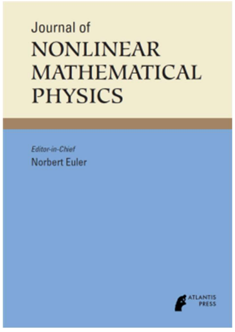

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Further Riccati Differential Equations with Elliptic Coefficients and Meromorphic Solutions

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We exhibit some families of Riccati differential equations in the complex domain having elliptic coefficients and study the problem of understanding the cases where there are no multivalued solutions. We give criteria ensuring that all the solutions to these equations are meromorphic functions defined in the whole complex plane, and highlight some cases where all solutions are, furthermore, doubly periodic.

Keywords: differential equations in the complex domain; Riccati equation; Lamé equation; meromorphic solutions.

2010 Mathematics Subject Classification: 34M05

1. Introduction

In the study of ordinary differential equations in the complex domain, a central problem is to determine and understand those, within a given class, which do not have multivalued solutions. Algebraic differential equations (including equations whose coefficients satisfy a differential equation with algebraic coefficients) give a natural setting for this problem. We will study it for some particular families of Riccati differential equations having elliptic (doubly periodic) coefficients. An instance of such a family is given by the equations

$$y'(t) = y^2(t) + \frac{1}{4}(1-p^2)\wp(t), \tag{1.1}$$

with $p \in \mathbf{Z}$ and $\wp$ a Weierstraß elliptic function (satisfying $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ for some $g_2, g_3 \in \mathbf{C}$). The above equation is closely related to the reduced Chazy XI equation [4, p. 337]

$$w''' = \frac{1-p^2}{2}ww'' + \left(\frac{1-p^2}{2}+6\right)(w')^2 - 3(1-p^2)w^2w' + \frac{3(1-p^2)^2}{8}w^4, \tag{1.2}$$

appearing in Chazy's program [4] to extend Painlevé's analysis of second-order equations [9] to third-order ones. Eq. (1.2) has the first integral $g_3 = 4z^3 - (z')^2$ for $z = w' - \frac{1}{4}(1-p^2)w^2$ and thus, if y is a solution to (1.1) for the particular case $g_2 = 0$, the function w defined by $y = \frac{1}{4}(1-p^2)w$ is a solution to (1.2). As for the instances of equations (1.1) that have only single-valued (univalent) solutions, we have that *if p is odd, all the solutions to (1.1) are meromorphic functions defined in the whole complex plane*. This follows from the fact that if y is a solution to (1.1) and $y = -u'/u$, then u

is a solution to the linear Lamé equation $u''(t) - n(n+1)\wp(t)u(t) = 0$ for $n = \frac{1}{2}(p-1)$, which is known to have meromorphic solutions when n is a positive integer [6, §15.62].

As observed in [7], there are not many concrete examples of Riccati equations having strictly meromorphic elliptic coefficients and such that all of their solutions are meromorphic functions defined in the whole plane (in particular, without multivalued solutions). Our aim is to exhibit some explicit two- and three-parameter families of equations and to give, for each family, conditions guaranteeing that all the solutions of a given equation are meromorphic.

The families of equations that we consider are:

$$y' = y^2 + \frac{1}{4}(1-p^2)\wp - \frac{1}{4}(1-q^2)\frac{g_3}{\wp^2}, \tag{I}$$

$$y' = y^2 - \frac{g_3}{16}(3-2p^2-2r^2+q^2)\frac{1}{\wp^2} + \frac{\sqrt{-g_3}}{8}(p^2-r^2)\frac{\wp'}{\wp^2} + \frac{1}{16}(1-q^2)\left(\frac{\wp'}{\wp}\right)^2, \tag{II}$$

$$y' = y^2 + \frac{g_3^2}{4}(1-q^2)\frac{1}{(\wp\wp')^2} + \frac{g_3}{4}(4+4q^2+r^2-9p^2)\frac{\wp}{(\wp')^2} + (1-r^2)\left(\frac{\wp^2}{\wp'}\right)^2, \tag{III}$$

$$y' = y^2 + \frac{1}{16}(1-p^2)[\mathsf{cn}(t)+i\mathsf{dn}(t)]^2 + \frac{1}{16}(1-q^2)[\mathsf{cn}(t)-i\mathsf{dn}(t)]^2, \tag{IV}$$

$$y' = y^2 + \frac{1}{8}(q^2+r^2-2)\mathsf{sn}^2(t) + \frac{1}{8}(q^2-r^2)i\mathsf{sn}'(t) + \frac{1}{4}(1-p^2)\mathsf{sn}^{-2}(t). \tag{V}$$

In equations I–III, $\wp$ is the Weierstraß elliptic function satisfying $(\wp')^2 = 4\wp^3 - g_3$, for $g_3 \in \mathbf{C}\setminus\{0\}$ (the actual value of g_3 is irrelevant); in equations IV–V, the coefficients are Jacobi elliptic functions with modulus $k^2 = -1$. In all of them, p, q and r are integers. Our results may be summarized as follows:

Theorem 1.1. *For the above equations, the following conditions guarantee that all the solutions are meromorphic:*

- *for* (I), $6 \nmid p$ *and* $3 \nmid q$*;*
- *for* (II), *in the special cases*
  - — $p^2 = r^2$, $3 \nmid r$ *and* $6 \nmid q$*;*
  - — $q^2 = p^2$, $3 \nmid p$ *and* $6 \nmid r$*;*
  - — $r^2 = q^2$, $3 \nmid q$ *and* $6 \nmid p$*;*

  in general, $3 \nmid p$, $3 \nmid q$ *and* $3 \nmid r$*;*
- *for* (III), $6 \nmid r$, $3 \nmid q$ *and* $2 \nmid p$*;*
- *for* (IV), *in the special case* $p^2 = q^2$, $4 \nmid q$*; in general,* p *and* q *are both odd;*
- *for* (V), *in the special case* $q^2 = r^2$, r *is odd and* $4 \nmid p$*; in general,* p, q *and* r *are all odd.*

When the previous conditions are satisfied, all the solutions are doubly periodic (but not necessarily with the same periods as the coefficients)

- *in* (I), *when* p *is even;*
- *in* (II), *when* $p+q+r$ *is even;*
- *in* (III), *when* r *is even;*
- *in* (V), *in the special case* $q^2 = r^2$, *when* p *is even.*

In [7], Ishizaki, Laine, Shimomura and Tohge studied Eq. (1.1) in the special case $g_2 = 0$, which is equation I for $q = 1$. They proved that if k is not divisible by 6 then all solutions are meromorphic (a fact already observed by Chazy [4, §11, p. 344]), established that there are always doubly periodic solutions, and proved that that if k is even, all the solutions are doubly periodic. They also exhibit explicit doubly periodic solutions in many cases when k is odd. In [8], they obtained similar results for the case $g_3 = 0$ of Eq. (1.1), which is equation V for $q = 1$ and $r = 1$ (the Weierstraß elliptic function $\wp$ such that $\wp' = 4\wp^3 - 4\wp$ and $\mathsf{sn}^{-2}(t)$ are the same function). Our results partially extend theirs to the larger families.

The families of Riccati equations here presented appear in our study of quadratic homogeneous systems of differential equations in three variables, of which we hope to soon give an account.

Many of the ideas for the proofs already appear in Chazy's analysis of Eq. (1.2); they may also be found, for instance, in [7]. We present them in Section 3, after recalling some facts about Riccati equations in Section 2. Theorem 1.1 will be proved in the last two sections.

2. Riccati equations

For the classical theory of Riccati differential equations in the complex domain, we refer the reader to Ince's and Hille's treatises [6, §12.51], [5, Chapter 4]. For a more geometric approach to the subject, including the birational point of view, we refer the reader to the first section in Chapter 4 of Brunella's text [3].

2.1. *Generalities*

Riccati equations are ordinary differential equations of the form

$$y' = A(t)y^2 + B(t)y + C(t), \tag{2.1}$$

where A, B and C are meromorphic functions defined in some domain $U \subset \mathbf{C}$. These equations may be compactified in the direction of the independent variable by adding a point $\{y = \infty\}$ for each value of t: in the variable $z = 1/y$, the equation reads $z' = -Cz^2 - Bz - A$, which is again a Riccati equation, holomorphic at the values of t where the original equation is holomorphic. The equation is thus naturally defined in $U \times \mathbf{CP}^1$. This implies that for the initial condition y_0, going from time t_0 to time t_1 following a solution to the equation (say, along a path $\gamma : [0,1] \to U$) is given by a projective (Möbius, fractional linear) transformation $y_0 \mapsto (ay_0 + b)/(cy_0 + d)$. Locally, the general solution to a Riccati equation may be written as

$$y(t) = \frac{a(t)y_0 + b(t)}{c(t)y_0 + d(t)}$$

for some functions a, b, c, d such that $ad - bc \equiv 1$.

If $U^* \subset U$ is the domain where the coefficients of the equation are holomorphic, we have a *monodromy* representation $\mu : \pi_1(U^*) \to \mathrm{PSL}(2,\mathbf{C})$ into the group of projective transformations. The global fixed points of the action of monodromy of a Riccati equation on $\mathbf{CP}^1$ are in correspondence with the solutions of the equation that do not present multivaluedness. The absence of multivalued solutions is equivalent to the triviality of the monodromy. Also, if there are three single-valued solutions, every solution is single-valued (for the only element of $\mathrm{PSL}(2,\mathbf{C})$ having three fixed points is the identity). We may also consider the *lifted* monodromy $\widetilde{\mu} : \pi_1(U^*) \to \mathrm{SL}(2,\mathbf{C})$, obtained by lifting paths in $\mathrm{PSL}(2,\mathbf{C})$ to $\mathrm{SL}(2,\mathbf{C})$.

2.2. *A Poincaré-Dulac normal form*

A Riccati equation of the form (2.1) is said to be *nondegenerate* at $t = 0$ if its coefficients A, B, C have at most simple poles at 0 and may be written as $ty' = P(t,y)$ with P a quadratic polynomial in y holomorphic in t. Such an equation is said to have *simple singularities at* $t = 0$ if the quadratic polynomial $P(0,y)$ has two different roots in $\mathbf{CP}^1$. We have a local (in time) and global (in space) normal form for such Riccati equations [3, Chapter 4, Section 1]:

Proposition 2.1. *For the nondegenerate Riccati equation $ty' = P(t,y)$ with simple singularities at $t = 0$, there exists $S(t) \in \mathrm{PSL}(2,\mathbf{C})$, defined in a neighborhood of $t = 0$, such that in the coordinate $z = S(t)y$, the equation is either $tz' = kz$ for some $k \in \mathbf{C}\setminus\{0\}$ or $tz' = kz + ct^k$, for some $k \in \mathbf{Z}$, $k > 0$, and some $c \in \mathbf{C}$.*

We include here a proof in the aim of making this article slightly more self-contained.

Proof. Up to a constant projective transformation, suppose that the roots of $P(0,y)$ are 0 and ∞. Let $k = \partial P/\partial y|_{(0,0)}$ and notice that, by the simplicity of the singularities, $k \neq 0$. Suppose that k is not a negative integer (otherwise consider the variable $1/y$ in which k is replaced by $-k$). In the variable $w = 1/y$ the equation reads $tw' = w^2P(t,w^{-1}) = Q(t,w)$. We have that $\partial Q/\partial w|_{(0,0)} = -k$. Since $-k$ is not a positive integer, by Briot and Bouquet's theorem [6, §12.6], there is a local solution $w_0(t)$ to this equation with $w_0(0) = 0$. In the variable $w - w_0(t)$ we have that the constant 0 is a solution. Thus, in the coordinate $y = (w - w_0(t))^{-1}$, the equation reads $ty' = B(t)y + C(t)$ for some holomorphic functions B and C. Up to replacing y by fy for f a function such that $tf' = (B(0) - B(t))f$, we may suppose that $B \equiv k$. In the variable $y = z - h(t)$, $h(0) = 0$, the equation reads $tz' = kz + (th' - kh + C)$. If $h = \sum_i h_i t^i$ and $C = \sum_i c_i t^i$, it becomes

$$th' - kh + C = \sum_i ([i-k]h_i + c_i)t^i.$$

By conveniently choosing h_i we obtain the desired result. □

In the normal form, the equations are easily integrated. For the first case, the equation $tz' = kz$, the solutions are given by $z_0 t^k$ and are not meromorphic in general (except those corresponding to $z_0 = 0$ and $z_0 = \infty$); the monodromy is $z \mapsto e^{2i\pi k}z$. Thus, *if all the solutions in the neighborhood of $t = 0$ are meromorphic, k is an integer*. For the second case, the equation $wz' = kz + ct^k$, the solutions are $z(t) = (c\log(t) + c_0)z^k$, and are not meromorphic at $t = 0$ unless $c = 0$. If $c \neq 0$, we do not have formal solutions vanishing at $t = 0$; actually, not even formal solutions up to order k exist.

2.3. *Transformations of Riccati equations*

There are some natural transformations between Riccati equations which leave the independent variable unchanged. The first ones, which we may call *holomorphic*, are given by time-dependent projective transformations: for a solution $y(t)$ of the Riccati equation (2.1) defined in a neighborhood of $t = 0$, if a, b, c and d are holomorphic functions such that $ad - bc \equiv 1$, $\dfrac{a(t)y(t) + b(t)}{c(t)y(t) + d(t)}$ is a solution to a Riccati equation which is independent of the chosen solution y [5, Chapter 5]. Other ones are given by *elementary transformations* (called *flips* by Brunella), the transformation

$$y \mapsto y/t \tag{2.2}$$

and all those which are conjugate to it by a holomorphic one. (See [3, Chapter 4] for an intrinsic description in terms of the birational geometry of surfaces.) Notice that, for fixed t, (2.2) is a projective transformation for $t \neq 0$ but not for $t = 0$. Notice also that the elementary transformation obtained by conjugating it by the projective transformation $y \mapsto 1/y$ is its inverse. Elementary transformations preserve the class of Riccati equations: for instance, under (2.2), for $z = y/t$, Eq. (2.1) is transformed into $z' = (tA)z^2 + (B - 1/t)z + C/t$ (an elementary transformation may create or destroy singularities of the equation).

Let us describe the effect of an elementary transformation upon the lifted monodromy of a Riccati equation (holomorphic transformations do not change neither the local monodromy nor its lift). In a disk Δ^* punctured at $t = 0$ consider a Riccati equation of the form (2.1). The general solution of the Riccati equation may be given by $y(t) = \dfrac{a(t)y_0 + b(t)}{c(t)y_0 + d(t)}$ for $N(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$, a multivalued function in Δ^* taking values in $\mathrm{SL}(2, \mathbf{C})$. Let $\gamma\colon [0,1] \to \Delta^*$ be a closed path around $t = 0$ starting at t_0. The lifted monodromy M around γ is given by $N|_{\gamma(1)}(N|_{\gamma(0)})^{-1}$. After the elementary transformation (2.2), the solutions are given by $z(t) = \left(\dfrac{1}{t}\right) \dfrac{a(t)y_0 + b(t)}{c(t)y_0 + d(t)}$, which corresponds to the path $\widehat{N}(t) = \sigma(t)N(t)$ for $\sigma(t) = \begin{pmatrix} t^{-1/2} & 0 \\ 0 & t^{1/2} \end{pmatrix}$. After the elementary transformation, the lift of the monodromy is given by

$$\widehat{N}|_{\gamma(1)}(\widehat{N}|_{\gamma(0)})^{-1} = \sigma|_{\gamma(1)}N|_{\gamma(1)}(\sigma|_{\gamma(0)}N|_{\gamma(0)})^{-1} = \sigma|_{\gamma(1)}M(\sigma|_{\gamma(0)})^{-1}.$$

Since $\sigma|_{\gamma(1)} = -\sigma|_{\gamma(0)}$, this monodromy is conjugate to $-M$, which has the same image than M in $\mathrm{PSL}(2, \mathbf{C})$: the monodromy is unchanged, but its lift is not.

3. Integrability criteria

Theorem 1.1 will be proved in the next two Sections (in Section 4 for equations I–III; in Section 5 for equations IV–V). All of the equations under consideration are of the form

$$y' = y^2 + E(t), \tag{3.1}$$

with E an elliptic function all of whose singularities are double poles. We will say that E has *index* k at $t = t_0$ if $E = \frac{1}{4}(1 - k^2)(t - t_0)^{-2} + \cdots$. We will present two criteria: Lemma 3.1, which will prove the first part of Theorem 1.1, and Theorem 3.1, which will prove the second one.

3.1. *A criterion ensuring the existence of meromorphic solutions*

We have the following result, present, in one way or another, in most approaches to the subject.

Lemma 3.1. *If ρ is a primitive n-th root of unity and $h(\rho t) = \rho^{n-2}h(t)$, then if k is an integer that does not divide n, all the solutions to*

$$y' = y^2 + \frac{1}{4}(1 - k^2)h(t), \quad h(t) = t^{-2} + \cdots \tag{3.2}$$

in the neighborhood of $t = 0$ are meromorphic.

Proof. In the variable

$$z = ty + \frac{1}{2}(1-k), \tag{3.3}$$

obtained after composing an elementary and a holomorphic transformation,

$$tz' = w^2 + kw + \frac{1}{4}(1-k^2)(t^2h-1). \tag{3.4}$$

Let $f(t) = t^2h(t) - 1$. Since $f(\rho t) = f(t)$, there exists a function F such that $f(t) = F(t^n)$. Since $f(0) = 0$, $F(0) = 0$. At $t = 0$ the above equation is nondegenerate and has simple singularities. According to the results in Section 2.2, if the equation has meromorphic solutions, up to a change of coordinates, the equation is of the form $tz' = kz + ct^k$, $k \in \mathbf{Z}$, $k > 0$. In order to prove that $c = 0$ (which implies that all the solutions are meromorphic), it is sufficient to give a local solution to Eq. (3.4) vanishing at 0. Set $\tau = t^n$. From Eq. (3.4),

$$n\tau\frac{dz}{d\tau} = z^2 + kz + \frac{1}{4}F(\tau).$$

According to Briot and Bouquet's theorem [6, §12.6], since $F(\tau)$ is a holomorphic function vanishing at 0 and, by hypothesis, k/n is not a positive integer, there is a local holomorphic solution to the above equation vanishing at 0, from which one can obtain a local holomorphic solution for (3.4) vanishing at 0. □

3.2. *Klein's four-group and periodic solutions*

The following result will allow us to guarantee that some of our equations have exclusively periodic solutions.

Theorem 3.1. *Suppose that in Eq.* (2.1) *all solutions are meromorphic, and that all the singular points are of the form* (3.2) *for integral indices. If there is an odd number of singularities with even indices in a fundamental domain of E then all the solutions are doubly periodic. In such a case there are three solutions with period lattices* Λ' *such that* $\Lambda \subset \Lambda' \subset 2\Lambda$, $[\Lambda' : \Lambda] = 2$ (*one for each of the three lattices* Λ' *satisfying these conditions*)*, and all the other solutions have* 2Λ *as their period lattice.*

This result appears in [7, Theorem 3.1] and, with the present generality, in [10, Theorem 2.2] (however, these only mention two of the three special solutions).

Proof. If we start with an equation of the form (3.2) with $k \in \mathbf{Z}$, $k > 1$ and we assume that all the solutions are meromorphic in a neighborhood of $t = 0$, it takes $k+1$ elementary transformations to transform the equation into the nonsingular one $y' = 0$: the elementary transformation (3.3), and then k more elementary transformations—under (2.2), equation $ty' = ky$ becomes $ty' = (k-1)y$. In the nonsingular situation ($k = 0$) the lifted monodromy is the identity $\mathbf{I}$. Thus, as discussed in Section 2.3, the lifted monodromy for the original equation around $t = 0$ is $(-1)^{k+1}\mathbf{I}$.

Suppose that, in Eq. (2.1), all the singular points are of the form (3.2) for integral indices and that all solutions are meromorphic. The lifted monodromy around each one of the fixed points is either $\mathbf{I}$ or $-\mathbf{I}$ according to the parity of the index. In all cases, it belongs to the center of $\mathrm{SL}(2,\mathbf{C})$. Let γ_1 and γ_2 be generators of $\pi_1(\mathbf{C}/\Lambda)$ and let M_1 and M_2 in $\mathrm{SL}(2,\mathbf{C})$ be the corresponding lifted monodromies. If there are p singularities with even indices in $\mathbf{C}/\Lambda$ then $[M_1, M_2] = (-1)^p\mathbf{I}$.

It is not difficult to classify the pairs of commuting elements of $\mathrm{PSL}(2,\mathbf{C})$. They

- are simultaneously conjugate to elements of the form $z \mapsto \alpha z$;
- are simultaneously conjugate to elements of the form $z \mapsto z+\tau$; or
- generate a group conjugate to *Klein's four-group*

$$V = \left\{ z \mapsto z,\ z \mapsto -z,\ z \mapsto \frac{1}{z},\ z \mapsto -\frac{1}{z} \right\}.$$

In particular, two commuting elements such that the commutator of (any) two of their lifts to $\mathrm{SL}(2,\mathbf{C})$ is $-\mathbf{I}$ generate a group conjugate to Klein's four-group (see also [7, Lemma 3.2]). In particular, this group is finite. All the solutions are periodic with respect to the sublattice of Λ associated to the kernel of the monodromy. The three special solutions correspond to the three orbits (for the action of this group on $\mathbf{CP}^1$) having nontrivial stabilizer. □

Since all pairs of commuting elements in $\mathrm{PSL}(2,\mathbf{C})$ generate a group having at least one finite orbit in $\mathbf{CP}^1$, *if all the solutions to a Riccati equation with elliptic coefficients are meromorphic, there is at least one doubly periodic solution.*

4. The equianharmonic cases

We will now prove Theorem 1.1 for equations I–III. The coefficients of these equations belong to the function field generated by the Weierstraß elliptic function $\wp$ such that

$$(\wp')^2 = 4\wp^3 - g_3, \tag{4.1}$$

and its derivative $\wp'$. Their module of periods Λ is invariant under multiplication by a primitive third root of unity ω (is *equianharmonic*). For the standard facts about Weierstraß elliptic functions that we will use we refer the reader to [1, Chapter 7]. The meromorphic maps that $\wp$ and $\wp'$ induce from $\mathbf{C}/\Lambda$ to $\mathbf{CP}^1$ have, respectively, degrees two and three. The meromorphic functions E we will consider have at most double poles at the points where $\wp$ and $\wp'$ have either zeros or poles, and are holomorphic at all other points.

Poles of $\wp$, $\wp'$. The origin is the only pole of $\wp$ and the only pole of $\wp'$ in $\mathbf{C}/\Lambda$, $\wp(t) = t^{-2} + \cdots$ and $\wp'(t) = -2t^{-3} + \cdots$. We have that $\wp$ is even, $\wp'$ is odd, and

$$\wp(\omega t) = \omega \wp(t), \quad \wp'(\omega t) = \wp'(t). \tag{4.2}$$

Zeros of $\wp$. There are two simple ones in $\mathbf{C}/\Lambda$. Let θ be such that $\wp(\theta) = 0$. By (4.2), $\wp(\omega\theta) = 0$, and since $\omega^3 = 1$, the two zeros of $\wp$ must be fixed under multiplication by ω and are thus located, when Λ is generated by 1 and ω, at the thirds-of-a-period $\theta = \frac{1}{3}\omega + \frac{2}{3}$ and $-\theta$. From (4.1), $(\wp'(\pm\theta))^2 = -g_3$. Since the sum of the residues of $\wp^{-1}(t)dt$ in $\mathbf{C}/\Lambda$ is zero, the values of $\wp'$ at θ and $-\theta$ have opposite signs. From the addition formula for Weierstraß functions,

$$\wp(\pm\theta + t) = \frac{(\wp'(t) - \wp'(\pm\theta))^2}{4\wp(t)^2} - \wp(t),$$

and from this and (4.2),

$$\wp(\pm\theta + \omega t) = \omega \wp(\pm\theta + t), \quad \wp'(\pm\theta + \omega t) = \wp'(\pm\theta + t). \tag{4.3}$$

Table 1. Coefficients of the leading term of the Taylor developments at the double poles.

	$\wp$	$\wp^{-2}$	$\wp^{-2}\wp'$	$\wp^{-2}\wp'^2$	$\wp^{-2}\wp'^{-2}$	$\wp\wp'^{-2}$	$\wp^4\wp'^{-2}$
poles of $\wp, \wp'$	1	$\times$	$\times$	4	$\times$	$\times$	$\frac{1}{4}$
zeros of $\wp$	$\times$	$-g_3^{-1}$	$\pm i g_3^{-1/2}$	1	g_3^{-2}	$\times$	$\times$
zeros of $\wp'$	$\times$	$\times$	$\times$	$\times$	$\frac{4}{9}g_3^{-2}$	$\frac{1}{9}g_3^{-1}$	$\frac{1}{36}$

Zeros of $\wp'$. There are three simple ones in $\mathbf{C}/\Lambda$, located at the three half-periods η_i. From the addition formula for Weierstraß functions,

$$\wp(\eta_i + t) = \frac{1}{4}\left(\frac{\wp'(t)}{\wp(t) - e_i}\right)^2 - \wp(t) - e_i,$$

where $e_i = \wp(\eta_i)$. Thus, since $\wp$ is even and $\wp'$ is odd,

$$\wp(\eta_i - t) = \wp(\eta_i + t), \;\; \wp'(\eta_i - t) = -\wp(\eta_i + t). \tag{4.4}$$

Table 1 gives the leading term of the Taylor development at the double poles of the summands of E. All the data in it follows from the previous discussion except the one concerning the zeros of $\wp'$, which follows from $\wp^3(\eta_i) = \frac{1}{4}g_3$, obtained from (4.1) and $\wp'' = 6\wp^2$, this last equality obtained by derivating (4.1).

4.1. *Equation* I

It is Eq. (3.1) for

$$E = \frac{1}{4}(1-p^2)\wp - \frac{1}{4}(1-q^2)\frac{g_3}{\wp^2}.$$

It has three poles in $\mathbf{C}/\Lambda$, all of them double, one at the pole of $\wp$ and one at each zero of $\wp$. At 0, pole of $\wp$, $E(t) = \frac{1}{4}(1-p^2)t^{-2} + \cdots$. For ρ a primitive sixth root of unity (say $\rho = -\omega$), $E(\rho t) = \rho^4 E(t)$ and thus, from Lemma 3.1, in the neighborhood of the poles of $\wp$, all the solutions to the Riccati equation are meromorphic if $6 \nmid p$. At the two zeros of $\wp$, $E = \frac{1}{4}(1-q^2)(t \mp \theta)^{-2} + \cdots$. By (4.3), $E(\pm\theta + \omega t) = \omega E(\pm\theta + t)$. Hence, by Lemma 3.1, in the neighborhood of these zeros, the solutions of the Riccati equation are meromorphic whenever $3 \nmid q$.

The hypothesis of Theorem 3.1 are fulfilled when p is even.

4.2. *Equation* II

It is Eq. (3.1) for

$$E = -\frac{g_3}{16}(3 - 2p^2 - 2r^2 + q^2)\frac{1}{\wp^2} + \frac{\sqrt{-g_3}}{8}(p^2 - r^2)\frac{\wp'}{\wp^2} + \frac{1}{16}(1-q^2)\left(\frac{\wp'}{\wp}\right)^2,$$

which has three poles in $\mathbf{C}/\Lambda$, all double, one at the pole of $\wp$ and one at each one of its zeros. At 0, pole of $\wp$, $E(t) = \frac{1}{4}(1-q^2)t^{-2} + \cdots$ and $E(\omega t) = \omega E(t)$, so by Lemma 3.1 the solutions are meromorphic in the neighborhood of $t = 0$ if $3 \nmid q$. For the two zeros θ_i of $\wp$, $E(t) = \frac{1}{4}(1-r^2)(t -$

$\theta_1)^{-2}+\cdots$ and $E(t)=\frac{1}{4}(1-p^2)(t-\theta_2)^{-2}+\cdots$. From (4.3), since $E(\theta_i+\omega t)=\omega E(\theta_i+t)$, by Lemma 3.1, all the solutions are meromorphic if $3\nmid p$ and $3\nmid r$.

The hypothesis of Theorem 3.1 are fulfilled when $p+q+r$ is even (when we have an odd number of even indices).

4.2.1. *The special case $p^2=r^2$*

In this case we further have that $E(\rho t)=\rho^4E(t)$; the solutions are meromorphic in the neighborhood of $t=0$ if $6\nmid q$.

4.2.2. *Symmetries and the other special cases*

The coefficient E is symmetric in p and r, which can be exchanged via the involution $t\mapsto -t$. The order three transformation $t\mapsto t+\theta$ induces a cyclic permutation of the parameters of E: q is replaced by r, r by p and p by q, but the equation remains otherwise unchanged. This can be checked directly using the relation

$$\wp(t+\theta)=\frac{[\wp'(t)-\wp'(\theta)]^2}{4\wp^2(t)}-\wp(t).$$

In this way the parameter space is symmetric under the full group of permutations of the three variables (this implies Theorem 1.1 for the other two special cases).

4.3. *Equation* III

It corresponds to Eq. (3.1) for

$$E=\frac{g_3^2}{4}(1-q^2)\frac{1}{(\wp\wp')^2}+\frac{g_3}{4}(4+4q^2+r^2-9p^2)\frac{\wp}{(\wp')^2}+(1-r^2)\left(\frac{\wp^2}{\wp'}\right)^2,$$

which has six double poles in $\mathbf{C}/\Lambda$: one at the pole of $\wp$, one at each zero of $\wp$ and one at each zero of $\wp'$. At 0, pole of $\wp$, $E=\frac{1}{4}(1-r^2)t^{-2}+\cdots$, and $E(\rho t)=\rho^4E(t)$. By Lemma 3.1, all the solutions are meromorphic if $6\nmid r$. At the zeros of $\wp$, $E(t)=\frac{1}{4}(1-q^2)(t\mp\theta)^{-2}+\cdots$. According to (4.3), $E(\pm\theta+\omega t)=\omega E(\pm\theta+t)$, and by Lemma 3.1 the solutions are meromorphic if $3\nmid q$. At all of the three zeros η_i of $\wp'$, $E=\frac{1}{4}(1-p^2)(t-\eta_i)^{-2}+\cdots$. From (4.4), $E(\eta_i-t)=E(\eta_i+t)$ and thus, by Lemma 3.1, the solutions are meromorphic if p is odd.

The hypothesis of Theorem 3.1 are fulfilled when r is even.

5. The lemniscatic cases

For all the standard facts about Jacobi elliptic functions that will be used in this Section, we refer to [2, Chapter 4]. The three principal Jacobi elliptic functions $\mathsf{sn}(k,t)$, $\mathsf{cn}(k,t)$, $\mathsf{dn}(k,t)$ are meromorphic elliptic functions in $\mathbf{C}$, where $k\in\mathbf{C}$ is a fixed parameter (the *modulus*). We will be exclusively concerned with the case where $k^2=-1$, and we will simply write $\mathsf{sn}(t)$, $\mathsf{cn}(t)$, $\mathsf{dn}(t)$. In this case the principal Jacobi elliptic functions are doubly periodic with respect to the lattice Λ generated by $4K$ and $4iK'$, where $K=\int_0^1(1-t^4)^{-1/2}dt$, $K'=(1-i)K$. This module of periods is invariant under multiplication by i (is *lemniscatic*).

We will now prove Theorem 1.1 for equations IV and V, whose coefficients belong to the elliptic function field generated by the principal Jacobi elliptic functions. Each one of the principal Jacobi

Table 2. Behavior at the poles.

u	iK'	$3iK'$	$iK'+2K$	$3iK'+2K$
$\mathsf{cn}(t+u)$	$-\dfrac{\mathsf{dn}(t)}{\mathsf{sn}(t)}$	$\dfrac{\mathsf{dn}(t)}{\mathsf{sn}(t)}$	$\dfrac{\mathsf{dn}(t)}{\mathsf{sn}(t)}$	$-\dfrac{\mathsf{dn}(t)}{\mathsf{sn}(t)}$
$\mathsf{dn}(t+u)$	$-i\dfrac{\mathsf{cn}(t)}{\mathsf{sn}(t)}$	$i\dfrac{\mathsf{cn}(t)}{\mathsf{sn}(t)}$	$-i\dfrac{\mathsf{cn}(t)}{\mathsf{sn}(t)}$	$i\dfrac{\mathsf{cn}(t)}{\mathsf{sn}(t)}$
$\mathsf{sn}(t+u)$	$-\dfrac{i}{\mathsf{sn}(t)}$	$-\dfrac{i}{\mathsf{sn}(t)}$	$\dfrac{i}{\mathsf{sn}(t)}$	$\dfrac{i}{\mathsf{sn}(t)}$

Table 3. Residues at the poles.

u	iK'	$3iK'$	$iK'+2K$	$3iK'+2K$
$\mathrm{Res}(\mathsf{cn},u)$	-1	1	1	-1
$\mathrm{Res}(\mathsf{dn},u)$	$-i$	i	$-i$	i
$\mathrm{Res}(\mathsf{sn},u)$	$-i$	$-i$	i	i

elliptic functions has an extra period of its own: $2iK'$ for sn, $2K+2iK'$ for cn and $2K$ for dn. The function sn is odd, while cn and dn are even. Their derivatives may be expressed in terms of the same functions; for example,

$$\mathsf{sn}'(t) = \mathsf{cn}(t)\mathsf{dn}(t). \tag{5.1}$$

The three principal functions have simple poles at iK', $iK'+2K$, $3iK'$, $3iK'+2K$. The corresponding residues, that may be obtained through the formulae in Table 2, appear in Table 3.

5.1. *Equation* IV

It corresponds to

$$E = \frac{1}{16}(1-p^2)[\mathsf{cn}(t)+i\mathsf{dn}(t)]^2 + \frac{1}{16}(1-q^2)[\mathsf{cn}(t)-i\mathsf{dn}(t)]^2.$$

It has four double poles in $\mathbf{C}/\Lambda$, located at the poles of cn and dn. According to Table 3, for u equal to iK' or $3iK'$, $E(t) = \frac{1}{4}(1-q^2)(t-u)^2 + \cdots$; for u equal to $iK'+2K$ or $3iK'+2K$, $E(t) = \frac{1}{4}(1-p^2)(t-u)^2 + \cdots$. From the data in Table 2 and from the fact that cn and dn are both even while sn is odd, for each pole u of E we have that $E(u-t) = E(u+t)$. Thus, by Lemma 3.1, in the neighborhood of the poles of E, all the solutions are meromorphic if both q and p are odd.

Since all the indices at the poles come by pairs, there is no situation where Theorem 3.1 can be applied.

5.1.1. *The special case* $p^2 = q^2$

In this case $E = -\frac{1}{8}(1-q^2)\mathsf{sn}(t)^2$. From the analysis of equation V that will follow (particularly the details given in Section 5.2.1), the equation will have meromorphic solutions as soon as $4 \nmid q$.

Table 4. Coefficients of the leading term of the Taylor developments at the double poles.

	$i\mathsf{sn}'$	sn^2	sn^{-2}
iK'	-1	-1	$\times$
$iK'+2K$	1	-1	$\times$
0	$\times$	$\times$	1
$2K$	$\times$	$\times$	1

5.2. *Equation* V

In this case

$$E = \frac{1}{8}(q^2+r^2-2)\mathsf{sn}^2(t) + \frac{1}{8}(q^2-r^2)i\mathsf{sn}'(t) + \frac{1}{4}(1-p^2)\mathsf{sn}^{-2}(t).$$

It has the module of periods Λ', $\Lambda \subset \Lambda'$, $[\Lambda' : \Lambda] = 2$ (sn has the extra period $2iK'$). The function E has double poles at the poles and zeros of sn.

In $\mathbf{C}/\Lambda'$, the poles of sn are simple and located at iK' and $iK'+2K$. In particular sn^2 has double poles at these points. The coefficient of the leading term of the Taylor development of E at them is -1; the function sn' has double poles at the poles of sn; the coefficients of the leading term of the Taylor development of E at these points, that may be obtained through Eq. (5.1) and Table 3, appear in Table 4. We have that $E(t) = \frac{1}{4}(1-r^2)(t-iK')^{-2}+\cdots$, $E(t) = \frac{1}{4}(1-q^2)(t-[iK'+2K])^{-2}+\cdots$. From Table 2 and from the fact that sn is even and sn' is odd, we have that for each pole u of E, $E(u-t) = E(u+t)$ and thus in the neighborhood of the poles of E the solutions to the Riccati equation are meromorphic if q and r are odd.

In $\mathbf{C}/\Lambda'$, the zeros of sn are simple and located at 0 and $2K$; we have that $\mathsf{sn}'(0) = 1$ and, since $\mathsf{sn}(2K+t) = -\mathsf{sn}(t)$, $\mathsf{sn}'(2K+t) = -\mathsf{sn}'(t)$ and $\mathsf{sn}'(2K) = -1$. We have that $E(t) = \frac{1}{4}(1-p^2)t^{-2}+\cdots$. Since $\mathsf{sn}(-t) = -\mathsf{sn}(t)$ and $\mathsf{sn}'(-t) = \mathsf{sn}'(t)$, $E(-t) = E(t)$. Thus, by Lemma 3.1, in a neighborhood of $t = 0$ we have meromorphic solutions if p is odd. We have that $E(t) = \frac{1}{4}(1-p^2)(t-2K)^{-2}+\cdots$. Since $\mathsf{sn}(2K-t) = -\mathsf{sn}(-t) = \mathsf{sn}(t) = -\mathsf{sn}(2K+t)$ and $\mathsf{sn}'(2K-t) = \mathsf{sn}'(2K+t)$, $E(2K-t) = E(2K+t)$. Thus, in a neighborhood of $t = 2K$, we have meromorphic solutions if p is odd.

All indices must be odd; the hypothesis of Theorem 3.1 are never satisfied.

5.2.1. *The special case $q^2 = r^2$*

In this case E has the extra period $2K$, for $\mathsf{sn}(t+2K) = -\mathsf{sn}(t)$. Since $\mathsf{sn}(it) = i\mathsf{sn}(t)$ (both functions give solutions to $(y')^2 = y^4-1$ vanishing at 0), $E(it) = -E(t)$ and thus, in a neighborhood of $t = 0$, we have meromorphic solutions as soon as $4 \nmid p$. If p is even, the hypothesis of Theorem 3.1 are satisfied.

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