\; n=1,2, \quad (1.8)$$

with

$$a_{nm}=dB_n\left[\alpha g_nB_n+S\beta(-1)^{n+m}B_{n+1}g_{m+1}\right],\;\; n=1,2,\;\text{mod}[2];\;\; m=1,2,\;\text{mod}[2]; \quad (1.9)$$

$$a_{n3}=d\,[2\alpha g_nB_1B_2+S\beta(-1)^nB_{n+1}\,(B_2g_2-B_1g_1)]\,, \quad (1.10)$$

with the parameters d and g_n defined as above (see (1.5)–(1.7)).

Remark 1.4. This case with $k=1$ is sufficiently interesting to deserve this additional remark. Its equations of motion (1.8)–(1.10) are written in terms of the 6 parameters a_{nj} ($n=1,2;\ j=1,2,3$) in terms of the 7 *a priori arbitrary* parameters α, β and B_n, C_j (with $n=1,2;\ j=1,2,3$). But this does *not* imply that these 6 parameters a_{nj} can be *arbitrarily* assigned: the fact that the right-hand sides of the 2 recursions (1.5) feature a *common zero*—they *both* vanish when $B_1x_1+B_2x_2$ vanishes—is easily seen to imply that these 6 parameters are constrained to satisfy (at least!) the following nonlinear relationship:

$$(a_{11}a_{22}-a_{21}a_{12})^2+(a_{13}a_{21}-a_{11}a_{23})\,(a_{13}a_{22}-a_{12}a_{23})=0 \quad (1.11)$$

(see, if need be, **Remark 5.3** of Ref. [8]). ■

For $k=-1$ the system (1.5)–(1.7) can of course be re-written as follows:

$$\tilde{x}_n=\frac{D_{n1}x_1+D_{n2}x_2}{B_1x_1+B_2x_2},\;\; n=1,2, \quad (1.12)$$

$$D_{nm}=d\left[\alpha g_nB_m+S\beta(-1)^{n+m}B_{n+1}g_{m+1}\right],\;\; n=1,2,\;\text{mod}[2];\;\; m=1,2,\;\text{mod}[2], \quad (1.13)$$

again with the parameters d and g_n defined as above in (1.5)–(1.7).

The *second main* result of the present paper is to provide the *explicit* solution of the initial-values problem for the system of 2 nonlinearly-coupled *discrete-time* evolution equations

$$\tilde{x}_n = (2x_1 + x_2)^k \left[(-1)^k a(2x_1 + x_2) + S(-1)^n nb(x_1 - x_2)\right], \quad n = 1, 2. \tag{1.14}$$

Note that in this case—differently from those reported above, see (1.1) (as well as (1.5)–(1.7) with $B_1 = B_2$ and $C_1 = C_2$)—the *discrete-time* evolutions of the 2 variables $x_1(\ell)$ and $x_2(\ell)$ are *essentially different*; hence these dependent variables are *no more* related to each other by just an exchange of their identities. Yet these evolution equations, (1.14), still have a somewhat analogous property to those discussed above: for any given pair of initial data, $x_1(0)$ and $x_2(0)$, only 2 different solutions, say the two pairs $x_1^{(+)}(\ell)$, $x_2^{(+)}(\ell)$ and $x_1^{(-)}(\ell)$, $x_2^{(-)}(\ell)$, emerge, not 2^ℓ as it might instead be inferred due to the indeterminacy of the signs $S(\ell)$ appearing in these evolution equations (1.14) at every step of the *discrete-time* evolution they yield. This is demonstrated by the explicit solution of the initial-values problem for this model, as reported below; but the interested reader may readily understand the origin of this remarkable phenomenon by noting that a simple iteration of (1.14) entails the (double-step) formula

$$x_n(\ell+2) = (-1)^k 3^{k+1} a^k \left[2x_1(\ell) + x_2(\ell)\right]^{k(k+2)} \cdot$$
$$\cdot \left\{a^2 \left[2x_1(\ell) + x_2(\ell)\right] - S(\ell)S(\ell+1)(-1)^n nb^2 \left[x_1(\ell) - x_2(\ell)\right]\right\}, \quad n = 1, 2; \tag{1.15}$$

indeed the quantity $S(\ell)S(\ell+1)$ is again just a sign, i.e. it can only take the 2 values $+1$ or -1, as implied by the very definition of $S(\ell)$, see **Notation 1.1**. Hence this formula clearly shows that—starting from the initial values $x_n(0)$—also at the $\ell = 2$ level (as at the $\ell = 1$ level)—this evolution yields *only two* (*not four*!) alternative values for the pair $x_1(2)$, $x_2(2)$, say $x_1^{(+)}(2)$, $x_2^{(+)}(2)$ (corresponding to $S(0)S(1) = +$) respectively $x_1^{(-)}(2)$, $x_2^{(-)}(2)$ (corresponding to $S(0)S(1) = -$); and this phenomenology prevails—see below—at every subsequent level $\ell > 2$ of the *discrete-time* evolution.

Additional results and proofs—including the *explicit* solutions of the initial-values problems for the 2 systems of 2 *discrete-time* evolution equations (1.5)–(1.7) respectively (1.14)—are provided in **Section 2**. **Section 3** contains some additional developments.

2. Additional results and proofs

The starting point of our treatment in this **Section 2** is the following *algebraically solvable* system of 2 *discrete-time* evolution equations in the 2 dependent variables $y_1 \equiv y_1(\ell)$ and $y_2 \equiv y_2(\ell)$:

$$\tilde{y}_1 = \alpha(y_1)^{1+k}, \quad \tilde{y}_2 = \beta^2 y_2(y_1)^q + \gamma(y_1)^r, \tag{2.1}$$

where the 6 parameters α, β, γ, k, q, r can be *a priori arbitrarily assigned* (see **Notation 1.1**; but see also below for eventual restrictions on these parameters). The fact that the initial-values problem for this system of 2 *discrete-time* evolution equations is *solvable* is demonstrated by exhibiting its solution:

$$y_1(\ell) = \alpha^{\left[(1+k)^\ell - 1\right]/k} \left[y_1(0)\right]^{(1+k)^\ell}, \tag{2.2}$$

$$y_2(\ell) = \beta^{2\ell} \alpha^{(q/k^2)\left[(1+k)^\ell - k\ell - 1\right]} \left[y_1(0)\right]^{(q/k)\left[(1+k)^\ell - 1\right]} Y_2(\ell), \tag{2.3}$$

$$Y_2(\ell) = y_2(0) + \gamma \sum_{s=0}^{\ell-1} \left\{ \beta^{-2(s+1)} \alpha^{u[(1+k)^s-1]/k^2+(q/k)s} \left[y_1(0)\right]^{[u(1+k)^s+q]/k} \right\}, \tag{2.4}$$

$$u \equiv kr - (1+k)q \; . \tag{2.5}$$

The interested reader will verify that this solution is consistent with the initial data $y_1(0)$, $y_2(0)$ and that it does satisfy the system of evolution equations (2.1). Note that the assumption that the 3 parameters k, q, r be *integers* (see **Notation 1.1**) is *necessary and sufficient* to guarantee that all exponents in these formulas are *integers*, thereby excluding any nonanalyticity/indeterminacy in the equations of motion (2.1) and in their solutions (2.2)–(2.5).

For reasons that shall be clear in the following, we are also interested in the solution of the system (2.1) in the particular case when the 2 parameters q and r are related to k as follows:

$$q = 2k, \quad r = 2(1+k). \tag{2.6}$$

Note that this assignment implies $u = 0$ (see (2.5)). In this case the sum in the right-hand side of (2.4) becomes a *geometric* sum, hence it can be performed *explicitly*; therefore in this special case the formulas (2.2)–(2.5) are replaced by the following, *more explicit*, versions:

$$y_1(\ell) = \alpha^{\left[(1+k)^\ell-1\right]/k} \left[y_1(0)\right]^{(1+k)^\ell}, \tag{2.7}$$

$$y_2(\ell) = \beta^{2\ell} \alpha^{2\left[(1+k)^\ell-k\ell-1\right]/k} \left[y_1(0)\right]^{2\left[(1+k)^\ell-1\right]} Y_2(\ell), \tag{2.8}$$

$$Y_2(\ell) = y_2(0) + \gamma\beta^{-2} \left[y_1(0)\right]^2 \left[\frac{(\alpha/\beta)^{2\ell}-1}{(\alpha/\beta)^2-1}\right]. \tag{2.9}$$

Our next task is to identify various systems—satisfied by 2 new dependent variables $x_1 \equiv x_1(\ell)$ and $x_2 \equiv x_2(\ell)$—the solutions of which can be identified via the solution of the system (2.1). To this end we set, to begin with,

$$y_1 = -(x_1 + x_2), \quad y_2 = x_1 x_2, \tag{2.10}$$

which clearly implies that x_1 and x_2 are the 2 zeros of the following second-degree monic polynomial:

$$z^2 + y_1 z + y_2 = (z - x_1)(z - x_2), \tag{2.11}$$

implying

$$(x_n)^2 + y_1 x_n + y_2 = 0, \;\; n = 1, 2, \tag{2.12}$$

$$x_n(\ell) = (1/2) \left\{ -y_1(\ell) + (-1)^n \left\{ \left[y_1(\ell)\right]^2 - 4y_2(\ell) \right\}^{1/2} \right\}, \;\; n = 1, 2. \tag{2.13}$$

Likewise (replacing ℓ with $\ell+1$)

$$(\tilde{x}_n)^2 + \tilde{y}_1 \tilde{x}_n + \tilde{y}_2 = 0, \;\; n = 1, 2, \tag{2.14}$$

$$\tilde{x}_n = (1/2 \left[-\tilde{y}_1 + (-1)^n \tilde{\Delta}_1\right], \quad \left(\tilde{\Delta}_1\right)^2 = (\tilde{y}_1)^2 - 4\tilde{y}_2. \tag{2.15}$$

We then use the evolution equations (2.1) to express, in the right-hand sides of (2.15), $\tilde{y}_1$ and $\tilde{y}_2$ in terms of y_1 and y_2, and then the relations (2.10) to express y_1 and y_2 in terms of x_1 and x_2; thereby getting the following system of *discrete-time* evolution equations for the 2 dependent variables $x_1(\ell)$ and $x_2(\ell)$:

$$\tilde{x}_n = (1/2)\left\{-\alpha\,[-(x_1+x_2)]^{k+1} + (-1)^n \Delta_1\right\}, \quad n = 1, 2, \tag{2.16}$$

$$\Delta_1 = S\left\{\alpha^2\,[-(x_1+x_2)]^{2(k+1)} - 4\beta^2 x_1 x_2\,[-(x_1+x_2)]^q - 4\gamma[-(x_1+x_2)]^r\right\}^{1/2}. \tag{2.17}$$

Remark 2.1. The $\pm$ sign $S \equiv S(\ell)$ in this definition (2.17) of Δ_1 might be considered pleonastic in view of the sign indeterminacy of the square-root in the right-hand side of this formula (2.17). We did put it there as a reminder of the fact that, for every value of the discrete time ℓ, the assignment of the labels 1 or 2 to the *solutions* of the second-degree evolution equation (2.16)–(2.17) is *optional*. Indeed the two variables $x_1(\ell)$ and $x_2(\ell)$—the *discrete-time* evolution of which is identified with the evolution of the 2 *zeros* of the second-degree ℓ-dependent (monic) polynomial (2.11) the 2 *coefficients* $y_1(\ell)$ and $y_2(\ell)$ of which evolve according to the *solvable* system (2.1)—should be considered *indistinguishable*. Note that this implies that this evolution equation is actually *not quite deterministic*; it is only deterministic for the couple of *indistinguishable* dependent variables $x_1 \equiv x_1(\ell)$, $x_2 \equiv x_2(\ell)$: a well-known phenomenon for this kind of evolution equations, as discussed above and in the past—see for instance [13] and **Chapter 7** ("Discrete Time") of the book [5] (in particular **Remark 7.1.2** there). ■

Remark 2.2. Of course when s is an *integer* the power $(-z)^s$ can be replaced by z^s respectively $-z^s$ for s *even* respectively *odd*. ■

The results obtained so far allow to formulate the following

Proposition 2.1. *The solution of the initial-values problem for the system of* discrete-time *evolution equations* (2.16)–(2.17) *is provided—up to the limitations implied by* **Remark 2.1**—*by the* 2 zeros $x_1 \equiv x_1(\ell)$ *and* $x_2 \equiv x_2(\ell)$ *of the polynomial* (2.11) (*see* (2.13)), *with its* coefficients $y_1 \equiv y_1(\ell)$ *and* $y_2 \equiv y_2(\ell)$ *given by the formulas* (2.2)–(2.5) *where of course* (*see* (2.10)) $y_1(0) = -x_1(0) - x_2(0)$ *and* $y_2(0) = x_1(0)x_2(0)$. ■

We believe that the interest—both theoretical and applicative—of the system (2.16)–(2.17) is modest, due to the appearance of a square root in the right-hand side of its equations of motion. Hence our next step is to restrict attention to the values identified by eq. (2.6). Indeed these assignments—beside allowing the more explicit solution of the system of recursions (2.1) characterizing the *discrete-time* evolution of the 2 dependent variables $y_1(\ell)$ and $y_2(\ell)$, see (2.7)–(2.9)—also allow (remarkably!) to get rid of the square-root in the right-hand side of (2.17), provided we moreover make the assignments

$$\gamma = a^2 - b^2, \quad \alpha = 2a, \quad \beta = 2b; \tag{2.18}$$

obtaining thereby just the system of evolution equations (1.5)–(1.7). This allows us to prove a subcase of our *first main* result, in the guise of the following:

Proposition 2.2. *The solution of the initial-values problem for the system of* discrete-time *evolution equations* (1.5)–(1.7)—*with* $B_1 = B_2 = -1$, $C_1 = C_2 = 0$, $C_3 = 1$ (*compare eq.* (3.15) *below with* (2.10))—*is provided by the* 2 zeros $x_1 \equiv x_1(\ell)$ *and* $x_2 \equiv x_2(\ell)$ *of the polynomial* (2.11) (*see* (2.13)), *with its* coefficients $y_1 \equiv y_1(\ell)$ *and* $y_2 \equiv y_2(\ell)$ *given by the formulas* (2.7)–(2.9) (*with the assignments* (2.18) *and* (2.6)), *where of course* (*see* (2.10)) $y_1(0) = -x_1(0) - x_2(0)$ *and* $y_2(0) = x_1(0)x_2(0)$.

Let us again emphasize that, for each value of ℓ, the assignment of the labels 1 *or* 2 *to the two zeros of the polynomial* (2.11) *is* optional. ■

This **Proposition 2.2** corresponds to the special case—with $B_1 = B_2 = -1$, $C_1 = C_2 = 0$, $C_3 = 1$—of the *first main* result reported in **Section 1** (see the paragraph including the eqs. (1.5)–(1.7)); a proof of the *first main* result in the general case with arbitrary parameters B_n and C_j is provided at the end of the paper.

Our next step is to modify the relationship among the variables $x_n(\ell)$ and $y_n(\ell)$ by replacing the *second-degree* (monic) polynomial (2.11) with the following (monic) *third-degree* polynomial:

$$z^3 + y_1 z^2 + y_2 z + y_3 = (z - x_1)^2 (z - x_2), \tag{2.19}$$

where of course now

$$y_1 = -(2x_1 + x_2), \quad y_2 = x_1(x_1 + 2x_2), \quad y_3 = -(x_1)^2 x_2. \tag{2.20}$$

Note that—following [6] and [8]—we are now focussing on a *special* polynomial of degree 3 which features only 2 *zeros*, the *zero* x_1 with multiplicity 2 and the *zero* x_2 with multiplicity 1: this of course implies that these 2 *zeros* are now quite *distinct*, and moreover that now only 2 of the 3 *coefficients* y_1, y_2, y_3 can be arbitrarily assigned, the unassigned one being determined in terms of the other two by the requirement that the 3 equations (2.20) be simultaneously satisfied.

Remark 2.3. It is for instance easy to check that the *coefficient* y_3 of the polynomial (2.19) is given by the following formula in terms of the other 2 *coefficients* y_1 and y_2:

$$y_3 = \left\{-2(y_1)^3 + 9y_1 y_2 + 2S\left[(y_1)^2 - 3y_2\right]^{3/2}\right\}/27. \tag{2.21}$$

Likewise, from the first 2 of the 3 formulas (2.20) one easily obtains the following expressions of the two zeros x_1 and x_2 in terms of the 2 coefficients y_1 and y_2:

$$x_n = (1/2)\left\{-y_1 + S(-1)^n n\left[(y_1)^2 - 3y_2\right]^{1/2}\right\}, \quad n = 1, 2. \tag{2.22}$$

Note the ambiguity in these formulas implied by the indeterminacy of the square-root sign, as evidenced by the presence of the sign S, see **Notation 1.1**. ■

It is now convenient to write again these expressions of the 2 *zeros* $x_n \equiv x_n(\ell)$, but with ℓ replaced by $\ell + 1$:

$$\tilde{x}_n = (1/2)\left\{-\tilde{y}_1 + \tilde{S}(-1)^n n\left[(\tilde{y}_1)^2 - 3\tilde{y}_2\right]^{1/2}\right\}, \quad n = 1, 2. \tag{2.23}$$

Our next step is to then assume again that the two coefficients $\tilde{y}_1 \equiv y_1(\ell + 1)$ and $\tilde{y}_2 \equiv y_2(\ell + 1)$ evolve according to the *solvable discrete-time* system (2.1). By proceeding in close analogy with the previous treatment—i.e., by replacing in the right-hand sides of (2.23) the variables $\tilde{y}_1$ and $\tilde{y}_2$ via the evolution equations (2.1) and then in the right-hand sides of the resulting equations y_1 and y_2 via (2.20)—we thereby obtain the following:

Proposition 2.3. *The solution of the initial-values problem for the following system of* discrete-time *evolution equations*

$$\tilde{x}_n = (1/2)\left\{-\alpha\,[-(2x_1+x_2)]^{k+1} + (-1)^n S\Delta\right\},\ \ n=1,2, \tag{2.24}$$

$$\Delta = \left\{\alpha^2\,[-(2x_1+x_2)]^{2(k+1)} - 3\beta^2 (x_1)^2 x_2\,[-(2x_1+x_2)]^q - 3\gamma[-(2x_1+x_2)]^r\right\}^{1/2}, \tag{2.25}$$

is provided by the 2 *zeros* $x_1 \equiv x_1(\ell)$ *and* $x_2 \equiv x_2(\ell)$ *of the polynomial* (2.19)–(2.20)—*i.e., by the formulas* (2.22)—*with the coefficients* $y_1 \equiv y_1(\ell)$ *and* $y_2 \equiv y_2(\ell)$ *given by the formulas* (2.2)–(2.5) *where of course now* (*see* (2.20)) $y_1(0) = -2x_1(0) - x_2(0)$ *and* $y_2(0) = x_1(0)\,[x_1(0) + 2x_2(0)]$.

Note that this system features the same kind of 2*-fold ambiguity as discussed in the previous* **Section 1** (*see after eq.* (1.14)). ■

However, a "defect" of this *solvable* system is the appearance in the right-hand side of its *discrete-time* equations of motion (2.24)–(2.25) of a square root; but this "defect" can now be eliminated by restricting the parameters q and r to satisfy the condition (2.6)—just the same condition that allows to replace the solution (2.2)–(2.5) with the *more explicit* solution (2.7)–(2.9)—and by moreover replacing the assignments (2.18) with the following assignments

$$\gamma = 3(a^2 - b^2), \quad \alpha = 3a, \quad \beta = 3b. \tag{2.26}$$

It is indeed easily seen that there thereby holds the following:

Proposition 2.4. *The solution of the initial-values problem for the system of* discrete-time *evolution equations* (1.14) *is provided by the* 2 *distinct* zeros $x_1 \equiv x_1(\ell)$ *and* $x_2 \equiv x_2(\ell)$ *of the polynomial* (2.19)–(2.20)—*i.e., by the formulas* (2.22)—*with the* coefficients $y_1 \equiv y_1(\ell)$ *and* $y_2 \equiv y_2(\ell)$ *given by the formulas* (2.7)–(2.9) *with* (2.26) *where of course now* (*see* (2.20)) $y_1(0) = -2x_1(0) - x_2(0)$ *and* $y_2(0) = x_1(0)\,[x_1(0) + 2x_2(0)]$.

Let us again emphasize that, for each value of the discrete-time ℓ*, this prescription yields* 2 *different solutions, say the* 2 *different pairs* $x_1^{(+)}(\ell)$, $x_2^{(+)}(\ell)$ *and* $x_1^{(-)}(\ell)$, $x_2^{(-)}(\ell)$. ■

This **Proposition 2.4** corresponds to the second *main* result reported in **Section 1** (see above the paragraph including eq. (1.14)).

3. Additional developments

An important issue is the possibility to generalize the *algebraically solvable* systems treated in the previous **Section 2**—which feature the 2 *arbitrary* (possibly *complex*) parameters a and b—to more general analogous models involving more free parameters. Following the treatment given in [1], let us outline how this can be done for the system (1.14).

The procedure is to introduce the simple invertible change of dependent variables

$$z_1 = A_{11}x_1 + A_{12}x_2, \quad z_2 = A_{21}x_1 + A_{22}x_2, \tag{3.1}$$

$$x_1 = (A_{22}z_1 - A_{12}z_2)/D, \quad x_2 = (-A_{21}z_1 + A_{11}z_2)/D, \tag{3.2}$$

$$D = A_{11}A_{22} - A_{12}A_{21}, \tag{3.3}$$

where of course the 4 parameters A_{nm} $(n = 1, 2\,;\, m = 1, 2)$ are 4, *a priori arbitrary*, time-independent constants (the restriction to time-independent constants is because we prefer in this paper to focus on *autonomous* systems of *discrete-time* evolutions).

It is then a matter of simple algebra to obtain the—of course *algebraically solvable*—evolution equations satisfied by the dependent variables $z_1 \equiv z_1(\ell)$ and $z_2 \equiv z_2(\ell)$:

$$\tilde{z}_n = D^{-(k+1)} \left[(2A_{22} - A_{21}) z_1 + (A_{11} - 2A_{12}) z_2 \right]^k \cdot \\ \cdot \left[A_{n1} f_1(z_1, z_2; k) + A_{n2} f_2(z_1, z_2; k) \right], \quad n = 1, 2, \tag{3.4}$$

$$f_n(z_1, z_2; k) = (\theta_{k;2,n;1} A_{22} - \theta_{k;1,n;1} A_{21}) z_1 + (\theta_{k;1,n;0} A_{11} + \theta_{k;2,n;0} A_{12}) z_2, \tag{3.5}$$

$$\theta_{k;n_1,n_2;n} = (-1)^k n_1 a + (-1)^n n_2 S b. \tag{3.6}$$

Let us also display these equations in the—possibly more relevant to applicative contexts—special cases with $k = \pm 1$.

For $k = 1$:

$$\tilde{z}_n = a_{n1} (z_1)^2 + a_{n2} (z_2)^2 + a_{n3} z_1 z_2, \quad n = 1, 2, \tag{3.7}$$

$$a_{n1} = D^{-2} \lambda_2 (\eta_1 A_{n1} + \eta_2 A_{n2}), \quad n = 1, 2, \tag{3.8}$$

$$a_{n2} = D^{-2} \lambda_1 (\eta_3 A_{n1} + \eta_4 A_{n2}), \quad n = 1, 2, \tag{3.9}$$

$$a_{n3} = D^{-2} \left[\lambda_1 (A_{n1} \eta_{11} + A_{n2} \eta_{21}) + \lambda_2 (A_{n1} \eta_{12} + A_{n2} \eta_{22}) \right], \quad n = 1, 2, \tag{3.10}$$

$$\lambda_n = (-1)^n (2A_{n2} - A_{n1}), \quad n = 1, 2, \tag{3.11}$$

$$\eta_{n1} = -\theta_{1;1,n;n} A_{21} - \theta_{0;2,n;n+1} A_{22}, \quad n = 1, 2, \tag{3.12}$$

$$\eta_{n2} = -\theta_{0;1,n;n} A_{11} + \theta_{0;2,n;n+1} A_{12}, \quad n = 1, 2. \tag{3.13}$$

For $k = -1$:

$$\tilde{z}_n = \frac{A_{n1} f_1(z_1, z_2; -1) + A_{n2} f_2(z_1, z_2; -1)}{\lambda_1 z_1 + \lambda_2 z_2}, \quad n = 1, 2, \tag{3.14}$$

with $f_n(z_1, z_2; -1)$ and λ_n ($n = 1, 2$) defined as above (see (3.5), (3.6) and (3.11)).

An alternative generalization is based on the replacement of the relations (2.10) and (2.20) and their generalization via (3.1)–(3.3) by the following more general relations:

$$y_1 = B_1 x_1 + B_2 x_2, \quad y_2 = C_1 (x_1)^2 + C_2 (x_2)^2 + C_3 x_1 x_2. \tag{3.15}$$

Note that these relations involve the 5 *a priori arbitrary* parameters B_n and C_j ($n = 1, 2$; $j = 1, 2, 3$), and that they are easily inverted:

$$x_1 = \frac{-g \pm \Gamma}{2f}, \quad x_2 = \frac{y_1 - B_1 x_1}{B_2}, \tag{3.16}$$

$$f = \frac{(B_2)^2 C_1 + (B_1)^2 C_2 - C_3 B_1 B_2}{(B_2)^2}, \quad g = \frac{(B_2 C_3 - 2B_1 C_2) y_1}{(B_2)^2}, \tag{3.17}$$

$$\Gamma^2 = g^2 - 4fh, \quad h = \frac{C_2 (y_1)^2 - (B_2)^2 y_2}{(B_2)^2}. \tag{3.18}$$

Starting from these formulas, and proceeding in close analogy with the treatment provided above—which involve of course the assumption that the quantities x_n and y_n ($n = 1, 2$) are ℓ-dependent while the parameters B_n and C_j ($n = 1, 2$; $j = 1, 2, 3$) are ℓ-independent, and moreover that the quantities y_n evolve according to the *solvable discrete-time* evolution equations (2.1) with (2.6)—one arrives at the equations (1.5)–(1.7). Note again the presence—in the right-hand side of eq. (1.5)—of the sign $S \equiv S(\ell)$, and that these equations—involving no square roots in their right-hand sides—have been obtained thanks to the assignments (2.6) and by moreover setting (in place of (2.18) respectively (2.26))

$$\gamma = \frac{\left[(C_3)^2 - 4C_1C_2\right]\left(\beta^2 - \alpha^2\right)}{4\left[(B_1)^2C_2 + (B_2)^2C_1 - B_1B_2C_3\right]}. \tag{3.19}$$

This concludes the proof of the *first main* result of this paper (see the paragraph including eqs. (1.5)–(1.7) in **Section 1**).

Assigning *solvable* evolutions to y_1 and y_3 or y_2 and y_3 (rather than to y_1 and y_2; in the case of the *third-degree* polynomial (2.19)) are possible further developments, but we postpone the relevant treatments to future papers.

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