



Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852

ISSN (Print): 1402-9251

Journal Home Page: <https://www.atlantis-press.com/journals/jnmp>

Pseudo-Hermitian Reduction of a Generalized Heisenberg Ferromagnet Equation. I. Auxiliary System and Fundamental Properties

A. B. Yanovski, T. I. Valchev

To cite this article: A. B. Yanovski, T. I. Valchev (2018) Pseudo-Hermitian Reduction of a Generalized Heisenberg Ferromagnet Equation. I. Auxiliary System and Fundamental Properties, Journal of Nonlinear Mathematical Physics 25:2, 324–350, DOI: <https://doi.org/10.1080/14029251.2018.1452676>

To link to this article: <https://doi.org/10.1080/14029251.2018.1452676>

Published online: 04 January 2021

Pseudo-Hermitian Reduction of a Generalized Heisenberg Ferromagnet Equation. I. Auxiliary System and Fundamental Properties

A. B. Yanovski

*Department of Mathematics & Applied Mathematics,
University of Cape Town,
Rondebosch 7700, Cape Town, South Africa
Alexander.Ianovsky@uct.ac.za*

T. I. Valchev

*Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences,
Acad. G. Bonchev Str., 1113 Sofia, Bulgaria
tiv@math.bas.bg*

Received 8 October 2017

Accepted 9 January 2018

We consider an auxiliary spectral problem originally introduced by Gerdjikov, Mikhailov and Valchev (GMV system) and its modification called pseudo-Hermitian reduction which is extensively studied here for the first time. We describe the integrable hierarchies of both systems in a parallel way and construct recursion operators. Using the concept of gauge equivalence, we construct expansions over the eigenfunctions of recursion operators. This permits us to obtain the expansions for both GMV systems with arbitrary constant asymptotic values of the potential functions in the auxiliary linear problems.

Keywords: Lax representation; gauge-equivalent equations; recursion operators; reductions.

2010 Mathematics Subject Classification: 37K10, 37K30, 35Q55, 35Q51

1. Introduction

We are going to study the auxiliary linear problem

$$\tilde{L}\psi = (i\partial_x - \lambda S)\psi = 0, \quad S = \begin{pmatrix} 0 & u & v \\ \varepsilon u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix}, \quad \lambda \in \mathbb{C} \quad (1.1)$$

and the theory of expansions over the adjoint solutions related to it. Above $\varepsilon = \pm 1$ and $*$ stands for complex conjugation. The complex valued functions u and v (the potentials) are assumed to be smooth for $x \in \mathbb{R}$ and they satisfy the relation:

$$\varepsilon|u|^2 + |v|^2 = 1. \quad (1.2)$$

The potentials must also satisfy some asymptotic conditions when $x \rightarrow \pm\infty$ to be discussed further in text.

The system (1.1) for $\varepsilon = 1$ was considered by Gerdjikov, Mikhailov and Valchev [6] with the sign $+$ before S . It is a particular case of a vector system introduced independently by Golubchik

and Sokolov [12]. For reasons that will become clear later we shall prefer this equivalent form (one just needs to change (u, v) to $(-u, -v)$).

Let us adopt the following convention: if ε appears in formulas then it will be $+1$ or -1 , if it is used as an index or label it will mean either $+$ or $-$. Thus we shall denote the original Gerdjikov-Mikhailov-Valchev system by GMV_+ while the general case will be referred to as GMV_ε system.

According to [6,7] the GMV_+ system arises naturally when one looks for integrable system having a Lax representation $[L, A] = 0$ with L and A subject to Mikhailov-type reduction requirements, see [16–18]. This is also true for both GMV_\pm systems. Indeed, the Mikhailov reduction group G_0 acting on the fundamental solutions of the system (1.1) could be defined as generated by the two elements g_1 and g_2 acting in the following way:

$$\begin{aligned} g_1(\psi)(x, \lambda) &= [Q_\varepsilon \psi(x, \lambda^*)^\dagger Q_\varepsilon]^{-1}, & Q_\varepsilon &= \text{diag}(1, \varepsilon, 1), \\ g_2(\psi)(x, \lambda) &= H\psi(x, -\lambda)H, & H &= \text{diag}(-1, 1, 1) \end{aligned} \tag{1.3}$$

where \dagger denotes Hermitian conjugation. Since $g_1 g_2 = g_2 g_1$ and $g_1^2 = g_2^2 = \text{id}$, $G_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$. We shall call the reduction defined by g_1, g_2 for $\varepsilon = 1$ Hermitian while the reduction defined by g_1, g_2 for $\varepsilon = -1$ pseudo-Hermitian. The requirement that G_0 is a reduction group immediately gives that the coefficients of the operators \tilde{L} and

$$\tilde{A} = i\partial_t + \sum_{k=0}^n \lambda^k \tilde{A}_k, \quad \tilde{A}_k \in \mathfrak{sl}(3, \mathbb{C}) \tag{1.4}$$

must satisfy:

$$\begin{aligned} HSH &= -S, & H\tilde{A}_k H &= (-1)^k \tilde{A}_k, \\ Q_\varepsilon S^\dagger Q_\varepsilon &= S, & Q_\varepsilon \tilde{A}_k^\dagger Q_\varepsilon &= \tilde{A}_k. \end{aligned} \tag{1.5}$$

One may consider a more general form of pseudo-Hermitian reduction, i.e. one with $Q_{\varepsilon_1 \varepsilon_2} = \text{diag}(1, \varepsilon_1, \varepsilon_2)$, $\varepsilon_1^2 = \varepsilon_2^2 = 1$. However, it is easily checked this does not give anything new compared to the pseudo-Hermitian reduction under consideration here.

As it can be checked the matrix S has constant eigenvalues. We have $g^{-1} S g = J_0$, where g is of the form:

$$g = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ \varepsilon u^* & \sqrt{2}v & \varepsilon u^* \\ v^* & -\sqrt{2}u & v^* \end{pmatrix}, \quad J_0 = \text{diag}(1, 0 - 1). \tag{1.6}$$

In case $\varepsilon = +1$ the matrix g belongs to the group $SU(3)$ ($g^\dagger = g^{-1}$) and when $\varepsilon = -1$ to the group $SU(2, 1)$ ($Q_- g^\dagger Q_- = g^{-1}$). Further on we shall use the general notation $SU(\varepsilon)$ referring to both cases, i.e. it implies $SU(\varepsilon) \equiv SU(3)$ when $\varepsilon = 1$ and $SU(\varepsilon) \equiv SU(2, 1)$ when $\varepsilon = -1$.

Since $g(x) \in SU(\varepsilon)$, the values of $S(x)$ will be in the orbit $\mathcal{O}_{J_0}(SU(\varepsilon))$ of J_0 with respect to $SU(\varepsilon)$ (it is a submanifold of $\text{isu}(\varepsilon)$). Thus $S(x) \in \mathcal{O}_{J_0}(SU(\varepsilon)) \cap \mathfrak{g}_1$ where \mathfrak{g}_1 is the space of the matrices X in $\mathfrak{sl}(3, \mathbb{C})$ such that $HXH = -X$, see (2.6) for the reason for this notation. Let us also note that conversely, if we assume that $S(x) \in \mathcal{O}_{J_0}(SU(\varepsilon)) \cap \mathfrak{g}_1$ then on the first place S has the form as in (1.1). Next, as easily checked, the eigenvalues of the matrix S are

$$\mu_1 = 0, \quad \mu_2 = -\mu_3 = \sqrt{\varepsilon|u|^2 + |v|^2}.$$

But since they coincide with $0, \pm 1$ we must have $\varepsilon|u|^2 + |v|^2 = 1$.

Our approach to the GMV_{\pm} system will be based on the fact that it is gauge equivalent to a generalized Zakharov-Shabat auxiliary systems (GZS systems) on the algebra $sl(3, \mathbb{C})$, see Sec. 3. Generalized Zakharov-Shabat systems, called Caudrey-Beals-Coifman (CBC) systems [2] when J is complex, are probably the best known auxiliary linear problems. In their most general form they are written on an arbitrary fixed simple Lie algebra \mathfrak{g} in some finite dimensional irreducible representation and have the form:

$$L\psi = (i\partial_x + q(x) - \lambda J)\psi = 0. \tag{1.7}$$

Here, $q(x)$ and J belong to \mathfrak{g} in some fixed representation, ψ belongs to the corresponding group. The element J must be such that the kernel of ad_J ($\text{ad}_J(X) \equiv [J, X], X \in \mathfrak{g}$) is a Cartan subalgebra \mathfrak{h}_J of \mathfrak{g} . The potential $q(x)$ belongs to the orthogonal complement \mathfrak{h}_J^{\perp} of \mathfrak{h} with respect to the Killing form:

$$\langle X, Y \rangle = \text{tr}(\text{ad}_X \text{ad}_Y) \quad X, Y \in \mathfrak{g}. \tag{1.8}$$

It is assumed that $q(x)$ is sufficiently smooth and it converges to 0 fast enough when $x \rightarrow \pm\infty$. The system (1.7) is called GZS (CBC) system over \mathfrak{g} in canonical gauge. When the algebra is understood from the context and as a representation is chosen the canonical one it is just called GZS (CBC) system. The system (1.7) is gauge equivalent to

$$\tilde{L}\tilde{\psi} = (i\partial_x - \lambda S(x))\tilde{\psi} = 0, \quad S(x) \in \mathcal{O}_J. \tag{1.9}$$

Usually, it is assumed that $\lim_{x \rightarrow \pm\infty} S(x) = J$ where the convergence is sufficiently fast. Here, \mathcal{O}_J stands for the orbit of J under the adjoint action of the group G corresponding to \mathfrak{g} : $\mathcal{O}_J = \{\tilde{X} : \tilde{X} = gJg^{-1}, g \in G\}$. The concept of gauge transformation, gauge equivalent auxiliary problems and gauge equivalent soliton equations originates from the famous work [27] where it has been employed to solve an equation that is a classical analogue of equations describing waves in magnetic chains (spin 1/2). It has been shown that one of the nonlinear evolution equations (NLEEs) related to \tilde{L} is the Heisenberg ferromagnet equation

$$S_t = \frac{1}{2i}[S, S_{xx}], \quad S \in \text{isu}(2), \quad S^2 = \mathbf{1}, \quad \lim_{x \rightarrow \pm\infty} S(x, t) = \text{diag}(1, -1)$$

being gauge equivalent to the famous nonlinear Schrödinger equation, see [4] for an extensive discussion on that issue. It should be mentioned that the soliton equations solvable through the auxiliary linear problem $\tilde{L}\tilde{\psi} = 0$ in the case $sl(3, \mathbb{C})$ is related to a classical analog of the equation describing spin 1 particle chains dynamics, see [3].

Later, the results of [27] were generalized to the soliton equations hierarchies associated with L and \tilde{L} , the conservation laws of those NLEEs, the hierarchies of their Hamiltonian structures etc. This was achieved by generalizing the so-called AKNS approach [1] (generating operators or recursion operators approach). Initially, this was done in the $sl(2, \mathbb{C})$ case, next in the case of GZS system on arbitrary semisimple Lie algebra [5, 9, 19]. Now, this theory is referred to as the gauge-covariant theory of the recursion operators related to the GZS (CBC) systems in canonical and pole gauge. For a detailed explanations of all these issues and more references (prior to 2008), see the monograph [8].

For GZS system in pole gauge most of the essential issues could be reformulated from the canonical gauge. Perhaps the main difficulty is to express explicitly all quantities depending on q

and its derivative through S and its derivatives. While there is a clear procedure how to achieve that goal, in each particular case the details could be different. The procedure has been developed in detail in the PhD thesis [19]. In the case of $\mathfrak{sl}(3, \mathbb{C})$ in general position (with no reductions) it has been carried out in [20].

Regarding now the system GMV_{\pm} , it was the Hermitian case, i.e. GMV_+ , that was mainly considered so far. The first paper in which GMV_+ was studied in detail for the asymptotic conditions $\lim_{x \rightarrow \pm\infty} u = 0$, $\lim_{x \rightarrow \pm\infty} v = \exp i\Phi_{\pm}$ was [7]. It contained discussion of the spectral properties of the GMV_+ system, two operators whose product play the role of a recursion operator were presented and expansions over the so-called adjoint solutions were derived. We shall also discuss these issues, however, we want to stress on the following: a) We shall be dealing with both GMV_{\pm} systems simultaneously; b) Our approach will be completely different, based on the gauge equivalence we mentioned above. Consequently, we shall be able to consider general asymptotic conditions – constant limits $\lim_{x \rightarrow \pm\infty} u$ and $\lim_{x \rightarrow \pm\infty} v$; c) Our point of view on the recursion operators when reductions are present is somewhat different from that adopted in [7].

2. Algebraic preliminaries

In order to proceed further, we shall need some information about the algebra $\mathfrak{sl}(3, \mathbb{C})$ and the involutions we are going to use which we introduce below. All facts from the theory of the semisimple Lie algebras we use are classical, see [13]. We also use the normalizations adopted in this monograph and most of its notation.

The Lie algebra $\mathfrak{sl}(3, \mathbb{C})$ is a simple Lie algebra of rank 2. We shall denote its Killing form $\text{tr}(\text{ad}_X \text{ad}_Y)$ by $\langle X, Y \rangle$ where as usual $\text{ad}_X(Y) := [X, Y]$. It is known that $\text{tr}(\text{ad}_X \text{ad}_Y) = 6 \text{tr}(XY)$ which simplifies considerably the calculations. A Cartan subalgebra could be introduced using any regular element $X \in \mathfrak{sl}(3, \mathbb{C})$ and constructing the space $\mathfrak{h}_X = \ker \text{ad}_X$. As it is known, the subalgebra of the diagonal matrices is a canonical choice for Cartan subalgebra. It is also equal to $\mathfrak{h} = \ker \text{ad}_J$ where J is any diagonal matrix $\text{diag}(\lambda_1, \lambda_2, \lambda_3)$ with distinct λ_i . In that case, we shall call \mathfrak{h} the *Cartan subalgebra*. For the canonical choice of the Cartan subalgebra the system of roots Δ for $\mathfrak{sl}(3, \mathbb{C})$ is

$$\Delta = \{\alpha_{i,j} = \varepsilon_i - \varepsilon_j, \quad i \neq j, \quad i, j = 1, 2, 3\} \tag{2.1}$$

where ε_i are functionals acting on \mathfrak{h} in the following way: $\varepsilon_i(\text{diag}(h_1, h_2, h_3)) = h_i$. Then the set of positive roots Δ_+ consists of the elements α_i :

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \quad \alpha_3 = \varepsilon_1 - \varepsilon_3 = \alpha_1 + \alpha_2. \tag{2.2}$$

The corresponding root vectors E_{α} , $\alpha \in \Delta$ together with the matrices $H_{\alpha_1}, H_{\alpha_2}$ written below:

$$\begin{aligned} E_{\alpha_1} &= \frac{1}{\sqrt{6}}e_{12}, & E_{\alpha_2} &= \frac{1}{\sqrt{6}}e_{23}, & E_{\alpha_3} &= \frac{1}{\sqrt{6}}e_{13}, \\ E_{-\alpha_1} &= \frac{1}{\sqrt{6}}e_{21}, & E_{-\alpha_2} &= \frac{1}{\sqrt{6}}e_{32}, & E_{-\alpha_3} &= \frac{1}{\sqrt{6}}e_{31}, \\ H_{\alpha_1} &= \frac{1}{6}(e_{11} - e_{22}), & H_{\alpha_2} &= \frac{1}{6}(e_{22} - e_{33}) \end{aligned} \tag{2.3}$$

form the Cartan-Weil basis of $\mathfrak{sl}(3, \mathbb{C})$ associated with the Cartan subalgebra \mathfrak{h} . Here as usual e_{ij} means a matrix whose only nonzero entry equal to one is located in the intersection of the i -th row and j -th column. The matrices $H_{\alpha_1}, H_{\alpha_2}$ span \mathfrak{h} and the matrices E_{α} span \mathfrak{h}^{\perp} .

The relations (1.5) have the following Lie algebraic meaning. First, the map $h : X \mapsto HXH = HXH^{-1}$ is involutive automorphism of $\mathfrak{sl}(3, \mathbb{C})$. Next, Q_ε defines a complex conjugation σ_ε of $\mathfrak{sl}(3, \mathbb{C})$:

$$\sigma_\varepsilon(X) = -Q_\varepsilon X^\dagger Q_\varepsilon. \quad (2.4)$$

The complex conjugation σ_ε defines the real form $\mathfrak{su}(3)$ ($\varepsilon = +1$) or the real form $\mathfrak{su}(2, 1)$ ($\varepsilon = -1$) of $\mathfrak{sl}(3, \mathbb{C})$. In order to treat both cases simultaneously we shall adopt the notation $\mathfrak{su}(\varepsilon)$ meaning $\mathfrak{su}(3)$ when $\varepsilon = +1$ and $\mathfrak{su}(2, 1)$ when $\varepsilon = -1$. Note that the complex conjugation σ_ε commutes with the automorphism h . Now, let us introduce the spaces:

$$\mathfrak{g}_j = \{X : h(X) = (-1)^j X\}, \quad j = 0, 1. \quad (2.5)$$

Then we shall have the splittings

$$\begin{aligned} \mathfrak{sl}(3, \mathbb{C}) &= \mathfrak{g}_0 \oplus \mathfrak{g}_1 \\ \mathfrak{su}(\varepsilon) &= (\mathfrak{g}_0 \cap \mathfrak{su}(\varepsilon)) \oplus (\mathfrak{g}_1 \cap \mathfrak{su}(\varepsilon)). \end{aligned} \quad (2.6)$$

The space \mathfrak{g}_1 consists of all off-diagonal matrices $X = (x_{ij})$ for which $x_{23} = x_{32} = 0$ and \mathfrak{g}_0 consists of all traceless matrices $X = (x_{ij})$ for which $x_{12} = x_{21} = x_{13} = x_{31} = 0$. Of course, S involved in the GMV_\pm satisfies $S \in \mathfrak{g}_1$. Also, since h is automorphism, the spaces \mathfrak{g}_0 and \mathfrak{g}_1 are orthogonal with respect to the Killing form.

Below we shall see that h we already introduced ($X \mapsto h(X) = HXH$) is closely related to the automorphism $X \mapsto k(X) = KXK$ where

$$K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (2.7)$$

So we shall need how h and k act on the Cartan-Weil basis. Note that similar to h the automorphism k is also involutive, that is $k^2 = \text{id}$. For h we get that

$$\begin{aligned} h(E_{\pm\alpha_1}) &= -E_{\pm\alpha_1}, & h(E_{\pm\alpha_2}) &= E_{\pm\alpha_2}, & h(E_{\pm\alpha_3}) &= -E_{\pm\alpha_3}, \\ h(H_1) &= H_1, & h(H_2) &= H_2. \end{aligned} \quad (2.8)$$

while for k we have

$$\begin{aligned} k(E_{\pm\alpha_1}) &= E_{\mp\alpha_2}, & k(E_{\pm\alpha_3}) &= E_{\mp\alpha_3}, & k(E_{\pm\alpha_2}) &= E_{\mp\alpha_1}, \\ k(H_1) &= -H_2, & k(H_2) &= -H_1. \end{aligned} \quad (2.9)$$

The above formulas become simpler if we introduce the action \mathcal{K} of k on the roots:

$$\mathcal{K}(\pm\alpha_1) = \mp\alpha_2, \quad \mathcal{K}(\pm\alpha_3) = \mp\alpha_3, \quad \mathcal{K}(\pm\alpha_2) = \mp\alpha_1. \quad (2.10)$$

We observe that \mathcal{K} maps positive roots into negative and vice-versa. Of course, \mathcal{K} just as k satisfies $\mathcal{K}^2 = \text{id}$. There is no need to introduce the action of h on the roots since this is simply the identity. With the above notation we have

$$k(E_\alpha) = E_{\mathcal{K}\alpha}, \quad h(E_\alpha) = r(\alpha)E_\alpha \quad (2.11)$$

where $r(\alpha) = 1$ if $\alpha = \pm\alpha_1, \pm\alpha_3$ and $r(\alpha) = -1$ if $\alpha = \pm\alpha_2$. Note that $r(\alpha)r(-\alpha) = 1$.

The invariance under the group generated by g_1, g_2 means that if ψ is the common G_0 -invariant fundamental solution of the linear problem

$$\tilde{L}\tilde{\psi} = (i\partial_x - \lambda S(x))\tilde{\psi} = 0, \quad S(x) \in \mathcal{O}_{J_0} \subset \mathfrak{sl}(3, \mathbb{C}) \quad (2.12)$$

and the linear problem of the type:

$$\tilde{A}\tilde{\psi} = i\partial_t\tilde{\psi} + \sum_{k=0}^n \lambda^k \tilde{A}_k \tilde{\psi} = 0, \quad \tilde{A}_k \in \mathfrak{sl}(3, \mathbb{C}) \quad (2.13)$$

we must have $S \in \mathfrak{g}_1 \cap \text{isu}(\varepsilon)$ and

$$\tilde{A}_{2k+1} \in \mathfrak{g}_1 \cap \text{isu}(\varepsilon), \quad \tilde{A}_{2k} \in \mathfrak{g}_0 \cap \text{isu}(\varepsilon), \quad k = 0, 1, 2, \dots \quad (2.14)$$

The conditions on \tilde{A}_k and S coincide with (1.5), in particular, we have that $h(S) = -S$ and $\sigma_\varepsilon(S) = -S$. This forces S to be in the form we introduced in (1.1). Also, as a direct consequence from the last relations we obtain that

$$h \circ \text{ad}_S = -\text{ad}_S \circ h, \quad \sigma_\varepsilon \circ \text{ad}_S = -\text{ad}_S \circ \sigma_\varepsilon. \quad (2.15)$$

Consequently, the spaces $\ker \text{ad}_S = \mathfrak{h}_S$ (which obviously is a Cartan subalgebra) and its orthogonal complement \mathfrak{h}_S^\perp are invariant under h . Each of them splits into two h -eigenspaces (for the eigenvalues $+1$ and -1 respectively):

$$\mathfrak{h}_S^\perp = \mathfrak{f}_0 \oplus \mathfrak{f}_1, \quad \mathfrak{h}_S = \mathfrak{h}_0 \oplus \mathfrak{h}_1. \quad (2.16)$$

Several consequences follow from the above :

- (1) ad_S and ad_S^{-1} interchange the spaces \mathfrak{f}_0 and \mathfrak{f}_1 :

$$\begin{aligned} \text{ad}_S \mathfrak{f}_0 &= \mathfrak{f}_1, & \text{ad}_S \mathfrak{f}_1 &= \mathfrak{f}_0, \\ \text{ad}_S^{-1} \mathfrak{f}_0 &= \mathfrak{f}_1, & \text{ad}_S^{-1} \mathfrak{f}_1 &= \mathfrak{f}_0. \end{aligned} \quad (2.17)$$

- (2) Since $S \in \mathfrak{g}_1$, $S_1 = S^2 - \frac{2}{3}\mathbf{1} \in \mathfrak{g}_0$, the spaces $\mathfrak{h}_0, \mathfrak{h}_1$ are 1-dimensional and are spanned by S and $S_1 = S^2 - \frac{2}{3}\mathbf{1}$ respectively.

- (3) From the previous item follows that

$$S_x \in \mathfrak{g}_1, \quad (S_1)_x = (S^2)_x \in \mathfrak{g}_0 \quad (2.18)$$

and since S_x and $(S_1)_x$ are orthogonal to \mathfrak{h}_S we have

$$S_x \in \mathfrak{f}_1, \quad S_{1x} \equiv (S_1)_x = (S^2)_x \in \mathfrak{f}_0. \quad (2.19)$$

Another issue we must discuss is the relation between h from one side and ad_S^{-1} and π_S from the other. Here π_S is the orthogonal projection on the space \mathfrak{h}_S^\perp with respect to the Killing form and ad_S^{-1} is defined only on the space \mathfrak{h}_S^\perp . As one can show, see [7, 22], ad_S^{-1} could be expressed as a polynomial $p_3(\text{ad}_S)$ in ad_S where only odd degrees of ad_S are present ($\text{ad}_{L_1}^{-1}$ through ad_{L_1} in the

notation of [7]). From the other side, this means that π_S could be expressed as a polynomial in ad_S where enter only the even powers of ad_S . Consequently, one obtains that

$$\text{ad}_S^{-1} \circ h = -h \circ \text{ad}_S^{-1}, \quad \pi_S \circ h = h \circ \pi_S \tag{2.20}$$

and in the same way

$$\text{ad}_S^{-1} \circ \sigma_\varepsilon = -\sigma_\varepsilon \circ \text{ad}_S^{-1}, \quad \pi_S \circ \sigma_\varepsilon = \sigma_\varepsilon \circ \pi_S. \tag{2.21}$$

Remark 2.1. If we understand ad_S^{-1} as polynomial $p_3(\text{ad}_S)$ it is clear that instead of $\text{ad}_S^{-1} \circ \pi_S$ one can simply write ad_S^{-1} which is usually done without mentioning.

For convenience we formulate these facts as a proposition:

Proposition 2.1. *The projection π_S commutes with h and σ_ε .*

3. The GMV $_{\pm}$ system and its gauge equivalent

It has been pointed out [7] that the GMV $_{+}$ system is gauge equivalent to a GZS type system. For the system GMV $_{\pm}$ the situation is the same. After the gauge transformation $\tilde{L} \mapsto L' = g^{-1}(x)\tilde{L}g(x)$ with $g(x)$ given in (1.6) we obtain the system

$$L'\psi' = (i\partial_x + q' - \lambda J_0)\psi' = 0 \tag{3.1}$$

where $q' = ig^{-1}g_x$. After expressing g with u and v , we obtain

$$q' = \frac{i}{2} \begin{pmatrix} \varepsilon uu_x^* + vv_x^* & \sqrt{2}(uv_x - vu_x) & \varepsilon uu_x^* + vv_x^* \\ \sqrt{2}\varepsilon(v^*u_x^* - u^*v_x^*) & 2(\varepsilon u^*u_x + v^*v_x) & \sqrt{2}\varepsilon(v^*u_x^* - u^*v_x^*) \\ \varepsilon uu_x^* + vv_x^* & \sqrt{2}(uv_x - vu_x) & \varepsilon uu_x^* + vv_x^* \end{pmatrix}. \tag{3.2}$$

As it should be for GZS system J_0 is a real diagonal matrix and its diagonal elements are ordered to decrease going down along the main diagonal. This is in fact our motivation when we choose the sign in front of S . As one could see, in the above q' is not off-diagonal as GZS requires and in [7] authors just stopped here. It turns out, however, that the issue could be easily addressed making one more gauge transformation. Indeed, due to the condition $\varepsilon|u|^2 + |v|^2 = 1$ the diagonal part of q' equals $\frac{i}{2}(\varepsilon uu_x^* + vv_x^*)J' = b(x)J'$ where $J' = \text{diag}(1, -2, 1)$. Moreover, the diagonal part is real, that is $b(x) = \frac{i}{2}(\varepsilon uu_x^* + vv_x^*)$ is real. Then the gauge transformation

$$L' \mapsto L = \left(\exp(-iJ' \int_{-\infty}^x b(y)dy) \right) L' \left(\exp(iJ' \int_{-\infty}^x b(y)dy) \right) \tag{3.3}$$

maps the spectral problem into

$$L\psi = (i\partial_x + q - \lambda J_0)\psi = 0 \tag{3.4}$$

where

$$q(x) = \exp \left[-iJ' \int_{-\infty}^x b(y)dy \right] q'(x) \exp \left[iJ' \int_{-\infty}^x b(y)dy \right].$$

One can easily see that q is off-diagonal and since the entries of q' decay when $x \rightarrow \pm\infty$ the entries of q also decay. Now, since the diagonal elements of J_0 are distinct we conclude that GMV $_{\pm}$ is a

problem of GZS type in pole gauge on $\mathfrak{sl}(3, \mathbb{C})$. We shall exploit this fact but we want to address first some other issues. The first one is whether we can be more precise about the potential q in the GZS system (3.4) to which GMV_\pm is equivalent. With direct calculation we are able to establish that the potential q in (3.4) has special properties, namely $Q_\varepsilon q^\dagger Q_\varepsilon = q$ and $KqK = q$ where K has been introduced in (2.7). We also notice that $KJ_0K = -J_0$ and $Q_\varepsilon J_0^\dagger Q_\varepsilon = J_0$.

The next issue is about the asymptotic conditions S must satisfy. Passing from GZS system in canonical gauge (3.4) to GZS in pole gauge (1.9) usually involves the Jost solution ψ'_0 of (3.4) such that $\lim_{x \rightarrow -\infty} \psi'_0(x) = \mathbf{1}$ at $\lambda = 0$. The gauge transformation is then $\psi \rightarrow \tilde{\psi} = \psi'^{-1}_0 \psi$. In that case we obtain $S = \psi'^{-1}_0 J_0 \psi'_0$ and $\lim_{x \rightarrow -\infty} S(x) = J_0$. However, this is not our case and such a condition is not compatible with the reductions. Natural for the GMV_\pm system are conditions when u, v tend to some constant values. In this work we shall assume that $\lim_{x \rightarrow \pm\infty} u(x) = u_\pm$, $\lim_{x \rightarrow \pm\infty} v(x) = v_\pm$. In that case the limit of g when $x \rightarrow \pm\infty$ is equal to

$$g_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ \varepsilon u_\pm^* & \sqrt{2} v_\pm & \varepsilon u_\pm^* \\ v_\pm^* & -\sqrt{2} u_\pm & v_\pm^* \end{pmatrix}. \tag{3.5}$$

Now we notice the following. If we come back to how q was constructed, we shall see that the function

$$\psi_0 = e^{-iJ' \int_{-\infty}^x b(y) dy} g^{-1} \tag{3.6}$$

(for the definitions of g and b see the explanations after (3.2)) is as a matter of fact a solution to the system (3.4) for $\lambda = 0$. It satisfies:

$$\lim_{x \rightarrow -\infty} \psi_0 = g^{-1}, \quad \lim_{x \rightarrow +\infty} \psi_0 = \exp(-i\Phi J') g_+^{-1} \tag{3.7}$$

where

$$\Phi = \int_{-\infty}^{+\infty} b(y) dy \pmod{2\pi}. \tag{3.8}$$

Remark 3.1. One can easily show that $b(x)$ is a conservation law density for all the NLEEs associated to the GMV_\pm system. Thus Φ is a conservation law and does not depend on time. Moreover, as it is well-known the conservation laws densities are local, this shows

$$B(x, t) = \int_{-\infty}^x b(y, t) dy$$

is a local expression (depends on x, t only through u, v and its derivatives).

So we can make a gauge transformation where instead of ψ'_0 we use ψ_0 . Since J_0 and J' commute, we obtain that $S(x) = \psi_0^{-1} J_0 \psi_0$ satisfies

$$\lim_{x \rightarrow -\infty} S(x) = g_- J_0 g_-^{-1} = S_-, \quad \lim_{x \rightarrow +\infty} S(x) = g_+ J_0 g_+^{-1} = S_+ \tag{3.9}$$

$$S_\pm = \begin{pmatrix} 0 & u_\pm & v_\pm \\ \varepsilon u_\pm^* & 0 & 0 \\ v_\pm^* & 0 & 0 \end{pmatrix}, \tag{3.10}$$

i.e. we have the correct asymptotic for $S(x)$. The last remark, which is in fact quite important, is that taking the explicit form of ψ_0 , see (3.6), one gets the following important formula:

$$K\psi_0 = \psi_0 H, \quad \psi_0^{-1} K = H\psi_0^{-1} \tag{3.11}$$

where H was introduced earlier in relation to the GMV_{\pm} system and K is given in (2.7). One could also obtain the same result in another way. Suppose one starts with the GZS system and assume it has the property that if ψ_0 is a fundamental solution, then $K\psi_0 K$ also is. Then naturally $K\psi_0$ is also a fundamental solution. So we must have $K\psi_0 = \psi_0 P$ where P is some non-degenerate matrix. Taking $x \rightarrow -\infty$ we get that $Kg^{-1} = g^{-1}P$. But, since $Kg^{-1} = g^{-1}H$ we have $P = H$. Finally, we formulate the results of the above discussions as a theorem:

Theorem 3.1. *The GMV_{\pm} system is gauge equivalent to a canonical GZS linear problem on $sl(3, \mathbb{C})$*

$$L\psi = (i\partial_x + q - \lambda J_0)\psi = 0 \tag{3.12}$$

subject to a Mikhailov reduction group generated by the two elements h_1 and h_2 . These elements act on the fundamental solutions ψ of the system (3.12) in the following way:

$$\begin{aligned} h_1(\psi)(x, \lambda) &= [Q_{\varepsilon}\psi(x, \lambda^*)^{\dagger}Q_{\varepsilon}]^{-1}, \quad Q_{\varepsilon} = \text{diag}(1, \varepsilon, 1), \quad \varepsilon = \pm 1, \\ h_2(\psi)(x, \lambda) &= K\psi(x, -\lambda)K. \end{aligned} \tag{3.13}$$

Since $h_1^2 = h_2^2 = id$ and $h_1 h_2 = h_2 h_1$ we have again a $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduction.

The properties of the fundamental solutions of the auxiliary systems play a paramount role in the spectral theory of such systems and for the theory of integration of the NLEEs associated with them. Since for the GZS systems with \mathbb{Z}_p reductions these questions have been studied in detail, see [11] (whether or not K is Coxeter automorphism does not make any difference here) the above theorem opens the possibility to study the GMV_{\pm} system using its gauge-equivalent GZS system with $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduction. In [11], it was considered much more complicated case of a CBC system and the automorphism has order $p > 1$ (in our case $p = 2$ since $K^2 = \mathbf{1}$). Then the complex plane is divided into p sectors by straight lines through the origin and in each sector there is a fundamental analytic solution to the corresponding CBC system. In our case the things are much simpler: we have the real line dividing \mathbb{C} into upper and lower half-plane. Then the system (3.12) possesses fundamental analytic solutions (FAS) $\chi^{\pm}(x, \lambda)$ of the form $m^{\pm}(x, \lambda) \exp(-i\lambda x J_0)$. The functions $m^{+}(x, \lambda)$ ($m^{-}(x, \lambda)$) are meromorphic in λ in the upper (lower) half plane \mathbb{C}_{+} (\mathbb{C}_{-}). Naturally, $\chi^{\pm}(x, \lambda)$ have the same analytic properties in λ as $m^{\pm}(x, \lambda)$. The points of \mathbb{C} that belong to the discrete spectrum are those at which $m^{\pm}(x, \lambda)$ and their inverse have singularities with respect to λ in \mathbb{C}_{+} (\mathbb{C}_{-}). It is usually considered the case when these singularities are poles, see for example [10] in the general case and [11] for the case of reductions. There is no difficulty to include the discrete spectrum for the GMV_{\pm} in our considerations but this will make all the formulas much more complicated so we shall do it elsewhere. Here we intend to explain mainly how our approach works and most of the issues are algebraic. Thus, in what follows we shall assume that there is no discrete spectrum for the GZS system gauge equivalent GMV_{\pm} and consequently no discrete spectrum for GMV_{\pm} .

Continuing with the properties of $m^\pm(x, \lambda)$, if the potential $q(x)$ has integrable derivatives up to the n -th order then

$$m^\pm(x, \lambda) = \mathbf{1} + \sum_{i=1}^n a_i(x) \lambda^{-i} + o(\lambda^{-n}) \tag{3.14}$$

when $|\lambda| \rightarrow \infty$ and λ remains in \mathbb{C}_+ (\mathbb{C}_- respectively). The asymptotic is uniform in $x \in \mathbb{R}$, and the coefficients $a_i(x)$ could be calculated through q and its x -derivatives. In particular, for absolutely integrable q we have $\lim_{|\lambda| \rightarrow \infty} m^\pm(x, \lambda) = \mathbf{1}$. The functions m^\pm allow extensions by continuity to the real line, these extensions will be denoted by the same letters. In addition, m^\pm satisfy:

$$\begin{aligned} Km^\pm(x, \lambda)K &= m^\mp(x, -\lambda), \\ \lim_{x \rightarrow -\infty} m^\pm(x, \lambda) &= \mathbf{1}. \end{aligned} \tag{3.15}$$

Since $KJ_0K = -J_0$, the solutions χ^\pm also allow extensions by continuity to the real line (denoted again by the same letters) and satisfy:

$$K\chi^\pm(x, \lambda)K = \chi^\mp(x, -\lambda). \tag{3.16}$$

It is simple to see that in our case χ^\pm will also satisfy:

$$Q_\varepsilon(\chi^\pm)^\dagger(x, \lambda)Q_\varepsilon = (\chi^\mp(x, \lambda^*))^{-1}. \tag{3.17}$$

We build from $\chi^\pm(x, \lambda)$ fundamental analytic solutions $\tilde{\chi}^\pm(x, \lambda)$ of the system (2.12) by setting:

$$\tilde{\chi}^\pm(x, \lambda) = \psi_0^{-1} \chi^\pm(x, \lambda). \tag{3.18}$$

Now we have:

Theorem 3.2. *The solutions $\tilde{\chi}^\pm(x, \lambda)$ satisfy^a:*

$$Q_\varepsilon(\tilde{\chi}^\pm(x, \lambda^*))^\dagger Q_\varepsilon = (\tilde{\chi}^\mp(x, \lambda))^{-1}, \quad H\tilde{\chi}^\pm(x, \lambda)H = \tilde{\chi}^\mp(x, -\lambda)KH. \tag{3.19}$$

Proof. The proof of the first statement is straightforward, one needs to see only that $Q_\varepsilon J_0^\dagger Q_\varepsilon = J_0$. The second one is obtained if one takes into account (3.11):

$$\begin{aligned} H\tilde{\chi}^\pm(x, \lambda) &= H\psi_0^{-1} \chi^\pm(x, \lambda) = \psi_0^{-1} K\chi^\pm(x, \lambda) \\ &= \psi_0^{-1} K\chi^\pm(x, \lambda)K^2 = \psi_0^{-1} \chi^\mp(x, -\lambda)K = \tilde{\chi}^\mp(x, -\lambda)K. \end{aligned}$$

Finally, we note that we have the same type of asymptotic behavior both for GMV_+ and GMV_- which allowed to treat both cases simultaneously. Of course, the solutions we speak of are different for different choices of ε but we denote them by the same letter since it will not cause ambiguities. As it is well known, the asymptotic of $\chi^\pm(x, \lambda)$ when $x \rightarrow \pm\infty$ are of the type $a_\pm^\pm \exp(-i\lambda J_0 x)$. So the asymptotic of $\tilde{\chi}^\pm(x, \lambda)$ are of the type $g_- a_\pm^\pm \exp(-i\lambda J_0 x)$ when $x \rightarrow -\infty$ and $g_+ \exp i(\Phi J') a_\pm^\pm \exp(-i\lambda J_0 x)$ when $x \rightarrow +\infty$.

^aIn [6, 7] FAS have mistakenly been claimed to satisfy $H\tilde{\chi}^\pm(x, \lambda)H = \tilde{\chi}^\mp(x, -\lambda)$.

4. Recursion operators for the GMV_± system

The recursion operators (Λ-operators) are theoretical tools that permit to describe [8]:

- The hierarchies of the NLEEs related to the auxiliary linear problem.
- The hierarchies of conservation laws for these NLEEs.
- The hierarchies of compatible Hamiltonian structures of these NLEEs.

The issue about the Hamiltonian structures is closely related to a beautiful interpretation of the recursion operators originating from the work [15]. In fact, the adjoint of the recursion operators could be interpreted as Nijenhuis tensors on the infinite dimensional manifold of ‘potentials’ (the functions $S(x)$), see [8].

Recursion operators arise naturally when one tries to find the hierarchy of Lax pairs related to a particular auxiliary linear problem [28]. So suppose we want to find the NLEEs having Lax representation $[\tilde{L}, \tilde{A}] = 0$ with $\tilde{L} = i\partial_x - \lambda S$ as in (2.12) and

$$\tilde{A} = i\partial_t + \sum_{k=0}^n \lambda^k \tilde{A}_k, \quad \tilde{A}_n \in \mathfrak{h}_S. \quad (4.1)$$

The first thing we notice is that we must have $i\partial_x \tilde{A}_n \in \mathfrak{h}_S^\perp$ and $\tilde{A}_0 = \text{const}$ in x . Using a gauge transformation depending only on t one can achieve $\tilde{A}_0 = 0$. Then one can show that the coefficients \tilde{A}_k for $k = 1, 2, \dots, n-1$ are calculated recursively. In order to see it we first recall that the elements S and $S_1 = S^2 - \frac{2}{3}\mathbf{1}$ span $\mathfrak{h}_S = \ker \text{ad}_S$ and that

$$\langle S, S \rangle = 12, \quad \langle S_1, S_1 \rangle = 4, \quad \langle S, S_1 \rangle = 0. \quad (4.2)$$

As before, denote by π_S the orthogonal projection (with respect to the Killing form) onto the space \mathfrak{h}_S^\perp . Then the orthogonal projection onto \mathfrak{h}_S will be $\mathbf{1} - \pi_S$ and for $X \in \mathfrak{sl}(3, \mathbb{C})$ we shall have

$$(\mathbf{1} - \pi_S)X = \frac{S}{12} \langle S, X \rangle + \frac{S_1}{4} \langle S_1, X \rangle. \quad (4.3)$$

Putting

$$\tilde{A}_s^d = (\mathbf{1} - \pi_S)\tilde{A}_s, \quad \tilde{A}_s^a = \pi_S\tilde{A}_s, \quad (4.4)$$

we obtain that the coefficients \tilde{A}_k for $k = 1, 2, \dots, n-1$ are obtained recursively in the following way:

$$\tilde{A}_{k-1}^a = \tilde{\Lambda}_\pm \tilde{A}_k^a, \quad (4.5)$$

$$\tilde{A}_k^d = i \left\{ \frac{S}{12} \int_{\pm\infty}^x \langle \tilde{A}_k^a, S_y \rangle dy + \frac{S_1}{4} \int_{\pm\infty}^x \langle \tilde{A}_k^a, S_{1y} \rangle dy \right\} \quad (4.6)$$

where

$$\tilde{\Lambda}_\pm(\tilde{Z}) = i \text{ad}_S^{-1} \pi_S \left(\partial_x \tilde{Z} + \frac{S_x}{12} \int_{\pm\infty}^x \langle \tilde{Z}, S_y \rangle dy + \frac{S_{1x}}{4} \int_{\pm\infty}^x \langle \tilde{Z}, S_{1y} \rangle dy \right) \quad (4.7)$$

and for the sake of brevity we write here and below S_{1x} instead of $(S_1)_x$.

The operators $\tilde{\Lambda}_\pm$ are recursion (generating) operators for GMV_± system. As one can see, they do not depend on the second reduction, that is on the choice of the real form. In fact, the recursion

operators for the $sl(3, \mathbb{C})$ -GZS system in general position were already known both in canonical and pole gauge [20, 21] and one can obtain $\tilde{\Lambda}$ for GMV from those as it was described in [22]. The expressions for the recursion operators for $sl(n, \mathbb{C})$ -CBC in pole gauge is also known [23]. From it one also could easily find $\tilde{\Lambda}_{\pm}$. Let us note that in [7] we already mentioned what is called recursion operator is in fact $\tilde{\Lambda}_{\pm}^2$ and it has been introduced as a product of two operators which differ from $\tilde{\Lambda}_{\pm}$. This complicates the picture (we shall discuss that a little further).

Let us continue with the calculation of the coefficients \tilde{A}_k . For the coefficient \tilde{A}_{n-1} one has that $i\partial_x \tilde{A}_n = [S, \tilde{A}_{n-1}]$ and since $\tilde{A}_n \in \mathfrak{h}_S$ there are scalar functions α, β such that $\tilde{A}_n = \alpha S + \beta S_1$. This gives

$$i\alpha_x S + i\beta_x S_1 + i\alpha S_x + i\beta(S_1)_x \in \mathfrak{h}_S^{\perp}$$

and therefore α and β are constants. Thus

$$\tilde{A}_{n-1} = i\alpha \text{ad}_S^{-1} S_x + i\beta \text{ad}_S^{-1} S_{1x}$$

and the hierarchy of NLEEs related to \tilde{L} in general position is

$$-\text{ad}_S^{-1} \partial_t S + (\tilde{\Lambda}_{\pm})^n (\text{ad}_S^{-1} (\alpha S_x + \beta(S_1)_x)) = 0. \tag{4.8}$$

In the case of GMV_{ε} system $\tilde{A}_{n-1} \in \text{isu}(\varepsilon)$ so α and β must be real. Next, if $\tilde{A}_n \in \mathfrak{h}_1$ and $n = 1(\text{mod } 2)$ because of the (4.5) and (4.6) we shall have automatically

$$\tilde{A}_{2k} \in (\mathfrak{h}_1 \oplus \mathfrak{f}_1) \cap \text{isu}(\varepsilon), \quad \tilde{A}_{2k-1} \in (\mathfrak{h}_0 \oplus \mathfrak{f}_0) \cap \text{isu}(\varepsilon). \tag{4.9}$$

If $\tilde{A}_n \in \mathfrak{h}_0$ and $n = 0(\text{mod } 2)$ we have again (4.9). Thus the Mikhailov type reductions are compatible with the action of recursion operator and the general form of the equations related to the GMV_{ε} system is:

$$\text{ad}_S^{-1} \partial_t S = \sum_{k=0}^r a_{2k} (\tilde{\Lambda}_{\pm})^{2k} \text{ad}_S^{-1} (S_x) + \sum_{k=1}^m a_{2k-1} (\tilde{\Lambda}_{\pm})^{2k-1} (\text{ad}_S^{-1} (S_{1x})) \tag{4.10}$$

where a_i are some real constants. This is the hierarchy found in [7] although it was presented there in a different form.

Let us denote the space of rapidly decreasing functions $\tilde{Z}(x)$ with values in \mathfrak{f}_0 (\mathfrak{f}_1) by $\mathfrak{f}_0[x]$ ($\mathfrak{f}_1[x]$). Since \mathfrak{f}_0 and \mathfrak{f}_1 are orthogonal with respect to the Killing form and taking into account Remark 2.1 we have

$$\begin{aligned} \tilde{Z} &\in \mathfrak{f}_0[x], \\ \tilde{\Lambda}_{\pm}(\tilde{Z}) &= i \text{ad}_S^{-1} \left(\partial_x \tilde{Z} + \frac{S_{1x}}{4} \int_{\pm\infty}^x \langle \tilde{Z}, S_{1y} \rangle dy \right) \in \mathfrak{f}_1[x], \end{aligned} \tag{4.11}$$

$$\begin{aligned} \tilde{Z} &\in \mathfrak{f}_1[x], \\ \tilde{\Lambda}_{\pm}(\tilde{Z}) &= i \text{ad}_S^{-1} \left(\partial_x \tilde{Z} + \frac{S_x}{12} \int_{\pm\infty}^x \langle \tilde{Z}, S_y \rangle dy \right) \in \mathfrak{f}_0[x]. \end{aligned} \tag{4.12}$$

Now, let us do the following:

- Put in the above expressions $S = -L_1$ (in order to express everything through L_1).

- Take into account that what is called Killing form in [7] and denoted $\langle X, Y \rangle$ is not the canonical definition of the Killing form, it is in fact $\text{tr}(XY)$ and as we know in our notation $\langle X, Y \rangle = 6 \text{tr}(XY)$.
- Take into account that because \tilde{Z} is orthogonal to \mathfrak{h}_S we have $\langle \tilde{Z}(x), S_x \rangle = -\langle \tilde{Z}_x, S(x) \rangle$ and $\langle \tilde{Z}(x), S_{1x} \rangle = -\langle \tilde{Z}_x, S_1(x) \rangle$.

Then one could convince himself that the operators $\Lambda_1^\pm, \Lambda_2^\pm$ introduced in [7] are defined by the expressions standing in the right hand sides of (4.11) and (4.12), see [22]. In other words, $\Lambda_1^\pm, \Lambda_2^\pm$ are restrictions of the operators $\tilde{\Lambda}_\pm$ on the spaces $\mathfrak{f}_0[x]$ and $\mathfrak{f}_1[x]$ respectively. If the hierarchy of the soliton equations (4.10) is written in terms of $\Lambda_1^\pm, \Lambda_2^\pm$ it acquires the form that was presented in [7].

5. Completeness relations for the GMV $_\pm$ system

We have seen that the consideration of the NLEEs hierarchy (4.10) shows that the operator ‘moving’ the equations along the hierarchy is $\tilde{\Lambda}_\pm^2$ and it does not depend on the real form. The geometric interpretation of the hierarchies and their conservation laws which is developed for the case of GMV $_+$ [24] also suggests that the appropriate operator we must consider is $\tilde{\Lambda}_\pm^2$. It remains to consider the third important aspect of the recursion operators – their relations to the so-called expansions over the adjoint solutions. In that approach one considers these expansions and then finds operators, for which the adjoint solutions entering into them are eigenfunctions. The importance of the adjoint solutions is related to the fact that they form complete systems, so roughly speaking, one can expand S and its variation over them and then have the famous interpretation of the inverse scattering method as a generalized Fourier transform, see [8, 14]. Recently, the theory of the expansions over adjoint solutions has been extended to the case when there are reductions. In [11], the theory for the CBC systems with \mathbb{Z}_p reductions was presented in full taking into account the discrete spectrum while in [25] is discussed also the gauge-covariant formulation of the expansions with \mathbb{Z}_p reductions. The geometric aspects of the NLEEs associated with CBC systems in canonical and pole gauge with reductions were presented recently in [26].

Our approach will be the following. We start from the spectral theory of the generating operators for (3.12) (a GZS system in canonical gauge) which is very well known and from it obtain the theory of the generating operators for GMV $_\pm$ (a GZS system in pole gauge). So let us sketch the theory for GZS in canonical gauge, for all the details see [8, 20]. The adjoint solutions for the system GZS system (3.12) are defined as follows:

$$\mathbf{e}_\alpha^\pm := \pi_0(\chi^\pm E_\alpha \hat{\chi}^\pm), \quad \alpha \in \Delta \tag{5.1}$$

where π_0 is the orthogonal projector on \mathfrak{h}^\perp where \mathfrak{h} is the Cartan subalgebra of $\mathfrak{sl}(3, \mathbb{C})$ and $\hat{\chi}$ stands for the matrix inverse to $\chi \in SL(3, \mathbb{C})$. The generating operators Λ_\pm then have the form:

$$\Lambda_\pm(Y(x)) = \text{ad}_{J_0}^{-1} \left[i\partial_x Y + \pi_0 \text{ad}_q Y(x) + \text{ad}_q \int_{\pm\infty}^x (\mathbf{1} - \pi_0) \text{ad}_q Y(y) dy \right]. \tag{5.2}$$

As mentioned, the above formula together with its derivation could be found in many sources, see for example [5, 8, 10, 11]. From the asymptotic behaviour of the solutions χ^\pm follows that for $\alpha > 0$

we have

$$\begin{aligned} \Lambda_-(\mathbf{e}_\alpha^+(x, \lambda)) &= \lambda \mathbf{e}_\alpha^+(x, \lambda), & \Lambda_-(\mathbf{e}_{-\alpha}^-(x, \lambda)) &= \lambda \mathbf{e}_{-\alpha}^-(x, \lambda) \\ \Lambda_+(\mathbf{e}_{-\alpha}^+(x, \lambda)) &= \lambda \mathbf{e}_{-\alpha}^+(x, \lambda), & \Lambda_+(\mathbf{e}_\alpha^-(x, \lambda)) &= \lambda \mathbf{e}_\alpha^-(x, \lambda). \end{aligned} \tag{5.3}$$

(It is of no importance whether or not we have some reductions). Let us define now the adjoint solutions for the GMV $_{\pm}$ in the following way:

$$\tilde{\mathbf{e}}_\alpha^\pm := \pi_S(\tilde{\chi}^\pm E_\alpha \hat{\chi}^\pm), \quad \alpha \in \Delta. \tag{5.4}$$

Let us recall now that we have a reduction defined by the automorphism h . Taking into account that h commutes with π_S (see Proposition (2.1)) and the properties of ψ_0 (see (3.11)) we see that $h(\tilde{\mathbf{e}}_\alpha^\pm)(x, \lambda)$ equals

$$\pi_S(H \tilde{\chi}^\pm E_\alpha \hat{\chi}^\pm H)(x, \lambda) = \pi_S(\tilde{\chi}^\mp K E_\alpha K \hat{\chi}^\mp)(x, -\lambda) = \pi_S(\tilde{\chi}^\mp E_{\mathcal{H}\alpha} \hat{\chi}^\mp)(x, -\lambda)$$

so we have:

$$h(\tilde{\mathbf{e}}_\alpha^\pm)(x, \lambda) = \tilde{\mathbf{e}}_{\mathcal{H}\alpha}^\mp(x, -\lambda). \tag{5.5}$$

One proves easily that $\pi_S = \text{Ad}(\psi_0^{-1}) \circ \pi_0 \circ \text{Ad}(\psi_0)$, and then one sees that for $\alpha \in \Delta$ we have that $\tilde{\mathbf{e}}_\alpha^\pm(x, \lambda) = \text{Ad}(\psi_0^{-1})\mathbf{e}_\alpha^\pm(x, \lambda)$. Here $\text{Ad}(r)$ denotes the adjoint action of the group on its algebra, that is $\text{Ad}(r)X = rXr^{-1}$. All the theory then develops in a similar way as in the case when we make a gauge transformation with a Jost solution which is very well known [10]. For example, one has that

$$\tilde{\Lambda}_\pm = \text{Ad}(\psi_0^{-1}) \circ \Lambda_\pm \circ \text{Ad}(\psi_0). \tag{5.6}$$

As a consequence, for $\alpha \in \Delta_+$ we have

$$\begin{aligned} \tilde{\Lambda}_-(\tilde{\mathbf{e}}_\alpha^+(x, \lambda)) &= \lambda \tilde{\mathbf{e}}_\alpha^+(x, \lambda), & \tilde{\Lambda}_-(\tilde{\mathbf{e}}_{-\alpha}^-(x, \lambda)) &= \lambda \tilde{\mathbf{e}}_{-\alpha}^-(x, \lambda), \\ \tilde{\Lambda}_+(\tilde{\mathbf{e}}_{-\alpha}^+(x, \lambda)) &= \lambda \tilde{\mathbf{e}}_{-\alpha}^+(x, \lambda), & \tilde{\Lambda}_+(\tilde{\mathbf{e}}_\alpha^-(x, \lambda)) &= \lambda \tilde{\mathbf{e}}_\alpha^-(x, \lambda). \end{aligned} \tag{5.7}$$

We are now ready to discuss the completeness of the adjoint solutions for the GZS system in pole gauge (which under reductions becomes our GMV $_{\pm}$ problem). In order to simplify the formulas we shall have further, let us adopt the following notation: for the functions $X(x), Y(x) : \mathbb{R} \rightarrow \mathfrak{sl}(3, \mathbb{C})$ we put

$$\langle\langle X, Y \rangle\rangle := \int_{-\infty}^{+\infty} \langle X(x), Y(x) \rangle dx. \tag{5.8}$$

First, we write the completeness relations for the 'adjoint' solutions of the canonical gauge GZS system (details can be found in [5, 8]). We also remind that we disregard the discrete spectrum. Then for the GZS system in canonical gauge (3.12) the following theorem holds:

Theorem 5.1. *Let \mathcal{S} be the space of sufficiently smooth functions defined on the real axis with values in \mathfrak{h}^\perp tending fast enough to zero when $x \rightarrow \pm\infty$. Then for every $Z(x) \in \mathcal{S}$ the following*

expansion formulas hold ($\eta = \pm$):

$$Z(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} \mathbf{e}_\alpha^\eta(x, \lambda) \langle \langle \mathbf{e}_{-\alpha}^\eta, [J_0, Z] \rangle \rangle - \mathbf{e}_{-\alpha}^{-\eta}(x, \lambda) \langle \langle \mathbf{e}_\alpha^{-\eta}, [J_0, Z] \rangle \rangle \right] d\lambda. \quad (5.9)$$

Let us make now a gauge transformation in the above completeness relations for the GZS system in canonical gauge, one will easily see that they transform to completeness relations for the GZS system in pole gauge (which becomes GMV $_{\pm}$ if we impose reductions):

Theorem 5.2. *Let $\tilde{\mathcal{S}}$ be the space of sufficiently smooth functions defined on the real axis with values in $\mathfrak{h}_S^\perp(x)$ tending fast enough to zero when $x \rightarrow \pm\infty$. Then for every $\tilde{Z}(x) \in \tilde{\mathcal{S}}$ the following expansion formulas hold ($\eta = \pm$):*

$$\tilde{Z}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} \tilde{\mathbf{e}}_\alpha^\eta(x, \lambda) \langle \langle \tilde{\mathbf{e}}_{-\alpha}^\eta, [S, \tilde{Z}] \rangle \rangle - \tilde{\mathbf{e}}_{-\alpha}^{-\eta}(x, \lambda) \langle \langle \tilde{\mathbf{e}}_\alpha^{-\eta}, [S, \tilde{Z}] \rangle \rangle \right] d\lambda. \quad (5.10)$$

Note that we have two expansions here – for $\eta = +$ and $\eta = -$. One is over the eigenfunctions of $\tilde{\Lambda}_-$, the other is over the eigenfunctions of $\tilde{\Lambda}_+$. In fact the above is the spectral theorem for these operators.

Now we assume that we have the reductions defined by h and σ_ε and we are going to transform these expansions so that it will be easier to see the action of h and σ_ε on these expansions when we expand functions that have values in the eigenspaces $\mathfrak{f}_0, \mathfrak{f}_1$ respectively or are ‘real’ or ‘imaginary’ with respect to the complex conjugation σ_ε .

Let us start with the reduction defined by h . We have seen that h acts on the adjoint solutions according to (5.5). Then using the invariance of the Killing form with respect to the action of h we get

$$\begin{aligned} \int_{-\infty}^{+\infty} \langle \tilde{\mathbf{e}}_{-\alpha}^\eta(x, -\lambda), [S, \tilde{Z}] \rangle dx &= \int_{-\infty}^{+\infty} \langle h(\tilde{\mathbf{e}}_{-\alpha}^\eta(x, -\lambda)), h([S, \tilde{Z}]) \rangle dx \\ &= \int_{-\infty}^{+\infty} \langle \tilde{\mathbf{e}}_{-\mathcal{H}\alpha}^{-\eta}(x, \lambda), H[S, \tilde{Z}]H \rangle dx = - \int_{-\infty}^{+\infty} \langle \tilde{\mathbf{e}}_{-\mathcal{H}\alpha}^{-\eta}(x, \lambda), [S, h(\tilde{Z})] \rangle dx. \end{aligned}$$

Therefore

$$\tilde{\mathbf{e}}_\alpha^\eta(x, -\lambda) \langle \langle \tilde{\mathbf{e}}_{-\alpha}^\eta(x, -\lambda), [S, \tilde{Z}] \rangle \rangle = -h(\tilde{\mathbf{e}}_{-\mathcal{H}\alpha}^{-\eta}(x, \lambda)) \int_{-\infty}^{+\infty} \langle \tilde{\mathbf{e}}_{-\mathcal{H}\alpha}^{-\eta}(x, \lambda), [S, h(\tilde{Z})] \rangle dx.$$

Suppose for example $h(\tilde{Z}) = \tilde{Z}$. Then the above is simply written as:

$$\tilde{\mathbf{e}}_\alpha^\eta(x, -\lambda) \langle \langle \tilde{\mathbf{e}}_{-\alpha}^\eta(x, -\lambda), [S, \tilde{Z}] \rangle \rangle = -h(\tilde{\mathbf{e}}_{-\mathcal{H}\alpha}^{-\eta}(x, \lambda)) \langle \langle \tilde{\mathbf{e}}_{-\mathcal{H}\alpha}^{-\eta}(x, \lambda), [S, \tilde{Z}] \rangle \rangle.$$

Since \mathcal{H} maps positive roots into negative and vice versa, we shall have:

$$\begin{aligned} \sum_{\alpha \in \Delta_+} \tilde{\mathbf{e}}_\alpha^\eta(x, -\lambda) \langle \langle \tilde{\mathbf{e}}_{-\alpha}^\eta(x, -\lambda), [S, \tilde{Z}] \rangle \rangle &= - \sum_{\alpha \in \Delta_+} h(\tilde{\mathbf{e}}_{-\alpha}^{-\eta}(x, \lambda)) \langle \langle \tilde{\mathbf{e}}_\alpha^{-\eta}(x, \lambda), [S, \tilde{Z}] \rangle \rangle, \\ \sum_{\alpha \in \Delta_+} \tilde{\mathbf{e}}_{-\alpha}^{-\eta}(x, -\lambda) \langle \langle \tilde{\mathbf{e}}_\alpha^{-\eta}(x, -\lambda), [S, \tilde{Z}] \rangle \rangle &= - \sum_{\alpha \in \Delta_+} h(\tilde{\mathbf{e}}_\alpha^\eta(x, \lambda)) \langle \langle \tilde{\mathbf{e}}_{-\alpha}^\eta(x, \lambda), [S, \tilde{Z}] \rangle \rangle. \end{aligned}$$

Now, let us make the change of variables $\lambda \mapsto -\lambda$ in the integral standing in the right hand side of (5.10) and use the above relations. We obtain:

$$\tilde{Z}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} h(\tilde{\mathbf{e}}_\alpha^\eta(x, \lambda)) \langle \langle \tilde{\mathbf{e}}_{-\alpha}^\eta, [S, \tilde{Z}] \rangle \rangle - h(\tilde{\mathbf{e}}_{-\alpha}^{-\eta}(x, \lambda)) \langle \langle \tilde{\mathbf{e}}_\alpha^{-\eta}, [S, \tilde{Z}] \rangle \rangle \right] d\lambda. \quad (5.11)$$

But then combining the original expansion with this one we immediately get:

$$\tilde{Z}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} \tilde{s}_\alpha^\eta(x, \lambda) \langle \tilde{\mathbf{e}}_{-\alpha}^\eta, [S, \tilde{Z}] \rangle - \tilde{s}_{-\alpha}^{-\eta}(x, \lambda) \langle \tilde{\mathbf{e}}_\alpha^\eta, [S, \tilde{Z}] \rangle \right] d\lambda \quad (5.12)$$

where

$$\tilde{s}_{\pm\alpha}^\eta(x, \lambda) = \frac{1}{2} (\tilde{\mathbf{e}}_{\pm\alpha}^\eta(x, \lambda) + h(\tilde{\mathbf{e}}_{\pm\alpha}^\eta(x, \lambda))). \quad (5.13)$$

In view of the formulas we shall write a little further, we also note, that one has

$$\begin{aligned} \langle \tilde{\mathbf{e}}_{\mp\alpha}^\eta, [S, \tilde{Z}] \rangle &= \langle h(\tilde{\mathbf{e}}_{\mp\alpha}^\eta), h([S, \tilde{Z}]) \rangle = \\ &= -\langle h(\tilde{\mathbf{e}}_{\mp\alpha}^\eta), [S, \tilde{Z}] \rangle \end{aligned}$$

so one can write

$$\langle \tilde{\mathbf{e}}_{\mp\alpha}^\eta, [S, \tilde{Z}] \rangle = \langle \tilde{\mathbf{a}}_{\mp\alpha}^\eta, [S, \tilde{Z}] \rangle$$

where

$$\tilde{\mathbf{a}}_{\pm\alpha}^\eta(x, \lambda) = \frac{1}{2} (\tilde{\mathbf{e}}_{\pm\alpha}^\eta(x, \lambda) - h(\tilde{\mathbf{e}}_{\pm\alpha}^\eta(x, \lambda))). \quad (5.14)$$

Thus finally we obtain:

$$\tilde{Z}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} \tilde{s}_\alpha^\eta(x, \lambda) \langle \tilde{\mathbf{a}}_{-\alpha}^\eta, [S, \tilde{Z}] \rangle - \tilde{s}_{-\alpha}^{-\eta}(x, \lambda) \langle \tilde{\mathbf{a}}_\alpha^\eta, [S, \tilde{Z}] \rangle \right] d\lambda. \quad (5.15)$$

If instead of $h(\tilde{Z}) = \tilde{Z}$ we assume that $h(\tilde{Z}) = -\tilde{Z}$ then in the same manner we shall obtain

$$\begin{aligned} \tilde{Z}(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} \tilde{\mathbf{a}}_\alpha^\eta(x, \lambda) \langle \tilde{\mathbf{e}}_{-\alpha}^\eta, [S, \tilde{Z}] \rangle - \tilde{\mathbf{a}}_{-\alpha}^{-\eta}(x, \lambda) \langle \tilde{\mathbf{e}}_\alpha^\eta, [S, \tilde{Z}] \rangle \right] d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} \tilde{\mathbf{a}}_\alpha^\eta(x, \lambda) \langle \tilde{\mathbf{s}}_{-\alpha}^\eta, [S, \tilde{Z}] \rangle - \tilde{\mathbf{a}}_{-\alpha}^{-\eta}(x, \lambda) \langle \tilde{\mathbf{s}}_\alpha^\eta, [S, \tilde{Z}] \rangle \right] d\lambda. \end{aligned} \quad (5.16)$$

Naturally,

$$h(\tilde{\mathbf{s}}_{\pm\alpha}^\eta(x, \lambda)) = \tilde{\mathbf{s}}_{\pm\alpha}^\eta(x, \lambda), \quad h(\tilde{\mathbf{a}}_{\pm\alpha}^\eta(x, \lambda)) = -\tilde{\mathbf{a}}_{\pm\alpha}^\eta(x, \lambda). \quad (5.17)$$

Thus in case $h(\tilde{Z}) = \tilde{Z}$ or $h(\tilde{Z}) = -\tilde{Z}$ the expansions could be written in terms of new sets of adjoint solutions, $\tilde{\mathbf{s}}_{\pm\alpha}^\eta(x, \lambda)$ in the first case and $\tilde{\mathbf{a}}_{\pm\alpha}^\eta(x, \lambda)$ in the second, that reflect the symmetry of \tilde{Z} .

Returning to the properties of the new families we introduced, we easily see that $\tilde{\mathbf{s}}_{\pm\alpha}^\eta(x, \lambda)$ and $\tilde{\mathbf{a}}_{\pm\alpha}^\eta(x, \lambda)$ are not eigenfunctions of $\tilde{\Lambda}_-$ or $\tilde{\Lambda}_+$. In order to see what happens, we look how $\tilde{\Lambda}_\pm$ is related to h and σ_ε .

Lemma 5.1. *The following relations hold:*

$$h \circ \tilde{\Lambda}_{\pm} = -\tilde{\Lambda}_{\pm} \circ h, \quad \sigma_{\varepsilon} \circ \tilde{\Lambda}_{\pm} = -\tilde{\Lambda}_{\pm} \circ \sigma_{\varepsilon}. \quad (5.18)$$

Proof. The proof of the first relation is straightforward, let us prove the second one. First, assume that $\sigma_{\varepsilon}(\tilde{Z}) = \tilde{Z}$ (the case when $\sigma_{\varepsilon}(\tilde{Z}) = -\tilde{Z}$ is treated in a similar way). First, we remark that $\sigma_{\varepsilon}(S_x) = -S_x$ and that $\sigma_{\varepsilon}(S_1) = -S_1$, consequently $\sigma_{\varepsilon}(S_{1x}) = -S_{1x}$. Next, we have

$$\langle \tilde{Z}, S_x \rangle^* = \langle \sigma_{\varepsilon}(\tilde{Z}), \sigma_{\varepsilon}(S_x) \rangle = -\langle \tilde{Z}, S_x \rangle$$

and similarly

$$\langle \tilde{Z}, S_{1x} \rangle^* = \langle \sigma_{\varepsilon}(\tilde{Z}), \sigma_{\varepsilon}(S_{1x}) \rangle = -\langle \tilde{Z}, S_{1x} \rangle.$$

So in case $\sigma_{\varepsilon}(\tilde{Z}) = \tilde{Z}$ (we call \tilde{Z} real in this case) the integrands in the formula for recursion operator are purely imaginary while for $\sigma_{\varepsilon}(\tilde{Z}) = -\tilde{Z}$ (we then say that \tilde{Z} is purely imaginary) these expressions will be real. Then, taking into account (2.21) we get

$$\begin{aligned} -\sigma_{\varepsilon} \tilde{\Lambda}_{\pm}(\tilde{Z}) &= \\ \text{iad}_S^{-1} \pi_S \left(\partial_x(\sigma_{\varepsilon} \tilde{Z}) + \frac{\sigma_{\varepsilon} S_x}{12} \int_{\pm\infty}^x \langle \tilde{Z}, S_y \rangle^* dy + \frac{(\sigma_{\varepsilon} S_{1x})}{4} \int_{\pm\infty}^x \langle \tilde{Z}, S_{1y} \rangle^* dy \right) &= \\ \text{iad}_S^{-1} \pi_S \left(\partial_x(\tilde{Z}) + \frac{S_x}{12} \int_{\pm\infty}^x \langle \tilde{Z}, S_y \rangle dy + \frac{S_{1x}}{4} \int_{\pm\infty}^x \langle \tilde{Z}, S_{1y} \rangle dy \right) &= \\ \tilde{\Lambda}_{\pm}(\tilde{Z}) &= \tilde{\Lambda}_{\pm}(\sigma_{\varepsilon} \tilde{Z}). \end{aligned}$$

As mentioned, one can show in a similar way that when $\sigma_{\varepsilon}(\tilde{Z}) = -\tilde{Z}$ we have the same relation. Finally, as each \tilde{Z} could be written in a unique way as $\tilde{Z} = \tilde{Z}_1 + \tilde{Z}_2$ where $\sigma_{\varepsilon}(\tilde{Z}_1) = -\tilde{Z}_1$, $\sigma_{\varepsilon}(\tilde{Z}_2) = \tilde{Z}_2$ then the second relation in (5.18) is proved for arbitrary \tilde{Z} .

Because of the relations (5.7) and the above lemma, for $\alpha \in \Delta_+$ one has

$$\begin{aligned} \tilde{\Lambda}_-(\tilde{\mathfrak{s}}_{\alpha}^+(x, \lambda)) &= \lambda \tilde{\mathfrak{a}}_{\alpha}^+(x, \lambda), & \tilde{\Lambda}_-(\tilde{\mathfrak{s}}_{-\alpha}^-(x, \lambda)) &= \lambda \tilde{\mathfrak{a}}_{-\alpha}^-(x, \lambda), \\ \tilde{\Lambda}_-(\tilde{\mathfrak{a}}_{\alpha}^+(x, \lambda)) &= \lambda \tilde{\mathfrak{s}}_{\alpha}^+(x, \lambda), & \tilde{\Lambda}_-(\tilde{\mathfrak{a}}_{-\alpha}^-(x, \lambda)) &= \lambda \tilde{\mathfrak{s}}_{-\alpha}^-(x, \lambda), \end{aligned} \quad (5.19)$$

$$\begin{aligned} \tilde{\Lambda}_+(\tilde{\mathfrak{s}}_{-\alpha}^+(x, \lambda)) &= \lambda \tilde{\mathfrak{a}}_{-\alpha}^+(x, \lambda), & \tilde{\Lambda}_+(\tilde{\mathfrak{s}}_{\alpha}^-(x, \lambda)) &= \lambda \tilde{\mathfrak{a}}_{\alpha}^-(x, \lambda), \\ \tilde{\Lambda}_+(\tilde{\mathfrak{a}}_{-\alpha}^+(x, \lambda)) &= \lambda \tilde{\mathfrak{s}}_{-\alpha}^+(x, \lambda), & \tilde{\Lambda}_+(\tilde{\mathfrak{a}}_{\alpha}^-(x, \lambda)) &= \lambda \tilde{\mathfrak{s}}_{\alpha}^-(x, \lambda) \end{aligned} \quad (5.20)$$

and one sees that the functions in the expansions when we have some symmetry with respect to h are eigenfunctions for $\tilde{\Lambda}_{\pm}^2$ ($\tilde{\Lambda}_{\pm}^2$) with eigenvalue λ^2 . This together with the fact that when recursively finding the coefficients for the pencil (4.1) one effectively uses $\tilde{\Lambda}_{\pm}^2$ has led to the interpretation that in case we have \mathbb{Z}_2 reduction defined by h the role of the Generating Operator is played by $\tilde{\Lambda}_{\pm}^2$.

Let us consider now the other reduction that we have in the GMV_{\pm} system, the one defined by the complex conjugation σ_{ε} . In accordance with Theorem 3.2 the FAS fulfill $Q_{\varepsilon}(\tilde{\chi}^{\mp}(x, \lambda))^{\dagger} Q_{\varepsilon} = (\tilde{\chi}^{\pm}(x, \lambda^*))^{-1}$ so on the real axis $Q_{\varepsilon}(\tilde{\chi}^{\mp}(x, \lambda))^{\dagger} Q_{\varepsilon} = (\tilde{\chi}^{\pm}(x, \lambda^*))^{-1}$. Then taking into account

Proposition (2.1) for $\alpha \in \Delta$

$$Q_\varepsilon \tilde{\mathbf{e}}_\alpha^\eta(x, \lambda)^\dagger Q_\varepsilon = \pi_S \tilde{\chi}^{-\eta}(x, \lambda) Q_\varepsilon E_\alpha^\dagger Q_\varepsilon \hat{\chi}^{-\eta}(x, \lambda^*)$$

and we need to take into account how $X \mapsto \sigma_\varepsilon(X) = -Q_\varepsilon X^\dagger Q_\varepsilon$ acts on the Cartan-Weil basis. One easily sees that

$$\begin{aligned} \sigma_+ E_{\pm\alpha_1} &= -E_{\mp\alpha_1}, & \sigma_+ E_{\pm\alpha_2} &= -E_{\mp\alpha_2}, & \sigma_+ E_{\pm\alpha_3} &= -E_{\mp\alpha_3} \\ \sigma_- E_{\pm\alpha_1} &= E_{\mp\alpha_1}, & \sigma_- E_{\pm\alpha_2} &= E_{\mp\alpha_2}, & \sigma_- E_{\pm\alpha_3} &= -E_{\mp\alpha_3} \\ \sigma_\varepsilon H_{\alpha_j} &= H_{\alpha_j}, & j &= 1, 2. \end{aligned} \tag{5.21}$$

In order to write down these relations in a concise way, we introduce the symbol $q_\varepsilon(\alpha)$ which is equal to -1 if $\varepsilon = 1$ for arbitrary $\alpha \in \Delta$ but if $\varepsilon = -1$ then $q_\varepsilon(\alpha) = 1$ for $\alpha = \pm\alpha_1, \pm\alpha_2$ and $q_\varepsilon(\pm\alpha_3) = -1$. Then $q_\varepsilon(\alpha)q_\varepsilon(-\alpha) = 1$ and we have

$$\sigma_\varepsilon E_\alpha = q_\varepsilon(\alpha)E_{-\alpha}, \quad \sigma_\varepsilon H_{\alpha_i} = -H_{\alpha_i}, \quad i = 1, 2. \tag{5.22}$$

With the above notation $Q_\varepsilon \tilde{\mathbf{e}}_\alpha^\eta(x, \lambda)^\dagger Q_\varepsilon = -q_\varepsilon(\alpha) \tilde{\mathbf{e}}_{-\alpha}^{-\eta}(x, \lambda)$ or if one prefers expression through the complex conjugation

$$\sigma_\varepsilon(\tilde{\mathbf{e}}_\alpha^\eta(x, \lambda)) = q_\varepsilon(\alpha) \tilde{\mathbf{e}}_{-\alpha}^{-\eta}(x, \lambda^*). \tag{5.23}$$

Naturally, we have

$$\sigma_\varepsilon(\tilde{\mathbf{e}}_\alpha^\eta(x, \lambda)) = q_\varepsilon(\alpha) \tilde{\mathbf{e}}_{-\alpha}^{-\eta}(x, \lambda), \quad \lambda \in \mathbb{R}. \tag{5.24}$$

Consider now the coefficients $\langle\langle \tilde{\mathbf{e}}_{-\alpha}^\eta, [S, \tilde{Z}] \rangle\rangle$ introduced in the above and let us assume that $\sigma_\varepsilon \tilde{Z} = \tilde{Z}$. We have

$$\langle\langle \tilde{\mathbf{e}}_{-\alpha}^\eta, [S, \tilde{Z}] \rangle\rangle = \langle\langle \sigma_\varepsilon \tilde{\mathbf{e}}_{-\alpha}^\eta, \sigma_\varepsilon [S, \tilde{Z}] \rangle\rangle^* = -q_\varepsilon(\alpha) \langle\langle \tilde{\mathbf{e}}_{-\alpha}^{-\eta}, [S, \tilde{Z}] \rangle\rangle^*.$$

Consequently:

$$\sigma_\varepsilon(\langle\langle \tilde{\mathbf{e}}_{-\alpha}^\eta(\lambda), [S, \tilde{Z}] \rangle\rangle \tilde{\mathbf{e}}_\alpha^\eta(x, \lambda)) = -\langle\langle \tilde{\mathbf{e}}_{-\alpha}^{-\eta}(\lambda), [S, \tilde{Z}] \rangle\rangle \tilde{\mathbf{e}}_{-\alpha}^{-\eta}(x, \lambda). \tag{5.25}$$

In case $\sigma_\varepsilon \tilde{Z} = -\tilde{Z}$ we obtain in a similar way

$$\sigma_\varepsilon(\langle\langle \tilde{\mathbf{e}}_{-\alpha}^\eta(\lambda), [S, \tilde{Z}] \rangle\rangle \tilde{\mathbf{e}}_\alpha^\eta(x, \lambda)) = \langle\langle \tilde{\mathbf{e}}_{-\alpha}^{-\eta}(\lambda), [S, \tilde{Z}] \rangle\rangle \tilde{\mathbf{e}}_{-\alpha}^{-\eta}(x, \lambda). \tag{5.26}$$

One could also see that in case $\sigma_\varepsilon \tilde{Z} = \tilde{Z}$ we have

$$\langle\langle \tilde{\mathbf{e}}_{-\alpha}^\eta(\lambda), [S, \tilde{Z}] \rangle\rangle^* = \langle\langle \sigma_\varepsilon \tilde{\mathbf{e}}_{-\alpha}^\eta(\lambda), \sigma_\varepsilon [S, \tilde{Z}] \rangle\rangle = -\langle\langle \sigma_\varepsilon \tilde{\mathbf{e}}_{-\alpha}^\eta(\lambda), [S, \tilde{Z}] \rangle\rangle.$$

Thus when $\sigma_\varepsilon \tilde{Z} = \tilde{Z}$ (respectively $\sigma_\varepsilon \tilde{Z} = -\tilde{Z}$) the expansions (5.10) acquire the form:

$$\tilde{Z}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} (\mathbf{1} + \sigma_\varepsilon) (\tilde{\mathbf{e}}_\alpha^\eta(x, \lambda) \langle\langle \tilde{\mathbf{e}}_{-\alpha}^\eta(\lambda), [S, \tilde{Z}] \rangle\rangle) \right] d\lambda \tag{5.27}$$

and

$$\tilde{Z}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} (\mathbf{1} - \sigma_\varepsilon) (\tilde{\mathbf{e}}_\alpha^\eta(x, \lambda) \langle\langle \tilde{\mathbf{e}}_{-\alpha}^\eta(\lambda), [S, \tilde{Z}] \rangle\rangle) \right] d\lambda \tag{5.28}$$

respectively.

In order to obtain expansions when both reductions are present, we notice that since σ_ε and h commute, in case $\sigma_\varepsilon \tilde{Z} = \tilde{Z}$ one has:

$$\begin{aligned} \sigma_\varepsilon(\langle \langle \tilde{\mathbf{e}}_{-\alpha}^\eta, [S, \tilde{Z}] \rangle \rangle \tilde{\mathbf{s}}_\alpha^\eta(x, \lambda)) &= -\langle \langle \tilde{\mathbf{e}}_{-\alpha}^{-\eta}, [S, \tilde{Z}] \rangle \rangle \tilde{\mathbf{s}}_{-\alpha}^{-\eta}(x, \lambda), \\ \sigma_\varepsilon(\langle \langle \tilde{\mathbf{e}}_{-\alpha}^\eta, [S, \tilde{Z}] \rangle \rangle \tilde{\mathbf{a}}_\alpha^\eta(x, \lambda)) &= -\langle \langle \tilde{\mathbf{e}}_{-\alpha}^{-\eta}, [S, \tilde{Z}] \rangle \rangle \tilde{\mathbf{a}}_{-\alpha}^{-\eta}(x, \lambda) \end{aligned} \tag{5.29}$$

and in case $\sigma_\varepsilon \tilde{Z} = -\tilde{Z}$ we have

$$\begin{aligned} \sigma_\varepsilon(\langle \langle \tilde{\mathbf{e}}_{-\alpha}^\eta, [S, \tilde{Z}] \rangle \rangle \tilde{\mathbf{s}}_\alpha^\eta(x, \lambda)) &= \langle \langle \tilde{\mathbf{e}}_{-\alpha}^{-\eta}, [S, \tilde{Z}] \rangle \rangle \tilde{\mathbf{s}}_{-\alpha}^{-\eta}(x, \lambda), \\ \sigma_\varepsilon(\langle \langle \tilde{\mathbf{e}}_{-\alpha}^\eta, [S, \tilde{Z}] \rangle \rangle \tilde{\mathbf{a}}_\alpha^\eta(x, \lambda)) &= \langle \langle \tilde{\mathbf{e}}_{-\alpha}^{-\eta}, [S, \tilde{Z}] \rangle \rangle \tilde{\mathbf{a}}_{-\alpha}^{-\eta}(x, \lambda). \end{aligned} \tag{5.30}$$

Then we obtain:

Theorem 5.3. *The following expansions hold:*

- In case $\sigma_\varepsilon \tilde{Z} = \tilde{Z}$, $h\tilde{Z} = \tilde{Z}$

$$\tilde{Z}(x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} (\mathbf{1} + \sigma_\varepsilon) (\tilde{\mathbf{s}}_\alpha^\eta(x, \lambda) \langle \langle \tilde{\mathbf{e}}_{-\alpha}^\eta, [S, \tilde{Z}] \rangle \rangle(\lambda)) \right] d\lambda. \tag{5.31}$$

- In case $\sigma_\varepsilon \tilde{Z} = \tilde{Z}$, $h\tilde{Z} = -\tilde{Z}$

$$\tilde{Z}(x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} (\mathbf{1} + \sigma_\varepsilon) (\tilde{\mathbf{a}}_\alpha^\eta(x, \lambda) \langle \langle \tilde{\mathbf{e}}_{-\alpha}^\eta, [S, \tilde{Z}] \rangle \rangle(\lambda)) \right] d\lambda. \tag{5.32}$$

- In case $\sigma_\varepsilon \tilde{Z} = -\tilde{Z}$, $h\tilde{Z} = \tilde{Z}$

$$\tilde{Z}(x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} (\mathbf{1} - \sigma_\varepsilon) (\tilde{\mathbf{s}}_\alpha^\eta(x, \lambda) \langle \langle \tilde{\mathbf{e}}_{-\alpha}^\eta, [S, \tilde{Z}] \rangle \rangle(\lambda)) \right] d\lambda. \tag{5.33}$$

- In case $\sigma_\varepsilon \tilde{Z} = -\tilde{Z}$, $h\tilde{Z} = -\tilde{Z}$

$$\tilde{Z}(x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} (\mathbf{1} - \sigma_\varepsilon) (\tilde{\mathbf{a}}_\alpha^\eta(x, \lambda) \langle \langle \tilde{\mathbf{e}}_{-\alpha}^\eta, [S, \tilde{Z}] \rangle \rangle(\lambda)) \right] d\lambda. \tag{5.34}$$

Corollary 5.1. *In view of Lemma 5.1, we see that the functions that stay in the integrands of the above expressions, that is $(\mathbf{1} \pm \sigma_\varepsilon) \tilde{\mathbf{s}}_\alpha^\eta(x, \lambda)$ and $(\mathbf{1} \pm \sigma_\varepsilon) \tilde{\mathbf{a}}_\alpha^\eta(x, \lambda)$ remain eigenfunctions of $\tilde{\Lambda}_\pm^2$ with eigenvalues λ^2 , that is, the reduction defined by a real form does not change the recursion operators.*

Let us make some final comments about the expansions we obtained. In the case GMV_+ the families $\tilde{\mathbf{s}}_{\pm\alpha}^\eta(x, \lambda)$ and $\tilde{\mathbf{a}}_{\pm\alpha}^\eta(x, \lambda)$ as well as the relations (5.19), (5.20) were introduced in [7]. However, they were written in terms of the restrictions of $\tilde{\Lambda}_\pm$ on the spaces $\mathfrak{f}_0[x]$ ($\mathfrak{f}_1[x]$), namely through the operators Λ_1^\pm , Λ_2^\pm we mentioned already. This complicates their form and obscures their meaning. We believe that the form in which we cast them now and in relation to Lemma 5.1 is are much easier to understand. As about the expansions (5.15), (5.16) and about expansions in Theorem 5.3 they were not presented until now in their general form but only for the specific cases of expansions of the functions S_x and $(S_1)_x$. Moreover, since in [7] the theory of the GMV_+ have been developed not in parallel (though by analogy) to the corresponding system in canonical gauge, the knowledge

about the GZS system in canonical gauge could not be exploited in full and everything ought to be developed from the beginning. In particular, there have been considered more restricted boundary conditions than necessary, namely the case $u_{\pm} = 0, v_{\pm} = \exp(i\Phi_{\pm})$. Consequently, all the results were obtained in less generality and with more effort.

The above expansions have big theoretical interest. They show that similar to the general position case (without reductions) the 'potential' and its variation could be expanded over a family of functions that are eigenfunctions of the operator that appears when one recursively solves the equations for the coefficients \tilde{A}_k . Of course, it is important also to know that when we have reductions the expansions could be written in a different form. Practically however, it is easier to adopt the following attitude. We could simply 'forget' that we have reductions. In other words, the whole Gauge Covariant Theory of the Generating Operators will proceed in the usual way, with or without reductions. Basically, all the formulas will remain true, one must simply take into account the following:

- Some additional symmetries of the scattering data (see the next Section) which lead to symmetries in the corresponding Riemann-Hilbert problem when one uses it for finding exact solutions.
- The operators that permit to 'move' along the hierarchies are $\tilde{\Lambda}_{\pm}^2$.
- Some of the conservation laws and Poisson structures in the corresponding hierarchies trivialize.

For the first of these items one can see [7], as for the second and the third, according to the general theory (see [8] or [20] specifically for the case of $\mathfrak{sl}(3, \mathbb{C})$) we have:

- The NLEEs (4.10) have the following series of conservation laws:

$$D_B^{(s)} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^y \langle S_y, \tilde{\Lambda}_{\pm}^s(\text{ad}_S^{-1} \tilde{B}_y) \rangle dy \right] dx, \quad B \in \mathfrak{h}, \quad B = \text{const}, \quad \tilde{B} = gBg^{-1} \quad (5.35)$$

where $s = 1, 2, \dots$. Now, if $k(B) = -B$, then $\text{ad}_S^{-1} \tilde{B}_x \in \mathfrak{h}_0$ and if s is even $\tilde{\Lambda}_{\pm}^s(\tilde{B}_x) \in \mathfrak{h}_0[x]$. Since $S_y \in \mathfrak{h}_1[x]$ we have $D_B^{(s)} = 0$. This of course does not happen if s is odd. Analogously, $D_B^{(s)} = 0$ if s is odd and $k(B) = B$. Thus indeed some of the conservation laws trivialize.

- The NLEEs (4.10) are Hamiltonian with respect to the hierarchy of symplectic forms:

$$\tilde{\Omega}^{(p)}(\delta S_1, \delta S_2) = \int_{-\infty}^{\infty} \langle \delta S_1, \tilde{\Lambda}^p \text{ad}_S^{-1} \delta S_2 \rangle dx, \quad p = 0, 1, 2, \dots \quad (5.36)$$

where $\delta_1 S, \delta_2 S$ are some variations of S and

$$\tilde{\Lambda} = \frac{1}{2}(\tilde{\Lambda}_+ + \tilde{\Lambda}_-). \quad (5.37)$$

Now, since $\langle S, S \rangle = 12$, $\delta_1 S, \delta_2 S$ are orthogonal to S and since $h(S) = -S$ we have $h(\delta_1 S) = -\delta_1 S, h(\delta_2 S) = -\delta_2 S$ so finally $\delta_1 S, \delta_2 S \in \mathfrak{f}_1[x]$. Further, $\text{ad}_S^{-1} \delta_2 S \in \mathfrak{f}_0[x]$ and therefore, if p is even $\tilde{\Omega}^{(p)}$ is identically zero. The simplest case is of course the case $p = 0$, that corresponds to the Kirillov Poisson structure ad_S (restricted to the manifold of the potentials) which obviously trivializes. On the contrary, for p odd, these structures are not

trivial. The symplectic structure for $p = 1$ corresponds to the Poisson tensor $i\partial_x$ (restricted to the manifold of the potentials) and does not trivialize, see [24] for the details.

6. Application. Scattering data

As an application we are going to discuss the implications of the reduction over the scattering data related to the $\mathfrak{sl}(3, \mathbb{C})$ -GZS system in canonical and in pole gauge. First of all, as well-known from the theory of the CBC (GZS) systems, there are essentially two types of scattering data: a) associated with the asymptotic of FAS b) associated with the coefficients in the expansions over the adjoint solutions. All these sets of scattering data are equivalent, but we are not going to enter into this issue here.

6.1. Scattering data related to the asymptotics

As it follows from the general theory of CBC (GZS) systems, the functions $m^\pm(x, \lambda)$ through which we introduced the FAS for (3.12) in case of the real λ have asymptotics, see for example [5] or [20]:

$$\begin{aligned} m^+(x, \lambda) &= e^{i\lambda J_0 x} \chi^+(x, \lambda) \rightarrow S^+(\lambda), & x \rightarrow -\infty, \\ m^+(x, \lambda) &= e^{i\lambda J_0 x} \chi^+(x, \lambda) \rightarrow T^-(\lambda)D^+(\lambda), & x \rightarrow +\infty, \\ m^-(x, \lambda) &= e^{i\lambda J_0 x} \chi^-(x, \lambda) \rightarrow S^-(\lambda), & x \rightarrow -\infty, \\ m^-(x, \lambda) &= e^{i\lambda J_0 x} \chi^-(x, \lambda) \rightarrow T^+(\lambda)D^-(\lambda), & x \rightarrow +\infty. \end{aligned} \tag{6.1}$$

The matrices S^+, T^+ are upper triangular with with diagonal elements equal to 1, the matrices S^-, T^- are lower triangular with with diagonal elements equal to 1 and the matrices D^\pm are diagonal with determinant 1. Besides, $D^+(\lambda)$ could be defined in the upper half-plane \mathbb{C}_+ while $D^-(\lambda)$ could be defined in the lower half-plane \mathbb{C}_- . They are meromorphic and the poles correspond to the discrete spectrum. As agreed, we shall assume that there is no discrete spectrum, so $D^+(\lambda)$ ($D^-(\lambda)$) are analytic in \mathbb{C}_+ (\mathbb{C}_-) respectively.

Since on the real line both $\chi^\pm(x, \lambda)$ exist, there is a non-degenerate matrix $R(\lambda)$ such that $\chi^+(x, \lambda) = \chi^-(x, \lambda)R(\lambda)$ and we get that

$$R(\lambda) = \widehat{S}^-(\lambda)S^+(\lambda) = \widehat{D}^-(\lambda)\widehat{T}^+(\lambda)T^-(\lambda)D^+(\lambda), \quad \lambda \in \mathbb{R}.$$

One could cast this in an equivalent form, introducing a matrix $T(\lambda)$

$$T(\lambda) = T^-(\lambda)D^+(\lambda)\widehat{S}^+(\lambda) = T^+(\lambda)D^-(\lambda)\widehat{S}^-(\lambda), \quad \lambda \in \mathbb{R}. \tag{6.2}$$

The matrices S^+, T^+ are upper-triangular with diagonal elements equal to 1, the matrices S^-, T^- are lower triangular with diagonal elements equal to 1 and the matrices D^\pm are diagonal. The above shows that the matrices we just introduced are factors of two Gauss decompositions of the matrix $T(\lambda)$, which is called the transition matrix. We have the following representations of the Gauss factors:

$$\begin{aligned} S^+(\lambda) &= \exp \sum_{\alpha \in \Delta_+} s_\alpha^+(\lambda) E_\alpha, & S^-(\lambda) &= \exp \sum_{\alpha \in \Delta_+} s_{-\alpha}^-(\lambda) E_{-\alpha}, \\ D^+(\lambda) &= \exp \sum_{i=1}^2 d_i^+(\lambda) H_i, & D^-(\lambda) &= \exp \sum_{i=1}^2 d_i^-(\lambda) H_i, \\ T^+(\lambda) &= \exp \sum_{\alpha \in \Delta_+} t_\alpha^+(\lambda) E_\alpha, & T^-(\lambda) &= \exp \sum_{\alpha \in \Delta_+} t_{-\alpha}^-(\lambda) E_{-\alpha}. \end{aligned} \tag{6.3}$$

As it is known, see [5, 8, 10] each one of the families of functions

$$\begin{aligned}\mathcal{F}_s &= \{s_\alpha^+(\lambda), s_\alpha^-(\lambda), \quad \lambda \in \mathbb{R}, \quad \alpha \in \Delta_+\}, \\ \mathcal{F}_t &= \{t_\alpha^+(\lambda), t_\alpha^-(\lambda), \quad \lambda \in \mathbb{R}, \quad \alpha \in \Delta_+\}\end{aligned}\tag{6.4}$$

could be taken as a scattering data, from which one could reconstruct the potential q (and hence the potential function S). We have reductions and they impose some restrictions on the scattering data. To start with, since $K\chi^\pm(x, \lambda)K = \chi^\mp(x, -\lambda)$, we get that

$$\begin{aligned}KS^+(-\lambda)K &= S^-(\lambda), & KT^-(-\lambda)K &= T^+(\lambda), \\ KD^+(-\lambda)K &= D^-(\lambda).\end{aligned}\tag{6.5}$$

Using the quantities introduced in (6.3), the above relations could be written in an equivalent form: for $\alpha \in \Delta_+$

$$\begin{aligned}s_\alpha^-(-\lambda) &= s_{\mathcal{K}\alpha}^+(\lambda), & t_\alpha^-(-\lambda) &= t_{\mathcal{K}\alpha}^+(\lambda), \\ s_\alpha^+(-\lambda) &= s_{\mathcal{K}\alpha}^-(\lambda), & t_\alpha^+(-\lambda) &= t_{\mathcal{K}\alpha}^-(\lambda), \\ d_1^+(-\lambda) &= -d_2^-(\lambda), & d_2^+(-\lambda) &= -d_1^-(\lambda)\end{aligned}\tag{6.6}$$

where \mathcal{K} is the action of the automorphism K on the roots. The second involution leads to the relations:

$$Q_\varepsilon(S^+(\lambda))^\dagger Q_\varepsilon = (S^-(\lambda))^{-1}, \quad Q_\varepsilon(T^+(\lambda))^\dagger Q_\varepsilon = (T^-(\lambda))^{-1},\tag{6.7}$$

$$\begin{aligned}(D^+(\lambda))^\dagger &= (D^-(\lambda))^{-1}, \\ (s_\alpha^+(\lambda))^* &= q_\varepsilon(\alpha)s_\alpha^-(\lambda), & (t_\alpha^+(\lambda))^* &= q_\varepsilon(\alpha)t_\alpha^-(\lambda), & \alpha \in \Delta_+ \\ (d_i^+(\lambda))^* &= -d_i^-(\lambda), & i &= 1, 2\end{aligned}\tag{6.8}$$

where in all the above relations λ is real.

Now it is not hard to find the corresponding relations for the GMV_\pm system. Indeed, because of (3.18) we get

$$\begin{aligned}e^{i\lambda J_0 x} g_-^{-1} \tilde{\chi}^+(x, \lambda) &\rightarrow S^+(\lambda), & x &\rightarrow -\infty, \\ e^{i\lambda J_0 x} g_+^{-1} \tilde{\chi}^+(x, \lambda) &\rightarrow e^{i\Phi J'} T^-(\lambda) D^+(\lambda), & x &\rightarrow +\infty, \\ e^{i\lambda J_0 x} g_-^{-1} \tilde{\chi}^-(x, \lambda) &\rightarrow S^-(\lambda), & x &\rightarrow -\infty, \\ e^{i\lambda J_0 x} g_+^{-1} \tilde{\chi}^-(x, \lambda) &\rightarrow e^{i\Phi J'} T^+(\lambda) D^-(\lambda), & x &\rightarrow +\infty.\end{aligned}\tag{6.9}$$

Since $J' = H_1 - H_2$ we easily get that

$$\begin{aligned}e^{i\Phi J'} T^-(\lambda) D^+(\lambda) &= \tilde{T}^-(\lambda) \tilde{D}^+(\lambda), \\ e^{i\Phi J'} T^+(\lambda) D^-(\lambda) &= \tilde{T}^+(\lambda) \tilde{D}^-(\lambda)\end{aligned}\tag{6.10}$$

where

$$\begin{aligned}\tilde{D}^+(\lambda) &= e^{i\Phi J'} D^+(\lambda), & \tilde{D}^-(\lambda) &= e^{i\Phi J'} D^-(\lambda), \\ \tilde{T}^\pm(\lambda) &= e^{i\Phi J'} T^\pm(\lambda) e^{-i\Phi J'}.\end{aligned}\tag{6.11}$$

We also put $\tilde{S}^\pm(\lambda) = S^\pm(\lambda)$. Since $KJ'K = J$ the factors $\tilde{D}^\pm(\lambda), \tilde{S}^\pm(\lambda), \tilde{T}^\pm(\lambda)$ obey the same relations (6.5) as the factors $D^\pm(\lambda), S^\pm(\lambda), T^\pm(\lambda)$. Also, we can write

$$\begin{aligned} \tilde{D}^+(\lambda) &= \exp \sum_{i=1}^2 \tilde{d}_i^+(\lambda) H_i, & \tilde{D}^-(\lambda) &= \exp \sum_{i=1}^2 \tilde{d}_i^-(\lambda) H_i \\ \tilde{T}^\pm(\lambda) &= \exp \sum_{\alpha \in \Delta_+} \tilde{t}_{\pm\alpha}^\pm(\lambda) E_{\pm\alpha}, & \tilde{S}^\pm(\lambda) &= \exp \sum_{\alpha \in \Delta_+} \tilde{s}_{\pm\alpha}^\pm(\lambda) E_{\pm\alpha} \end{aligned} \tag{6.12}$$

where $\tilde{s}_{\pm\alpha} = s_{\pm\alpha}$ and

$$\tilde{d}_1^\pm = d_1^\pm + i\Phi, \quad \tilde{d}_2^\pm = d_2^\pm - i\Phi, \quad \tilde{t}_{\pm\alpha}^\pm = t_{\pm\alpha}^\pm \exp \pm i\alpha(J')\Phi. \tag{6.13}$$

So setting $\tilde{S}^\pm(\lambda) = S^\pm(\lambda)$ and $\tilde{s}_{\pm\alpha}^\pm = s_{\pm\alpha}^\pm$ one could put (6.9) in a more symmetric form

$$\begin{aligned} e^{i\lambda J_0 x} g_-^{-1} \tilde{\chi}^+(x, \lambda) &\rightarrow \tilde{S}^+(\lambda), & x &\rightarrow -\infty, \\ e^{i\lambda J_0 x} g_+^{-1} \tilde{\chi}^+(x, \lambda) &\rightarrow \tilde{T}^-(\lambda) \tilde{D}^+(\lambda), & x &\rightarrow +\infty, \\ e^{i\lambda J_0 x} g_-^{-1} \tilde{\chi}^-(x, \lambda) &\rightarrow \tilde{S}^-(\lambda), & x &\rightarrow -\infty, \\ e^{i\lambda J_0 x} g_+^{-1} \tilde{\chi}^-(x, \lambda) &\rightarrow \tilde{T}^+(\lambda) \tilde{D}^-(\lambda), & x &\rightarrow +\infty. \end{aligned} \tag{6.14}$$

One can see that again for $\alpha \in \Delta_+$

$$\begin{aligned} \tilde{s}_{-\alpha}^-(\lambda) &= \tilde{s}_{-\mathcal{H}\alpha}^+(\lambda), & \tilde{t}_{-\alpha}^-(\lambda) &= \tilde{t}_{-\mathcal{H}\alpha}^+(\lambda), \\ \tilde{s}_{\alpha}^+(\lambda) &= \tilde{s}_{\mathcal{H}\alpha}^-(\lambda), & \tilde{t}_{\alpha}^+(\lambda) &= \tilde{t}_{\mathcal{H}\alpha}^-(\lambda), \\ \tilde{d}_1^+(\lambda) &= -\tilde{d}_2^-(\lambda), & \tilde{d}_2^+(\lambda) &= -\tilde{d}_1^-(\lambda) \end{aligned} \tag{6.15}$$

and we get that the families

$$\begin{aligned} \tilde{\mathcal{F}}_s &= \{\tilde{s}_{\alpha}^+(\lambda), \tilde{s}_{-\alpha}^-(\lambda), \lambda \in \mathbb{R}, \alpha \in \Delta_+\}, \\ \tilde{\mathcal{F}}_t &= \{\tilde{t}_{\alpha}^+(\lambda), \tilde{t}_{-\alpha}^-(\lambda), \lambda \in \mathbb{R}, \alpha \in \Delta_+\} \end{aligned} \tag{6.16}$$

could be considered as scattering data for the GMV_{\pm} system. Taking into account that Φ is integral of motion for the NLEEs associated to the GMV_{\pm} system (see Remark 3.1) we notice that passing from quantities without tilde to quantities with tilde is essentially the same thing, that is S could be reconstructed using the same scattering data as the scattering data for q , something that of course could be expected.

6.2. Scattering data related to the expansions over the adjoint solutions

Now let us turn our attention to the scattering data associated with the expansions over the adjoint solutions of the GMV_{\pm} . First we make the following observation. Suppose $B = \text{const} \in \mathfrak{h}$. Then the function $\tilde{B} = gBg^{-1}$ obviously satisfies $[S, \tilde{B}] = 0$, so \tilde{B} takes values in \mathfrak{h}_S . Moreover, we have that $\langle \tilde{B}, S \rangle = \langle B, J_0 \rangle = \text{const}$ and $\langle \tilde{B}, S_1 \rangle = \langle B, J_1 \rangle = \text{const}$ where $J_1 = J_0^2 - \frac{2}{3}\mathbf{1}$. But J_0 and J_1 span \mathfrak{h} , so B is of the form $B = a_0 J_0 + a_1 J_1$ where a_0, a_1 are some numbers. Then $\tilde{B} = a_0 S + a_1 S_1$. From the other side, we have seen that S_x and $S_{1,x}$ are orthogonal to \mathfrak{h}_S , so we see that \tilde{B}_x is also orthogonal to

\mathfrak{h}_S , that is, belongs to \mathfrak{h}_S^\perp . Take now any fundamental solution $\tilde{\chi}$ and a any root vector E_α . We have

$$\langle \tilde{B}_x, \tilde{\chi} E_\alpha \tilde{\chi}^{-1} \rangle = \langle \tilde{B}_x, \pi_S(\tilde{\chi} E_\alpha \tilde{\chi}^{-1}) \rangle = \langle [S, \text{ad}_S^{-1} \tilde{B}_x], \pi_S(\tilde{\chi} E_\alpha \tilde{\chi}^{-1}) \rangle.$$

From the other side, since $i\partial_x(\tilde{\chi} E_\alpha \tilde{\chi}^{-1}) = \lambda [S, \tilde{\chi} E_\alpha \tilde{\chi}^{-1}]$, we have that

$$\partial_x(\langle \tilde{B}, \tilde{\chi} E_\alpha \tilde{\chi}^{-1} \rangle) = \langle \tilde{B}_x, \tilde{\chi} E_\alpha \tilde{\chi}^{-1} \rangle.$$

Combining these relations we obtain that

$$\int_{-\infty}^{\infty} \langle [S, \text{ad}_S^{-1} \tilde{B}_x], \pi_S(\tilde{\chi} E_\alpha \tilde{\chi}^{-1}) \rangle dx = \langle \tilde{B}, \tilde{\chi} E_\alpha \tilde{\chi}^{-1} \rangle|_{-\infty}^{\infty}, \quad \lambda \in \mathbb{R}, \quad \alpha \in \Delta \quad (6.17)$$

provided the integral exists, that is the right hand exists. Putting in the above formulas $\tilde{\chi} = \tilde{\chi}^\pm$ we see that we can calculate the coefficients in the expansions from Theorem 5.2 for the functions $\text{ad}_S^{-1} \tilde{B}_x$. We present the result of these calculations as a theorem.

Theorem 6.1. *The following expansion formulas hold:*

$$\begin{aligned} \text{a) } -\text{ad}_S^{-1} \tilde{B}_x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} \left(\tilde{\rho}_{B,-\alpha}^+ \tilde{\mathbf{e}}_\alpha^+ - \tilde{\rho}_{B,\alpha}^- \tilde{\mathbf{e}}_{-\alpha}^- \right) \right] d\lambda \\ \text{b) } -\text{ad}_S^{-1} \tilde{B}_x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} \left(\tilde{\sigma}_{B,\alpha}^+ \tilde{\mathbf{e}}_{-\alpha}^+ - \tilde{\sigma}_{B,-\alpha}^- \tilde{\mathbf{e}}_\alpha^- \right) \right] d\lambda. \end{aligned} \quad (6.18)$$

The coefficients $\tilde{\rho}$ and $\tilde{\sigma}$ (of course we have $\alpha \in \Delta_+$ and $\lambda \in \mathbb{R}$) are equal to

$$\begin{aligned} \tilde{\rho}_{B,\mp\alpha}^\pm &= \langle (\tilde{S}^\pm)^{-1} B \tilde{S}^\pm, E_{\mp\alpha} \rangle = \langle (S^\pm)^{-1} B S^\pm, E_{\mp\alpha} \rangle, \\ \tilde{\sigma}_{B,\pm\alpha}^\pm &= \langle (\tilde{D}^\pm)^{-1} (\tilde{T}^\pm)^{-1} B \tilde{T}^\pm \tilde{D}^\pm, E_{\pm\alpha} \rangle = \\ &= \langle (D^\pm)^{-1} (T^\pm)^{-1} B T^\pm D^\pm, E_{\pm\alpha} \rangle. \end{aligned} \quad (6.19)$$

The functions

$$\begin{aligned} \rho_{B,\mp\alpha}^\pm &= \langle (S^\pm)^{-1} B S^\pm, E_{\mp\alpha} \rangle, \\ \sigma_{B,\pm\alpha}^\pm &= \langle (D^\pm)^{-1} (T^\pm)^{-1} B T^\pm D^\pm, E_{\pm\alpha} \rangle \end{aligned} \quad (6.20)$$

are exactly the coefficients one obtains expanding $i \text{ad}_{J_0}^{-1} [B, q]$ over the adjoint solutions of (3.12), see for example [5, 20]. Thus we have

$$\tilde{\rho}_{B,\mp\alpha}^\pm = \rho_{B,\mp\alpha}^\pm, \quad \tilde{\sigma}_{B,\pm\alpha}^\pm = \sigma_{B,\pm\alpha}^\pm. \quad (6.21)$$

It is known that provided that B is a regular element, the families

$$\begin{aligned} \mathcal{F}_\rho &= \{ \rho_{B,-\alpha}^+(\lambda), \rho_{B,\alpha}^-(\lambda) : \lambda \in \mathbb{R}, \alpha \in \Delta_+ \}, \\ \mathcal{F}_\sigma &= \{ \sigma_{B,\alpha}^+(\lambda), \sigma_{B,-\alpha}^-(\lambda) : \lambda \in \mathbb{R}, \alpha \in \Delta_+ \} \end{aligned} \quad (6.22)$$

could also be taken as sets of scattering data for GZS (3.12), that is using it one can construct the potential q . Then the families

$$\begin{aligned} \tilde{\mathcal{F}}_\rho &= \{\tilde{\rho}_{B,-\alpha}^+(\lambda), \tilde{\rho}_{B,\alpha}^-(\lambda) : \lambda \in \mathbb{R}, \alpha \in \Delta_+\}, \\ \tilde{\mathcal{F}}_\sigma &= \{\tilde{\sigma}_{B,\alpha}^+(\lambda), \tilde{\sigma}_{B,-\alpha}^-(\lambda) : \lambda \in \mathbb{R}, \alpha \in \Delta_+\} \end{aligned} \tag{6.23}$$

could be treated as scattering data for the GMV_\pm system. In particular, if one chooses $B = J_0$ and $B = J_1$ one will obtain expansions for the functions $\text{ad}_S^{-1}S_x$ and $\text{ad}_S^{-1}S_{1x}$. However, depending on the properties of B the scattering data have different symmetries. Indeed, suppose that $h(\tilde{B}) = \eta\tilde{B}$, where $\eta = \pm 1$. This as easily seen is equivalent to $k(B) = \eta B$. Then of course we shall have also $h(\tilde{B}_x) = \eta\tilde{B}_x$ and

$$\begin{aligned} &\langle\langle [S, \text{ad}_S^{-1}\tilde{B}_x], \pi_S \tilde{\chi}^\pm(x, \lambda) E_\alpha(\tilde{\chi}^\pm)^{-1}(x, \lambda) \rangle\rangle = \\ &\eta \langle\langle [S, \text{ad}_S^{-1}\tilde{B}_x], \pi_S \tilde{\chi}^\mp E_{\mathcal{H}\alpha}(\tilde{\chi}^\mp)^{-1}(x, -\lambda) \rangle\rangle. \end{aligned}$$

Consequently, for $\alpha \in \Delta_+$ we have

$$\begin{aligned} \tilde{\rho}_{B,-\alpha}^+(\lambda) &= \eta \tilde{\rho}_{B,-\mathcal{H}\alpha}^-(\lambda), & \tilde{\rho}_{B,\alpha}^-(\lambda) &= \eta \tilde{\rho}_{B,\mathcal{H}\alpha}^+(\lambda), \\ \tilde{\sigma}_{B,\alpha}^+(\lambda) &= \eta \tilde{\sigma}_{B,\mathcal{H}\alpha}^-(\lambda), & \tilde{\sigma}_{B,-\alpha}^-(\lambda) &= \eta \tilde{\sigma}_{B,-\mathcal{H}\alpha}^+(\lambda). \end{aligned} \tag{6.24}$$

Since $k(J_0) = -J_0$ and $k(J_1) = J_1$, taking for example $B = J_0$, one obtains

$$\begin{aligned} \tilde{\rho}_{J_0,-\alpha}^+(\lambda) &= -\tilde{\rho}_{J_0,-\mathcal{H}\alpha}^-(\lambda), & \tilde{\rho}_{J_0,\alpha}^-(\lambda) &= -\tilde{\rho}_{J_0,\mathcal{H}\alpha}^+(\lambda), \\ \tilde{\sigma}_{J_0,\alpha}^+(\lambda) &= -\tilde{\sigma}_{J_0,\mathcal{H}\alpha}^-(\lambda), & \tilde{\sigma}_{J_0,-\alpha}^-(\lambda) &= -\tilde{\sigma}_{J_0,-\mathcal{H}\alpha}^+(\lambda) \end{aligned} \tag{6.25}$$

and one sees that

$$\begin{aligned} \text{a) } -\text{ad}_S^{-1}\tilde{S}_x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} \left(\tilde{\rho}_{J_0,-\alpha}^+ \tilde{s}_\alpha^+ - \tilde{\rho}_{J_0,\alpha}^- \tilde{s}_\alpha^- \right) \right] d\lambda, \\ \text{b) } -\text{ad}_S^{-1}\tilde{S}_x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_{\alpha \in \Delta_+} \left(\tilde{\sigma}_{J_0,\alpha}^+ \tilde{s}_\alpha^+ - \tilde{\sigma}_{J_0,-\alpha}^- \tilde{s}_\alpha^- \right) \right] d\lambda \end{aligned} \tag{6.26}$$

as it should be according to the theory we developed. If we expand $\text{ad}_S^{-1}S_{1x}$ then the expansion could be reformulated as expansion over the functions $\tilde{\mathbf{a}}_\alpha^\pm$.

7. Conclusions

In the present work we have been able to do the following:

- We have showed that the results for the $\text{GMV} = \text{GMV}_+$ system about the expansions over the adjoint solutions could be generalized in two directions: i) To apply for the system GMV_- corresponding to a pseudo-Hermitian reduction of a certain $\text{sl}(3, \mathbb{C})$ -GZS system in pole gauge subject to $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduction of Mikhailov type. ii) Both for GMV_+ and GMV_- we have been able to develop the theory of expansions for arbitrary constant asymptotic conditions of the potentials.
- We have showed how the expansions over adjoint solutions should be modified if one takes into account possible symmetries of the functions we expand.

The above was achieved using the gauge equivalence between the GZS system in pole gauge and the one in canonical gauge, see Theorem 3.1. This has permitted to avoid repetition of cumbersome proofs and frequently to reduce the things to purely algebraic arguments.

Since the expansions over the adjoint solutions of an auxiliary linear problem are the main tool of the so-called recursion operator method to the soliton equations, it would be natural to develop the theory of these equations, their conservation laws, hierarchies of compatible Poisson structures etc., for the GZS system in pole gauge and in canonical gauge, making all the approach gauge-covariant. As mentioned in the introduction this task that has been successfully achieved in the case of the Heisenberg Ferromagnet hierarchy and nonlinear Schrödinger hierarchy and later generalized for systems on arbitrary simple Lie algebra with no reductions.

Also, one must consider the discrete spectrum which we have not done here for the sake of brevity.

Another direction in which we would like to proceed is construct special exact solutions for the first equations in the hierarchies since they are likely to have some physical applications.

Including all these issues in one article is, of course, not possible, so we intend to address them in future works.

Acknowledgments

The work has been supported by the NRF incentive grant of South Africa and grant DN 02-5 of Bulgarian Fund "Scientific Research" .

References

- [1] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, The Inverse scattering problem — Fourier analysis for nonlinear problems, *Studies in Appl. Math.* **53** (1974) 249–315.
- [2] B. Beals and R. R. Coifman, Scattering and inverse scattering for first order systems, *Commun. Pure & Appl. Math.* **37** (1984) 39–89.
- [3] A. E. Borovik and V. Yu. Popkov, Completely integrable spin - 1 chains, *Sov. Phys. JETPH* **71** (1990) 177–85.
- [4] L. D. Faddeev and L. A. Takhtadjan, *Hamiltonian Method in the Theory of Solitons* (Springer Verlag, Berlin, 1987).
- [5] V. S. Gerdjikov, Generalized Fourier Transforms for the Soliton Equations. Gauge-covariant Formulation, *Inv. Problems* **2** (1986) 51–74.
- [6] V. S. Gerdjikov, A. V. Mikhailov and T. I. Valchev, Reductions of integrable equations on **A.III**-symmetric spaces *J. Phys. A* **43** (2010) 434015.
- [7] V. S. Gerdjikov, G. G. Grahovski, A. V. Mikhailov and T. I. Valchev, Polynomial bundles and generalized Fourier transforms for integrable equations on **A.III**-type symmetric spaces, *SIGMA* **7** 096 (2011) 48 pages.
- [8] V. S. Gerdjikov, G. Vilasi and A. B. Yanovski, *Integrable Hamiltonian Hierarchies — Spectral and Geometric Methods* (Springer, Heidelberg, 2008).
- [9] V. S. Gerdjikov and A. B. Yanovski, Gauge-covariant theory of the generating operator. I, *Commun. Math. Phys.* **103** (1986) 549–68.
- [10] V. S. Gerdjikov and A. B. Yanovski, Completeness of the eigenfunctions for the Caudrey-Beals-Coifman system, *J. Math. Phys.* **35** (1994) 3687–3721.
- [11] V. S. Gerdjikov and A. B. Yanovski, CBC systems with Mikhailov reductions by Coxeter automorphism: I. Spectral theory of the recursion operators, *Studies in Appl. Maths* **134** Issue 2 (2014) 145–180.
- [12] I. Z. Golubchik and V. V. Sokolov, Multicomponent generalization of the hierarchy of the Landau-Lifshitz equation *Theor. Math. Phys.* **124** n. 1 (2000) 909–917.

- [13] M. Goto and F. Grosshans, *Semisimple Lie Algebras*, Lecture Notes in Pure and Applied Mathematics **38** (M. Dekker Inc., New-York & Basel, 1978).
- [14] I. D. Iliev, E. Kh. Khristov and K. P. Kirchev, *Spectral Methods in Soliton Equations*, Pitman Monographs and Surveys in Pure and Appl. Mathematics **73** (John Wiley & Sons Inc., New York, 1994).
- [15] F. Magri, A simple model of the integrable Hamiltonian equations, *J. Math. Phys.* **19** (1978) 1156–1162.
- [16] A. V. Mikhailov, Reduction in the integrable systems. Reduction groups, *Lett. JETF* **32** (1979) 187–192.
- [17] A. V. Mikhailov, The reduction problem and inverse scattering method, *Physica D* **3** (1981) 73–117.
- [18] A. V. Mikhailov and S. Lombardo, Reductions of integrable equations. Dihedral group, *J. Phys. A* **37** (2004) 7727–7742.
- [19] A. B. Yanovski, Gauge-covariant approach to the theory of the generating operators for soliton equations, *PhD thesis: 5-87-222*, Joint Institute for Nuclear Research (JINR) 1987.
- [20] A. B. Yanovski, Generating operators for the generalized Zakharov-Shabat system and its gauge equivalent system in $\mathfrak{sl}(3, \mathbb{C})$ case, *Preprint: Universität Leipzig, Naturwissenschaftlich Theoretisches Zentrum Report N20* (1993) <http://cdsweb.cern.ch/record/256804/files/P00019754.pdf>
- [21] A. B. Yanovski, Geometric interpretation of the recursion operators for the generalized Zakharov-Shabat system in pole gauge on the algebra A_2 , *Jour. Geom. and Symm. Phys.* **23** 97–111.
- [22] A. B. Yanovski, On the recursion operators for the Gerdjikov, Mikhailov and Valchev system, *J. Math. Phys.* **52**, 8 (2011) 082703.
- [23] A. B. Yanovski and G. Vilasi, Geometric theory of the recursion operators for the generalized Zakharov-Shabat system in pole gauge on the algebra $\mathfrak{sl}(n; \mathbb{C})$ with and without reductions, *SIGMA* **8**, 087 (2012) 23 pages.
- [24] A. B. Yanovski and G. Vilasi, Geometry of the recursion operators for the GMV system, *Jour. Nonl. Math. Phys.* **19** N3 (2012) 1250023-1/18.
- [25] A. B. Yanovski, Gauge-covariant theory of the generating operators associated with linear problems of Caudrey-Beals-Coifman type in canonical and in pole gauge with and without reductions, *Proc. Int. Conf. BGSIAM14 (Sofia)* (2015) 2–43.
- [26] A. B. Yanovski and G. Vilasi, Recursion operators for CBC system with reductions. Geometric theory, *Il Nuovo Cimento* **38 C** (2015) 172.
- [27] V. E. Zakharov and L. A. Takhtadjan, Equivalence between nonlinear Schrödinger equation and Heisenberg ferromagnet equation, *TMF* **38** (1979) 26–35.
- [28] V. E. Zakharov and B. G. Konopelchenko, On the theory of recursion operator, *Commun. Math. Phys.* **94** (1985) 483–510.