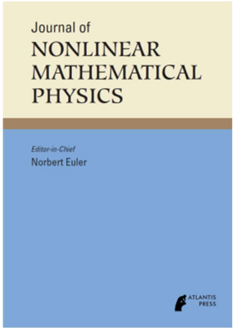

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On the discretization of Laine equations

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We consider the discretization of Darboux integrable equations. For each of the integrals of a Laine equation we constructed either a semi-discrete equation which has that integral as an n-integral, or we proved that such an equation does not exist. It is also shown that all constructed semi-discrete equations are Darboux integrable.

Keywords: Semi-discrete chain; Darboux integrability; x-integral, n-integral; discretization.

2000 Mathematics Subject Classification: 35Q51, 37K60

1. Introduction

When considering hyperbolic type equations

$$u_{xy} = g(x,y,u,u_x,u_y) \tag{1.1}$$

one finds an important special subclass, so called Darboux integrable equations, that is described in terms of x- and y-integrals. Recall that a function $W(x,y,u,u_x,u_{xx},...)$ is called a $y-$integral of equation (1.1) if $D_yW(x,y,u,u_x,...)|_{(1.1)} = 0$, where D_y represents the total derivative with respect to y (see [2] and [8]). An x-integral $\bar{W} = \bar{W}(x,y,u,u_y,u_{yy},...)$ for equation (1.1) is defined in a similar way. Equation (1.1) is said to be Darboux integrable if it admits a nontrivial x-integral and a nontrivial $y-$integral.

The classification problem for Darboux integrable equations was considered by Goursat, Zhiber and Sokolov (see [2] and [8]). In his paper Goursat obtained a supposedly complete list of Darboux integrable equations of the form (1.1). A detailed discussion of the subject and corresponding references can be found in the survey [9].

Later Laine [7] published two Darboux integrable hyperbolic equations, which were absent in Goursat's list. The first equation found by Laine is

$$u_{xy} = u_x\left(\frac{\sqrt{u_y}+u_y}{u-y} + \frac{u_y}{u-x}\right). \tag{1.2}$$

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It has a second order y-integral

$$W_1 = \frac{u_{xx}}{u_x} - \frac{1}{2}u_x\left(\frac{1}{u-y} + \frac{3}{u-x}\right) + \frac{1}{u-x} \tag{1.3}$$

and a third order x-integral

$$\bar{W} = \left(u_{yyy} - \frac{u_{yy}^2}{2u_y} - u_{yy}\frac{1+5u_y^{\frac{1}{2}}+4u_y}{u-y}\right)\left(u_{yy} - 2\frac{u_y+2u_y^{\frac{3}{2}}+u_y^2}{u-y}\right)^{-1} - \left(\frac{2u_y+2u_y^{\frac{3}{2}}-6u_y^2-10u_y^{\frac{5}{2}}-4u_y^3}{(u-y)^2}\right)\left(u_{yy} - 2\frac{u_y+2u_y^{\frac{3}{2}}+u_y^2}{u-y}\right)^{-1}. \tag{1.4}$$

The second equation found by Laine is

$$u_{xy} = 2\left((u+X)^2 + u_x + (u+X)\sqrt{(u+X)^2+u_x}\right)\left(\frac{\sqrt{u_y}+u_y}{u-y} - \frac{u_y}{\sqrt{(u+X)^2+u_x}}\right). \tag{1.5}$$

It has a second order y-integral

$$W_2 = \frac{u_{xx}}{2u_x}\left(1 - \frac{u+X}{\sqrt{(u+X)^2+u_x}}\right) + u + \frac{(u+X)^2+2u_x}{\sqrt{(u+X)^2+u_x}} - \frac{(u+X)^2+u_x+(u+X)\sqrt{(u+X)^2+u_x}}{u-y} \tag{1.6}$$

and a third order x-integral (1.4). For the second equation Laine assumed X to be an arbitrary function of x. However Kaptsov (see [6]) has shown that X must be a constant function if equation (1.5) admits the integrals (1.6) and (1.4). Thus it can be assumed, without loss of generality, that $X = 0$.

One can also consider a semi-discrete analogue of Darboux integrable equations (see [1]). The notion of Darboux integrability for semi-discrete equations was developed by Habibullin (see [3]). For a function $t = t(n,x)$ of the continuous variable x and discrete variable n we introduce notations

$$t_k = t(n+k,x), \quad k \in \mathbb{Z}, \qquad t_{[m]} = \frac{d^m}{dx^m}t(n,x), \quad m \in \mathbb{N}.$$

Then a hyperbolic type semi-discrete equation can be written as

$$t_{1x} = f(x,n,t,t_1,t_x). \tag{1.7}$$

A function F of variables x, n, and $t,t_1,\ldots,t_k$ is called an x-integral of equation (1.7) if $D_xF|_{(1.7)} = 0$. A function I of variables x, n, $t,t_{[1]},\ldots,t_{[m]}$ is called an n-integral of equation (1.7) if $DI|_{(1.7)} = I$, where D is a shift operator. Equation (1.7) is said to be Darboux integrable if it admits a nontrivial n-integral and a nontrivial x-integral. In what follows we consider the equalities $D_xF = 0$ and $DI = I$, which define x- and n-integrals F and I, only on solutions of the corresponding equations. For more information on semi-discrete Darboux integrable equations see [3], [4] and [5].

The interest in the continuous and discrete Darboux integrable models is stimulated by exponential type systems. Such systems are connected with semi-simple and affine Lie algebras which have applications in Liouville and conformal field theories.

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The discretization of equations from Goursat's list was considered by Habibullin and Zheltukhina in [5]. In the present paper we find semi-discrete versions of Laine equations (1.2) and (1.5). In particular we find semi-discrete equations that admit functions (1.3) or (1.6) as n-integrals, and show that these equations are Darboux integrable. This is the main result of our paper given in Theorem 1.1 and Theorem 1.2 below.

Theorem 1.1. *The semi-discrete chain* (1.7)*, which admits a minimal order n-integral*

$$I_1=\frac{t_{xx}}{t_x}-\frac{1}{2}t_x\left(\frac{1}{t-\varepsilon(n)}+\frac{3}{t-x}\right)+\frac{1}{t-x}, \tag{1.8}$$

where $\varepsilon(n)$ is an arbitrary function of n, is

$$t_{1x}=t_x\frac{(t_1-x)}{(t-x)}B(n,t,t_1)\,, \tag{1.9}$$

where B is a function of n, t, t_1, satisfying the following equation

$$(t_1-\varepsilon)(t_1-\varepsilon_1)-2(t-\varepsilon)(t_1-\varepsilon_1)B+(t-\varepsilon)(t-\varepsilon_1)B^2=0\,. \tag{1.10}$$

Moreover, chain (1.9) admits an x-integral of minimal order 3.

Theorem 1.2. *The semi-discrete chain* (1.7)*, which admits a minimal order n-integral*

$$I_2=\frac{t_{xx}}{2t_x}\left(1-\frac{t}{\sqrt{t^2+t_x}}\right)+t+\frac{t^2+2t_x}{\sqrt{t^2+t_x}}-\frac{t^2+t_x+t\sqrt{t^2+t_x}}{t-\varepsilon(n)}, \tag{1.11}$$

where $\varepsilon(n)$ is an arbitrary function of n, is

$$t_{1x}=2A(tA-t_1)\sqrt{t^2+t_x}+A^2t_x+2tA(tA-t_1)\,, \tag{1.12}$$

where A is a function of n, t, t_1, satisfying the following system of equations

$$\begin{cases} A_t &= \dfrac{-2t_1(t_1-\varepsilon_1)A+(-\varepsilon+2t)(t_1-\varepsilon_1)A^2-\varepsilon_1(t-2\varepsilon)A^3}{2(t_1-\varepsilon_1)(t-\varepsilon)(t_1-tA)}\,, \\[2ex] A_{t_1} &= \dfrac{\varepsilon(t_1-\varepsilon_1)+(t-\varepsilon)(2t_1-\varepsilon_1)A-2t(t-2\varepsilon)A^2}{2(t_1-\varepsilon_1)(t-\varepsilon)(t_1-tA)}\,. \end{cases} \tag{1.13}$$

Moreover, chain (1.12) admits an x-integral of minimal order 2.

The paper is organized as follows. In Sections 2 and 3 we give proofs of Theorems 1.1 and 1.2 respectively. In Section 4 we show that function (1.4) can not be a minimal order n-integral for any equation (1.7).

2. Proof of Theorem 1.1

Discretization by n-integral: Let us find $f(x,n,t,t_1,t_x)$ such that $DI_1 = I_1$, where I_1 is defined by (1.8). Equality $DI_1 = I_1$ implies

$$\frac{f_x + f_t t_x + f_{t_1} f + f_{t_x} t_{xx}}{f} - \frac{f}{2}\left(\frac{1}{t_1-\varepsilon_1} + \frac{3}{t_1 - x}\right) + \frac{1}{t_1 - x} = \frac{t_{xx}}{t_x} - \frac{t_x}{2}\left(\frac{1}{t-\varepsilon} + \frac{3}{t-x}\right) + \frac{1}{t-x}, \quad (2.1)$$

where $\varepsilon = \varepsilon(n)$ and $\varepsilon_1 = \varepsilon(n+1)$.

By comparing the coefficients before t_{xx} in (2.1), we get $\frac{f_{t_x}}{f} = \frac{1}{t_x}$, which implies that $f = A(x,n,t,t_1)t_x$. We substitute this expression for f in (2.1) and get

$$\frac{A_x + A_t t_x + A_{t_1} A t_x}{A} - \frac{A t_x}{2}\left(\frac{1}{t_1-\varepsilon_1} + \frac{3}{t_1-x}\right) + \frac{1}{t_1-x} = -\frac{t_x}{2}\left(\frac{1}{t-\varepsilon} + \frac{3}{t-x}\right) + \frac{1}{t-x}. \quad (2.2)$$

The above equation is equivalent to a system of two equations

$$\begin{cases} \dfrac{A_x}{A} + \dfrac{1}{t_1-x} = \dfrac{1}{t-x}, \\ \dfrac{A_t}{A} + A_{t_1} - \dfrac{A}{2}\left(\dfrac{1}{t_1-\varepsilon_1} + \dfrac{3}{t_1-x}\right) = \dfrac{-1}{2}\left(\dfrac{1}{t-\varepsilon} + \dfrac{3}{t-x}\right). \end{cases} \quad (2.3)$$

The first equation of system (2.3) can be written as $\frac{\partial}{\partial x}(\ln|A| - \ln|t_1 - x| + \ln|t-x|) = 0$ which implies that

$$A(x,n,t,t_1) = \frac{t_1 - x}{t-x} B(n,t,t_1) \quad (2.4)$$

for some function B of variables n, t, t_1. Substituting expression (2.4) for A into the second equation of system (2.3), we get

$$-\frac{1}{t-x} + \frac{B_t}{B} + \frac{B}{t-x} + \frac{B_{t_1}(t_1-x)}{t-x} - \frac{B(t_1-x)}{2(t-x)}\left(\frac{1}{t_1-\varepsilon_1} + \frac{3}{t_1-x}\right) = -\frac{1}{2}\left(\frac{1}{t-\varepsilon} + \frac{3}{t-x}\right). \quad (2.5)$$

Thus

$$(t-x)\frac{B_t}{B} + (t_1-x)B_{t_1} - \frac{B}{2}\left(1 + \frac{t_1-x}{t_1-\varepsilon_1}\right) = -\frac{1}{2}\left(1 + \frac{t-x}{t-\varepsilon}\right). \quad (2.6)$$

We compare the coefficients before x and x^0 in (2.6) and obtain

$$\begin{cases} -\dfrac{B_t}{B} - B_{t_1} + \dfrac{B}{2(t_1-\varepsilon_1)} = \dfrac{1}{2(t-\varepsilon)}, \\ \dfrac{tB_t}{B} + t_1 B_{t_1} - \dfrac{B}{2} - \dfrac{t_1 B}{2(t_1-\varepsilon_1)} = \dfrac{-1}{2} - \dfrac{t}{2(t-\varepsilon)}, \end{cases} \quad (2.7)$$

which is equivalent to

$$\begin{cases} B_t = \dfrac{B(\varepsilon - 2t + t_1 - \varepsilon B + tB)}{2(t-\varepsilon)(t-t_1)}, \\[2ex] B_{t_1} = \dfrac{-\varepsilon_1 + t_1 + \varepsilon_1 B + tB - 2t_1 B}{2(t_1-\varepsilon_1)(t-t_1)}. \end{cases} \tag{2.8}$$

The last system is compatible, that is $B_{tt_1} = B_{t_1t}$, if and only if equality (1.10) is satisfied.

Existence of an x-integral: Let us show that equation (1.9) where function B satisfies (1.10) has a finite dimensional x-ring. We have,

$$t_{1x} = \frac{t_1-x}{t-x}Bt_x, \quad t_{2x} = \frac{t_2-x}{t-x}BB_1t_x, \quad \text{and} \quad t_{3x} = \frac{t_3-x}{t-x}BB_1B_2t_x, \tag{2.9}$$

where $B = B(n,t,t_1)$, $B_1 = B(n+1,t_1,t_2)$ and $B_2 = B(n+2,t_2,t_3)$. We are looking for a function $F(x,n,t,t_1,t_2,t_3)$ such that $D_xF = 0$, that is

$$F_x + F_tt_x + F_{t_1}t_{1x} + F_{t_2}t_{2x} + F_{t_3}t_{3x} = 0. \tag{2.10}$$

Thus

$$F_x + F_tt_x + F_{t_1}\frac{t_1-x}{t-x}Bt_x + F_{t_2}\frac{t_2-x}{t-x}BB_1t_x + F_{t_3}\frac{t_3-x}{t-x}BB_1B_2t_x = 0, \tag{2.11}$$

which is equivalent to

$$\begin{cases} F_x = 0, \\ (t-x)F_t + (t_1-x)BF_{t_1} + (t_2-x)BB_1F_{t_2} + (t_3-x)BB_1B_2F_{t_3} = 0. \end{cases} \tag{2.12}$$

By comparing the coefficients of x^0 and x in the last equality we get the following system

$$\begin{cases} tF_t + t_1BF_{t_1} + t_2BB_1F_{t_2} + t_3BB_1B_2F_{t_3} = 0, \\ -F_t - BF_{t_1} - BB_1F_{t_2} - BB_1B_2F_{t_3} = 0. \end{cases} \tag{2.13}$$

After diagonalization this system becomes

$$\begin{cases} F_t + \frac{BB_1(t_2-t_1)}{t-t_1}F_{t_2} + \frac{BB_1B_2(t_3-t_1)}{t-t_1}F_{t_3} = 0, \\[2ex] F_{t_1} + \frac{B_1(t-t_2)}{t-t_1}F_{t_2} + \frac{B_1B_2(t-t_3)}{t-t_1}F_{t_3} = 0. \end{cases} \tag{2.14}$$

We introduce vector fields

$$\begin{aligned} V_1 &= \frac{\partial}{\partial t} + \frac{BB_1(t_2-t_1)}{t-t_1}\frac{\partial}{\partial t_2} + \frac{BB_1B_2(t_3-t_1)}{t-t_1}\frac{\partial}{\partial t_3}, \\ V_2 &= \frac{\partial}{\partial t_1} + \frac{B_1(t-t_2)}{t-t_1}\frac{\partial}{\partial t_2} + \frac{B_1B_2(t-t_3)}{t-t_1}\frac{\partial}{\partial t_3}. \end{aligned} \tag{2.15}$$

and $V = [V_1, V_2]$. Then, we have

$$\frac{2(t-t_1)^2}{B_1}V = (t_1 - t_2 + B(t_2 - t + (t-t_1)B_1)\frac{\partial}{\partial t_2} + B_2(t_1 - t_3 + B(t_3 - t + (t-t_1)B_1B_2))\frac{\partial}{\partial t_3}.$$

Direct calculation show that

$$[V_1,V]=\frac{3\varepsilon-4t+t_1}{2(\varepsilon-t)(t-t_1)}V \quad \text{and} \quad [V_2,V]=\frac{3\varepsilon_1+t-4t_1}{2(\varepsilon_1-t_1)(t_1-t)}V. \tag{2.16}$$

Hence vector fields V_1, V_2 and V form a finite-dimensional ring. By the Jacobi Theorem the system of three equations $V_1(F)=0, V_2(F)=0, V(F)=0$ has a nonzero solution $F(t,t_1,t_2,t_3)$. The function $F(t,t_1,t_2,t_3)$ is an x-integral of equation (1.9).

3. Proof of Theorem 1.2

Discretization by n-integral: Let us find a function $f(x,n,t,t_1,t_x)$ such that $DI_2=I_2$, where I_2 is given by (1.11). The equality $DI_2=I_2$ implies that

$$\frac{f_x+f_t t_x+f_{t_1}f+f_{t_x}t_{xx}}{2f}\left(1-\frac{t_1}{\sqrt{t_1^2+f}}\right)-\frac{t_1^2+f+t_1\sqrt{t_1^2+f}}{t_1-\varepsilon_1}+t_1+\frac{t_1^2+2f}{\sqrt{t_1^2+f}}$$
$$=\frac{t_{xx}}{2t_x}\left(1-\frac{t}{\sqrt{t^2+t_x}}\right)-\frac{t^2+t_x+t\sqrt{t^2+t_x}}{t-\varepsilon}+t+\frac{t^2+2t_x}{\sqrt{t^2+t_x}}, \tag{3.1}$$

where $\varepsilon=\varepsilon(n)$ and $\varepsilon_1=\varepsilon(n+1)$. Comparing the coefficients before t_{xx} in equality (3.1), we get

$$\frac{f_{t_x}}{f}\left(1-\frac{t_1}{\sqrt{t_1^2+f}}\right)=\frac{1}{t_x}\left(1-\frac{t}{\sqrt{t^2+t_x}}\right). \tag{3.2}$$

This can be written as

$$\frac{\partial}{\partial t_x}\ln\left(f\frac{\sqrt{f+t_1^2}+t_1}{\sqrt{f+t_1^2}-t_1}\right)=\frac{\partial}{\partial t_x}\ln\left(t_x\frac{\sqrt{t_x+t^2}+t}{\sqrt{t_x+t^2}-t}\right). \tag{3.3}$$

Thus

$$\sqrt{f+t_1^2}+t_1=(\sqrt{t_x+t^2}+t)A(x,n,t,t_1), \tag{3.4}$$

where A is some function of variables x, n, t and t_1. The last equality is equivalent to

$$f=(2A^2t-2At_1)\sqrt{t_x+t^2}+A^2t_x+t(2A^2t-2At_1). \tag{3.5}$$

We substitute f given by (3.5) into equality (3.1), use (3.4) and equality

$$\sqrt{f+t_1^2}-t_1=\frac{f(\sqrt{t_x+t^2}-t)}{At_x}$$

to get

$$\frac{1}{\sqrt{t_x+t^2}\sqrt{f+t_1^2}}\left(\Lambda_1t_x^2+\Lambda_2t_x\sqrt{t_x+t^2}+\Lambda_3t_x+\Lambda_4\sqrt{t_x+t^2}++\Lambda_5t^2\right)=0, \tag{3.6}$$

where

$$\Lambda_i=\alpha_{i1}A_x+\alpha_{i2}A_t+\alpha_{i3}A_{t_1}+\alpha_{i4}, \qquad 1\le i\le 5 \tag{3.7}$$

and

$$\alpha_{11}=0,\, \alpha_{12}=1,\, \alpha_{13}=A^2,\, \alpha_{14}=\frac{A}{t-\varepsilon}-\frac{A^3}{t_1-\varepsilon_1},$$

$$\alpha_{21}=0,\, \alpha_{22}=t-\frac{t_1}{A},\, \alpha_{23}=-3t_1A+3tA^2, \alpha_{24}=\frac{-t_1+2tA}{t-\varepsilon}+\frac{2t_1A^2-3tA^3}{t_1-\varepsilon_1}+A^2-A,$$

$$\alpha_{31}=1,\, \alpha_{32}=t^2,\, \alpha_{33}=2t_1^2+5t^2A^2-6t_1tA,$$
$$\alpha_{34}=\frac{-t_1t+t(t+2\varepsilon)A}{t-\varepsilon}+\frac{-5t^2A^3+4t_1tA^2-t_1^2A}{t_1-\varepsilon_1}+t_1+2tA^2-t_1A,$$

$$\alpha_{41}=t-\frac{t_1}{A},\, \alpha_{42}=0,\, \alpha_{43}=4t^3A^2-6t_1t^2A+2t_1^2t\,,$$
$$\alpha_{44}=\frac{2\varepsilon t^2A+\varepsilon tt_1}{t-\varepsilon}+\frac{-4t^3A^3+4t_1t^2A^2-t_1^2tA}{t_1-\varepsilon_1}+2t^2A^2-t_1tA\,,$$

$$\alpha_{51}=1,\, \alpha_{52}=0,\, \alpha_{53}=2t_1^2+4t^2A^2-6t_1tA\,,$$
$$\alpha_{54}=\frac{-t_1t+2\varepsilon t}{t-\varepsilon}+\frac{-4t^2A^3+4t_1tA^2-t_1^2A}{t_1-\varepsilon_1}+t_1+2tA^2-t_1A.$$

We can solve the overdetermined system of linear equations $\Lambda_i=0$, $i=1,2\dots5$, with respect to A_x, A_t, A_{t_1} and obtain

$$\begin{cases} A_x=0\,,\\ A_t=-\dfrac{A}{t-\varepsilon}+\dfrac{A^2}{2(t_1-tA)}\left(\dfrac{A\varepsilon_1}{t_1-\varepsilon_1}-\dfrac{\varepsilon}{t-\varepsilon}\right),\\ \\ A_{t_1}=\dfrac{A}{t_1-\varepsilon_1}-\dfrac{1}{2(t_1-tA)}\left(\dfrac{A\varepsilon_1}{t_1-\varepsilon_1}-\dfrac{\varepsilon}{t-\varepsilon}\right). \end{cases} \tag{3.8}$$

By direct calculations one can check that $A_{tt_1}=A_{t_1t}$, so the above system has a solution.
Existence of an x-integral: We are looking for a function $F(t,t_1,t_2)$ such that $D_xF=0$ that is

$$F_tt_x+F_{t_1}t_{1x}+F_{t_2}t_{2x}=0, \tag{3.9}$$

where t satisfies equation (1.7) with function f given by (3.5). We use

$$t_{1x}=A^2(t,t_1)t_x+2A(t,t_1)(tA(t,t_1)-t_1)(\sqrt{t_x+t^2}+t)$$

and

$$\sqrt{f+t_1^2}=(\sqrt{t_x+t^2}+t)A-t_1\,,$$

to get

$$t_{2x} = A^2(t,t_1)A^2(t_1,t_2)t_x + 2(\sqrt{t_x+t^2}+t)(tA(t,t_1)-t_1)A(t,t_1)A^2(t_1,t_2)+ \\ 2(\sqrt{t_x+t^2}+t)(t_1A(t_1,t_2)-t_2)A(t,t_1)A(t_1,t_2).$$

By substituting these expressions for t_{1x} and t_{2x} into equality (3.9) and comparing the coefficients of $\sqrt{t_x+t^2}$, t_x and t_x^0, we obtain the following system of equations

$$\begin{cases} 2A(t,t_1)(tA(t,t_1)-t_1)F_{t_1} + 2A(t,t_1)A(t_1,t_2)(tA(t,t_1)A(t_1,t_2)-t_2)F_{t_2} = 0, \\ F_t + A^2(t,t_1)F_{t_1} + A^2(t,t_1)A^2(t_1,t_2)F_{t_2} = 0, \\ 2tA(t,t_1)(tA(t,t_1)-t_1)F_{t_1} + 2tA(t,t_1)A(t_1,t_2)(tA(t,t_1)A(t_1,t_2)-t_2)F_{t_2} = 0. \end{cases}$$

To check for the existence of a solution we transform the above system to its row reduced form

$$\begin{cases} F_t + \dfrac{A^2(t,t_1)A(t_1,t_2)(t_2-t_1A(t_1,t_2))}{tA(t,t_1)-t_1}F_{t_2} = 0, \\ F_{t_1} + \dfrac{A(t_1,t_2)(t_2-tA(t,t_1)A(t_1,t_2))}{-tA(t,t_1)+t_1}F_{t_2} = 0. \end{cases} \tag{3.10}$$

The corresponding vector fields

$$V_1 = \frac{\partial}{\partial t} + \frac{A^2(t,t_1)A(t_1,t_2)(t_2-t_1A(t_1,t_2))}{tA(t,t_1)-t_1}\frac{\partial}{\partial t_2},$$
$$V_2 = \frac{\partial}{\partial t_1} + \frac{A(t_1,t_2)(t_2-tA(t,t_1)A(t_1,t_2))}{-tA(t,t_1)+t_1}\frac{\partial}{\partial t_2}$$

commute, that is $[V_1,V_2]=0$, provided A satisfies system (3.8). Thus by the Jacobi theorem, system (3.10) has a solution. To solve the system define a function $E(t,t_1,t_2)$ by

$$E_t = \frac{A^2}{tA-t_1},\ E_{t_2} = \frac{1}{A_1(t_1A_1-t_2)},\ E_{t_1} = \frac{t_2-tAA_1}{(tA-t_1)(t_1A_1-t_2)} + \frac{1}{t_1-\varepsilon_1}E,$$

where $A=A(t,t_1)$ and $A_1=A(t_1,t_2)$.
One can check that $E_{tt_1}=E_{t_1t}$ and $E_{t_1t_2}=E_{t_2t_1}$, so such a function E exists. Function E is a first integral of the first equation of system (3.10). We write system (3.10) using new variables

$$\tilde{t}=t,\ \tilde{t}_1=t_1,\ \tilde{t}_2=E(t,t_1,t_2)$$

and obtain

$$\begin{cases} F_{\tilde{t}}=0 \\ F_{\tilde{t}_1} + \frac{\tilde{t}_2}{\tilde{t}_1-\varepsilon_1}F_{\tilde{t}_2}=0. \end{cases} \tag{3.11}$$

Therefore one of the x-integrals is $F(t,t_1,t_2)=E(t,t_1,t_2)/(t_1-\varepsilon(n+1))$ where function E defined above.

4. Nonexistence of a chain (1.7) admitting the minimal order n-integral (1.4)

Let us find a function $f(x,n,t,t_1,t_x)$ such that equation (1.7) has the n-integral

$$I=\frac{t_{xxx}-\frac{t_{xx}^2}{2t_x}-t_{xx}\frac{1+5\sqrt{t_x}+4t_x}{t-x}-\frac{2t_x+2t_x\sqrt{t_x}-6t_x^2-10t_x^2\sqrt{t_x}-4t_x^3}{(t-x)^2}}{t_{xx}-\frac{2t_x+4t_x\sqrt{t_x}+2t_x^2}{t-x}}\,.$$

We have,

$$t_{1x}=f(x,n,t,t_1,t_x)\,,$$

$$t_{1xx}=f_x+f_tt_x+f_{t_1}f+f_{t_x}t_{xx}\,,$$

$$t_{1xxx}=(f_{xx}+f_{xt}t_x+f_{xt_1}f+f_{xt_x}t_{xx})+t_x(f_{xt}+f_{tt}t_x+f_{tt_1}f+f_{tt_x}t_{xx})+f_tt_{xx}$$

$$+f(f_{xt_1}+f_{tt_1}t_x+f_{t_1t_1}f+f_{t_1t_x}t_{xx})+f_{t_1}(f_x+f_tt_x+f_{t_1}f+f_{t_x}t_{xx})$$

$$+t_{xx}(f_{xt_x}+f_{tt_x}t_x+f_{t_1t_x}f+f_{t_xt_x}t_{xx})+f_{t_x}t_{xxx}\,.$$

Equality $DI=I$ is equivalent to $J:=L(DL)(DI-I)=0$, where $L=\sqrt{2}t_x(t-x)\{t_{xx}(t-x)-2t_x(\sqrt{t_x}+1)^2\}$. We have,

$$J=\Lambda_1t_{xxx}+\Lambda_2t_{xx}^3+\Lambda_3t_{xx}^2+\Lambda_4t_{xx}+\Lambda_5\,,$$

where Λ_k, $1\le k\le 5$, are some functions of variables x, n, t, t_1, t_x. In particular,

$$\frac{\Lambda_1}{2(t-x)(t_1-x)t_xf}=2(t-x)f(1+\sqrt{f})^2-2(t_1-x)t_xf_{t_x}(1+\sqrt{t_x})^2-(t_1-x)(t-x)(f_x+f_tt_x+f_{t_1}f)\,,$$

$$\Lambda_2=(t-x)^2(t_1-x)^2\{ff_{t_x}-t_xf_{t_x}^2+2t_xff_{t_xt_x}\}\,,$$

$$\frac{\Lambda_3}{(t-x)(t_1-x)}=(t-x)f[4f^{3/2}+2f^2+(x-t_1)f_x+f(2+(x-t_1)f_{t_1})]+10(x-t_1)t_x^{3/2}ff_{t_x}$$

$$+t_x[10(t-x)f^{3/2}f_{t_x}+2(t-x)(t_1-x)f_{t_x}f_x+4(t-x)f^2(2f_{t_x}+(x-t_1)f_{t_1t_x})]$$

$$+t_xf(2(t-t_1)f_{t_x}+(t-x)(x-t_1)(3f_t+4f_{xt_x}))$$

$$-2(t_1-x)t_x^2[2f(2f_{t_x}-f_{t_xt_x}+(t-x)f_{tt_x})+f_{t_x}(f_{t_x}+(x-t)f_t)]$$

$$-4(f_{t_x}^2-2ff_{t_xt_x})(t_1-x)t_x^{5/2}-2(f_{t_x}^2-2ff_{t_xt_x})(t_1-x)t_x^3\,.$$

Equality $\Lambda_2=0$ implies that $ff_{t_x}-t_xf_{t_x}^2+2t_xff_{t_xt_x}=0$, thus

$$\frac{f^2}{f_{t_x}}\frac{\partial}{\partial t_x}\left\{\frac{t_xf_{t_x}^2}{f}\right\}=0\,.$$

Hence, $\frac{t_x f_{t_x}^2}{f} = A^2(x,n,t,t_1)$ for some function A depending on x, n, t, t_1 only. Therefore, $\frac{f_{t_x}}{\sqrt{f}} = \frac{A}{\sqrt{t_x}}$ and hence $\frac{\partial}{\partial t_x}\{\sqrt{f} - A\sqrt{t_x} = 0\}$. We have,

$$\sqrt{f} = A\sqrt{t_x} + B$$

where $A = A(x,n,t,t_1)$ and $B = B(x,n,t,t_1)$. We substitute $f = A^2 t_x + 2AB\sqrt{t_x} + B^2$ into $\Lambda_1 = 0$ and get

$$\alpha_1 t_x^2 + \alpha_2 t_x^{3/2} + \alpha_3 t_x + \alpha_4\sqrt{t_x} + \alpha_5 = 0.$$

We solve the system of equations $\alpha_k = 0$, $1 \le k \le 5$, and obtain $B = 0$, that is

$$\begin{cases} A_x = \frac{B}{2A}B_t - \frac{3}{2}ABB_{t_1} + \frac{2(t_1-x)B + A\{2(t-t_1)+6(t-x)B+3(t-x)B^2\}}{2(t-x)(x-t_1)}, \\ A_t = \frac{A}{2B}B_t + \frac{A^3}{2B}B_{t_1} + \frac{A\{2(t_1-x)A+2(x-t_1)B-(t-x)A^2(2+B)\}}{2(t-x)(x-t_1)B}, \\ A_{t_1} = -\frac{1}{2AB}B_t - \frac{A}{2B}B_{t_1} + \frac{2(x-t_1)+(t-x)A(2+3B)}{2(t-x)(x-t_1)B}, \\ B_x = -B^2 B_{t_1} - \frac{B(1+B)^2}{t_1-x}. \end{cases} \tag{4.1}$$

We substitute $f = A^2 t_x + 2AB\sqrt{t_x} + B^2$ into $\Lambda_3 = 0$ and get

$$\beta_1 t_x^3 + \beta_2 t_x^{5/2} + \beta_3 t_x^2 + \beta_4 t_x^{3/2} + \beta_5 t_x + \beta_6\sqrt{t_x} + \beta_7 = 0.$$

We solve the system of equations $\beta_k = 0$, $1 \le k \le 7$, and obtain $B = 0$, or

$$\begin{cases} A_x = \frac{3B}{8A}B_t - \frac{23}{24}ABB_{t_1} + \frac{21(t_1-x)B + A\{16(t-t_1)+51(t-x)B+23(t-x)B^2\}}{24(t-x)(x-t_1)}, \\ A_t = \frac{3A}{8B}B_t + \frac{3A^3}{8B}B_{t_1} + \frac{A\{7(t_1-x)A+8(x-t_1)B-(t-x)A^2(7+3B)\}}{8(t-x)(x-t_1)B}, \\ A_{t_1} = -\frac{3}{8AB}B_t - \frac{3A}{8B}B_{t_1} + \frac{7(x-t_1)+(t-x)A(7+11B)}{8(t-x)(x-t_1)B}, \\ B_x = -B^2 B_{t_1} - \frac{B(1+B)^2}{t_1-x}. \end{cases} \tag{4.2}$$

We equate expressions for A_x and A_t from (4.1) and (4.2) and find

$$\begin{cases} B_t = -\frac{A\{2(t_1-x)B + A((t-t_1)+(t-x)B)\}}{2(t-x)(x-t_1)B}, \\ B_{t_1} = \frac{t-t_1+3(t-x)B+2(t-x)B^2}{2(t-x)(x-t-1)B}. \end{cases} \tag{4.3}$$

Then, it follows from (4.1) that

$$\begin{cases} A_x = \dfrac{(t_1 - x + (t-x)A)B}{2(t-x)(x-t_1)}, \\ A_t = \dfrac{A((t_1-x)A + (x-t)A^2 + 2(x-t_1)B)}{2(t-x)(x-t_1)B}, \\ A_{t_1} = \dfrac{x - t_1 + (t-x)A(1+2B)}{2(t-x)(x-t_1)B}, \\ B_x = \frac{B(t+t_1-2x+(t-x)B)}{2(t_1-x)(x-t)}. \end{cases} \tag{4.4}$$

Equality $A_{tt_1} - A_{t_1 t} = 0$ becomes $\dfrac{(t_1-x)^2 - (t-x)^2A^3}{(t-x)^2(t_1-x)^2B} = 0$, thus

$$A^3 = \frac{(t_1-x)^2}{(t-x)^2}. \tag{4.5}$$

Equality $A_{xt_1} - A_{t_1 x} = 0$ becomes $\dfrac{-(t_1-x)^2 + (t-x)^2A(1+B)^2}{(t-x)^2(t_1-x)^2B} = 0$, thus

$$A(1+B)^2 = \frac{(t_1-x)^2}{(t-x)^2}. \tag{4.6}$$

Equality $A_{xt} - A_{tx} = 0$ becomes $\dfrac{(t_1-x)^2(A-B)^2 - (t-x)^2A^3}{(t-x)^2(t_1-x)^2B} = 0$. It implies that

$$\frac{A^3}{(A-B)^2} = \frac{(t_1-x)^2}{(t-x)^2}, \tag{4.7}$$

or $A = B$, that leads to $A = B = 0$ and $f = 0$. It follows from (4.5) and (4.7) that $A - B = 1$ or $A - B = -1$. It follows from (4.5) and (4.6) that $1 + B = A$ or $1 + B = -A$. This gives rise to four possibilities:
1) $A - B = 1$;
2) $A - B = 1$ and $A + B = -1$ which gives $A = 0$, $B = -1$ and therefore $f = 1$;
3) $A - B = -1$ and $A - B = 1$ which is an inconsistent system;
4) $A - B = -1$ and $A + B = -1$ which gives $A = -1$, $B = 0$ and therefore $f = t_x$.

We have to study case 1) only. In this case we get $B = A - 1$ and equation $\sqrt{t_{1x}} = A\sqrt{t_x} + B$ becomes $\sqrt{t_{1x}} + 1 = A(\sqrt{t_x} + 1)$, that can be written as well as

$$(\sqrt{t_{1x}} + 1)^3 = A^3(\sqrt{t_x} + 1)^3. \tag{4.8}$$

Due to (4.5), our equation (4.8) becomes

$$\frac{(\sqrt{t_{1x}} + 1)^3}{(t_1-x)^2} = \frac{(\sqrt{t_x} + 1)^3}{(t-x)^2}.$$

The last equation admits an n-integral $I = \dfrac{(\sqrt{t_x}+1)^3}{(t-x)^2}$ of order one.

Let us consider case $B=0$. We write $DI-I=0$ for the chain $t_{1x}=C(x,n,t,t_1)t_x$ and get

$$\Lambda_1 t_{xxx}+\Lambda_2 t_{xx}^2+\Lambda_3 t_{xx}+\Lambda_4=0$$

where $\Lambda_k=\Lambda_k(x,n,t,t_1,t_x)$, $1\le k\le 4$. Equation $\Lambda_1=0$ implies

$$\alpha_1 t_x+\alpha_2\sqrt{t_x}+\alpha_3=0$$

where $\alpha_k=\alpha_k(x,n,t,t_1)$, $1\le k\le 3$. In particular, $\alpha_2=4C(-(t_1-x)+(t-x)\sqrt{C})$. Since $\alpha_2=0$, we have $C=(t_1-x)^2(t-x)^{-2}$. The chain becomes $t_{1x}=(t_1-x)^2(t-x)^{-2}t_x$. It admits the n-integral $I=(t-x)^{-2}t_x$ of order one.

Therefore, if equation (1.7) admits n-integral (1.4) then (1.4) is not a minimal order integral.

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