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On the discretization of Laine equations

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We consider the discretization of Darboux integrable equations. For each of the integrals of a Laine equation we constructed either a semi-discrete equation which has that integral as an $n$-integral, or we proved that such an equation does not exist. It is also shown that all constructed semi-discrete equations are Darboux integrable.

Keywords: Semi-discrete chain; Darboux integrability; $x$-integral, $n$-integral; discretization.

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1. Introduction

When considering hyperbolic type equations

$$u_{xy} = g(x,y,u,u_x,u_y)$$ (1.1)

one finds an important special subclass, so called Darboux integrable equations, that is described in terms of $x$- and $y$-integrals. Recall that a function $W(x,y,u,u_x,u_{xx},\ldots)$ is called a $y$–integral of equation (1.1) if $D_y W(x,y,u,u_x,\ldots)|_{(1.1)} = 0$, where $D_y$ represents the total derivative with respect to $y$ (see [2] and [8]). An $x$-integral $\bar{W} = \bar{W}(x,y,u,u_y,u_{yy},\ldots)$ for equation (1.1) is defined in a similar way. Equation (1.1) is said to be Darboux integrable if it admits a nontrivial $x$-integral and a nontrivial $y$–integral.

The classification problem for Darboux integrable equations was considered by Goursat, Zhiber and Sokolov (see [2] and [8]). In his paper Goursat obtained a supposedly complete list of Darboux integrable equations of the form (1.1). A detailed discussion of the subject and corresponding references can be found in the survey [9].

Later Laine [7] published two Darboux integrable hyperbolic equations, which were absent in Goursat’s list. The first equation found by Laine is

$$u_{xy} = u_x \left( \sqrt{u_x + u_y} + \frac{u_y}{u - x} \right).$$ (1.2)
It has a second order $y$-integral
\[ W_1 = \frac{u_{xx}}{u_x} - \frac{1}{2} u_x \left( \frac{1}{u - y} + \frac{3}{u - x} \right) + \frac{1}{u - x} \]
and a third order $x$-integral
\[
\bar{W} = \left( u_{yxy} - \frac{u_{xyy}^2}{2u_y} + u_{yy} \frac{1 + 5u_y^2 + 4u_y}{u - y} \right) \left( u_{xyy} - 2u_y + 2u_y^3 + u_y^5 \right)^{-1} \\
- \left( \frac{2u_y + 2u_y^3 - 6u_y^2 - 10u_y^2 - 4u_y^3}{(u - y)^2} \right) \left( u_{xyy} - 2u_y + 2u_y^3 + u_y^5 \right)^{-1}.
\]
(1.4)
The second equation found by Laine is
\[ u_{xy} = 2 \left( (u + X)^2 + u_x + (u + X) \sqrt{(u + X)^2 + u_x} \right) \left( \frac{\sqrt{u_y} + u_y}{u - y} - \frac{u_y}{\sqrt{(u + X)^2 + u_x}} \right). \]
(1.5)
It has a second order $y$-integral
\[ W_2 = \frac{u_{xx}}{2u_x} \left( 1 - \frac{u + X}{\sqrt{(u + X)^2 + u_x}} \right) + u + \frac{(u + X)^2 + 2u_x}{\sqrt{(u + X)^2 + u_x}} \\
- \frac{(u + X)^2 + u_x + (u + X)\sqrt{(u + X)^2 + u_x}}{u - y} \]
(1.6)
and a third order $x$-integral (1.4). For the second equation Laine assumed $X$ to be an arbitrary function of $x$. However Kaptsov (see [6]) has shown that $X$ must be a constant function if equation (1.5) admits the integrals (1.6) and (1.4). Thus it can be assumed, without loss of generality, that $X = 0$.

One can also consider a semi-discrete analogue of Darboux integrable equations (see [11]). The notion of Darboux integrability for semi-discrete equations was developed by Habibullin (see [3]). For a function $t = t(n, x)$ of the continuous variable $x$ and discrete variable $n$ we introduce notations
\[ t_k = t(n + k, x), \quad k \in \mathbb{Z}, \quad t_{[m]} = \frac{d^m}{dx^m} t(n, x), \quad m \in \mathbb{N}. \]
Then a hyperbolic type semi-discrete equation can be written as
\[ t_{k+1} = f(x, n, t, t_1, t_k). \]
(1.7)
A function $F$ of variables $x$, $n$, and $t, t_1, \ldots, t_k$ is called an $x$-integral of equation (1.7) if $D_x F|_{(1.7)} = 0$. A function $I$ of variables $x$, $n$, $t, t_1, \ldots, t_m$ is called an $n$-integral of equation (1.7) if $D_I|_{(1.7)} = I$, where $D$ is a shift operator. Equation (1.7) is said to be Darboux integrable if it admits a nontrivial $n$-integral and a nontrivial $x$-integral. In what follows we consider the equalities $D_x F = 0$ and $D I = I$, which define $x$- and $n$-integrals $F$ and $I$, only on solutions of the corresponding equations. For more information on semi-discrete Darboux integrable equations see [3], [4] and [5].

The interest in the continuous and discrete Darboux integrable models is stimulated by exponential type systems. Such systems are connected with semi-simple and affine Lie algebras which have applications in Liouville and conformal field theories.
The discretization of equations from Goursat’s list was considered by Habibullin and Zheltukhina in [5]. In the present paper we find semi-discrete versions of Laine equations (1.2) and (1.5). In particular we find semi-discrete equations that admit functions (1.3) or (1.6) as \( n \)-integrals, and show that these equations are Darboux integrable. This is the main result of our paper given in Theorem 1.1 and Theorem 1.2 below.

**Theorem 1.1.** The semi-discrete chain (1.7), which admits a minimal order \( n \)-integral

\[
I_1 = \frac{t_{xx}}{t_x} - \frac{1}{2} t_x \left( \frac{1}{t - \varepsilon(n)} + \frac{3}{t - x} \right) + \frac{1}{t - x},
\]

where \( \varepsilon(n) \) is an arbitrary function of \( n \), is

\[
t_{1x} = t_x \frac{(t_1 - x)}{(t - x)} B(n, t, t_1),
\]

where \( B \) is a function of \( n, t, t_1 \), satisfying the following equation

\[
(t_1 - \varepsilon)(t_1 - \varepsilon_1) - 2(t - \varepsilon)(t_1 - \varepsilon_1)B + (t - \varepsilon)(t - \varepsilon_1)B^2 = 0.
\]

Moreover, chain (1.9) admits an \( x \)-integral of minimal order 3.

**Theorem 1.2.** The semi-discrete chain (1.7), which admits a minimal order \( n \)-integral

\[
I_2 = \frac{t_{xx}}{2 t_x} \left( 1 - \frac{t}{\sqrt{t^2 + t_x}} \right) + \frac{t}{\sqrt{t^2 + t_x}} \left( \frac{t^2 + 2 t_x}{\sqrt{t^2 + t_x}} \right) - \frac{t_x + t \sqrt{t^2 + t_x}}{t - \varepsilon(n)},
\]

where \( \varepsilon(n) \) is an arbitrary function of \( n \), is

\[
t_{1x} = 2A(tA - t_1) \sqrt{t^2 + t_x} + A^2 t_x + 2tA(tA - t_1),
\]

where \( A \) is a function of \( n, t, t_1 \), satisfying the following system of equations

\[
\begin{align*}
A_t &= \frac{-2 t_1 (t_1 - \varepsilon_1) A + (t - \varepsilon + 2t)(t_1 - \varepsilon_1)A^2 - \varepsilon_1 (t - 2\varepsilon)A^3}{2(t_1 - \varepsilon_1)(t - \varepsilon)(t_1 - tA)}, \\
A_{t_1} &= \frac{\varepsilon(t_1 - \varepsilon_1) + (t - \varepsilon)(2t_1 - \varepsilon_1)A - 2t(t - 2\varepsilon)A^2}{2(t_1 - \varepsilon_1)(t - \varepsilon)(t_1 - tA)}.
\end{align*}
\]

Moreover, chain (1.12) admits an \( x \)-integral of minimal order 2.

The paper is organized as follows. In Sections 2 and 3 we give proofs of Theorems 1.1 and 1.2 respectively. In Section 4 we show that function (1.4) can not be a minimal order \( n \)-integral for any equation (1.7).
2. Proof of Theorem 1.1

**Discretization by n-integral**

Let us find \( f(x, n, t, t_1, t_x) \) such that \( DI_1 = I_1 \), where \( I_1 \) is defined by (1.8). Equality \( DI_1 = I_1 \) implies

\[
\frac{f_t + f_t s + f_t f + f_t s x}{f} - \frac{f}{2} \left( \frac{1}{t_1 - \varepsilon_1} + \frac{3}{t_1 - x} \right) + \frac{1}{t_1 - x} = \frac{t_x - t_s}{2} \left( \frac{1}{t - \varepsilon} + \frac{3}{t - x} \right) + \frac{1}{t - x},
\]

(2.1)

where \( \varepsilon = \varepsilon(n) \) and \( \varepsilon_1 = \varepsilon(n + 1) \).

By comparing the coefficients before \( t_x \) in (2.1), we get \( \frac{f_t}{f} = \frac{1}{t_x} \), which implies that \( f = A(x, n, t, t_1) t_x \). We substitute this expression for \( f \) in (2.1) and get

\[
\frac{A_x + A_t x + A_t A_t s}{A} - \frac{A_t s}{2} \left( \frac{1}{t_1 - \varepsilon_1} + \frac{3}{t_1 - x} \right) + \frac{1}{t_1 - x} = -\frac{t_s}{2} \left( \frac{1}{t - \varepsilon} + \frac{3}{t - x} \right) + \frac{1}{t - x},
\]

(2.2)

The above equation is equivalent to a system of two equations

\[
\begin{cases}
\frac{A_x}{A} + \frac{1}{t_1 - x} = \frac{1}{t - x}, \\
\frac{A_t}{A} + A_t - \frac{A}{2} \left( \frac{1}{t_1 - \varepsilon_1} + \frac{3}{t_1 - x} \right) = -\frac{1}{2} \left( \frac{1}{t - \varepsilon} + \frac{3}{t - x} \right).
\end{cases}
\]

(2.3)

The first equation of system (2.3) can be written as \( \frac{\partial}{\partial t} (\ln |A| - \ln |t_1 - x| + \ln |t - x|) = 0 \) which implies that

\[
A(x, n, t, t_1) = \frac{t_1 - x}{t - x} B(n, t, t_1)
\]

(2.4)

for some function \( B \) of variables \( n, t, t_1 \). Substituting expression (2.4) for \( A \) into the second equation of system (2.3), we get

\[
-\frac{1}{t - x} + \frac{B_t}{B} + \frac{B}{t - x} + \frac{B_t (t_1 - x)}{t - x} - \frac{B (t_1 - x)}{2 (t - x)} \left( \frac{1}{t_1 - \varepsilon_1} + \frac{3}{t_1 - x} \right) = -\frac{1}{2} \left( \frac{1}{t - \varepsilon} + \frac{3}{t - x} \right).
\]

(2.5)

Thus

\[
(t - x) \frac{B_t}{B} + (t_1 - x) B_{t_1} - \frac{B}{2} \left( 1 + \frac{t_1 - x}{t_1 - \varepsilon_1} \right) = -\frac{1}{2} \left( 1 + \frac{t - x}{t - \varepsilon} \right).
\]

(2.6)

We compare the coefficients before \( x \) and \( x^0 \) in (2.6) and obtain

\[
\begin{cases}
-\frac{B_t}{B} - B_{t_1} + \frac{B}{2 (t_1 - \varepsilon_1)} = \frac{1}{2 (t - \varepsilon)}, \\
\frac{t B_t}{B} + t B_{t_1} - \frac{B}{2} - \frac{t_1 B}{2 (t_1 - \varepsilon_1)} = -\frac{1}{2} - \frac{t}{2 (t - \varepsilon)}.
\end{cases}
\]

(2.7)
After diagonalization this system becomes

\[
\begin{cases}
B_t = \frac{B(\varepsilon - 2t + t_1 - \varepsilon B + tB)}{2(t - \varepsilon)(t - t_1)}, \\
B_{t_1} = -\varepsilon_1 + t_1 + \varepsilon_1 B + tB - 2t_1 B \\
2(t_1 - \varepsilon_1)(t - t_1).
\end{cases}
\]

(2.8)

The last system is compatible, that is \(B_{t_1} = B_{t_{11}}\), if and only if equality (1.10) is satisfied.

**Existence of an \(x\)-integral:** Let us show that equation (1.9) where function \(B\) satisfies (1.10) has a finite dimensional \(x\)-ring. We have,

\[
t_{1x} = \frac{t_1 - x}{t - x}, \quad t_{2x} = \frac{t_2 - x}{t - x} BB_1 t_x, \quad \text{and} \quad t_{3x} = \frac{t_3 - x}{t - x} BB_1 BB_2 t_x,
\]

(2.9)

where \(B = B(n, t, t_1)\), \(B_1 = B(n + 1, t_1, t_2)\) and \(B_2 = B(n + 2, t_2, t_3)\). We are looking for a function \(F(x, n, t, t_1, t_2, t_3)\) such that \(D_x F = 0\), that is

\[
F_x + F_{t_1} t_x + F_{t_2} t_{2x} + F_{t_3} t_{3x} = 0.
\]

(2.10)

Thus

\[
F_x + F_{t_1} t_x + F_{t_1} t_{1x} - x BB_1 t_x + F_{t_2} t_{2x} = 0
\]

(2.11)

which is equivalent to

\[
\begin{cases}
F_x = 0, \\
(t - x)F_{t_1} + (t_1 - x)BB_1 F_{t_1} + (t_2 - x)BB_1 BB_2 F_{t_2} + (t_3 - x)BB_1 BB_2 F_{t_3} = 0.
\end{cases}
\]

(2.12)

By comparing the coefficients of \(x^0\) and \(x\) in the last equality we get the following system

\[
\begin{cases}
tF_{t_1} + t_1 BF_{t_1} + t_2 BBB_1 F_{t_2} + t_3 BB_1 BB_2 F_{t_3} = 0, \\
-tF_{t_1} - BB_1 F_{t_1} - BB_1 BB_2 F_{t_3} = 0.
\end{cases}
\]

(2.13)

After diagonalization this system becomes

\[
\begin{cases}
F_{t_1} + \frac{BB_1 (t_2 - t_1)}{t - t_1} F_{t_2} + \frac{BB_1 B_2 (t_3 - t_1)}{t - t_1} F_{t_3} = 0, \\
F_{t_1} + \frac{B_1 (t_2 - t_1)}{t - t_1} F_{t_2} + \frac{B_1 B_2 (t_3 - t_1)}{t - t_1} F_{t_3} = 0.
\end{cases}
\]

(2.14)

We introduce vector fields

\[
V_1 = \frac{\partial}{\partial t_1} + \frac{BB_1 (t_2 - t_1)}{t - t_1} \frac{\partial}{\partial t_2} + \frac{BB_1 B_2 (t_3 - t_1)}{t - t_1} \frac{\partial}{\partial t_3},
\]

\[
V_2 = \frac{\partial}{\partial t_2} + \frac{B_1 (t_2 - t_1)}{t - t_1} \frac{\partial}{\partial t_2} + \frac{B_1 B_2 (t_3 - t_1)}{t - t_1} \frac{\partial}{\partial t_3},
\]

(2.15)

and \(V = [V_1, V_2]\). Then, we have

\[
\frac{2(t - t_1)^2}{B_1} V = (t_1 - t_2 + B(t_2 - t + (t - t_1)B_1) \frac{\partial}{\partial t_2} + B_2(t_1 - t_3 + B(t_3 - t + (t - t_1)B_1 B_2)) \frac{\partial}{\partial t_3}.
\]
Direct calculation show that

\[ [V_1, V] = \frac{3\varepsilon - 4t + t_1}{2(\varepsilon - t)(t - t_1)} V \quad \text{and} \quad [V_2, V] = \frac{3\varepsilon + t - 4t_1}{2(\varepsilon - t)(t_1 - t)} V, \]

(2.16)

Hence vector fields \( V_1, V_2 \) and \( V \) form a finite-dimensional ring. By the Jacobi Theorem the system of three equations \( V_1(F) = 0, V_2(F) = 0, V(F) = 0 \) has a nonzero solution \( F(t, t_1, t_2, t_3) \). The function \( F(t, t_1, t_2, t_3) \) is some function of variables \( t, t_1, t_2, t_3 \).

3. Proof of Theorem 1.2

**Discretization by \( n \)-integral:** Let us find a function \( f(x, n, t_1, t_2, t_3) \) such that \( DI_2 = I_2 \), where \( I_2 \) is given by (1.11). The equality \( DI_2 = I_2 \) implies that

\[ \frac{f_x + f_t x + f_t f + f_t t}{2f} \left( 1 - \frac{t_1}{\sqrt{t_1^2 + f}} \right) - \frac{t_1^2 + f + t_1\sqrt{t_1^2 + f}}{t_1 - \varepsilon_1} + t + \frac{t_1^2 + 2f}{\sqrt{t_1^2 + f}} = \frac{t_2 x}{2t} \left( 1 - \frac{t}{\sqrt{t^2 + t_x}} \right) - \frac{t_2 + t_x + t\sqrt{t^2 + t_x}}{t - \varepsilon} + t + \frac{t^2 + 2t_x}{\sqrt{t^2 + t_x}}, \]

(3.1)

where \( \varepsilon = \varepsilon(n) \) and \( \varepsilon_1 = \varepsilon(n + 1) \). Comparing the coefficients before \( t_2 x \) in equality (3.1), we get

\[ \frac{f_x}{f} \left( 1 - \frac{t_1}{\sqrt{t_1^2 + f}} \right) = \frac{1}{t_2} \left( 1 - \frac{t}{\sqrt{t^2 + t_x}} \right), \]

(3.2)

This can be written as

\[ \frac{\partial}{\partial t_2} \ln \left( \frac{f\sqrt{f + t_1^2 + t_1}}{\sqrt{f + t_1^2 - t_1}} \right) = \frac{\partial}{\partial t} \ln \left( \frac{t_2\sqrt{t_2 + t^2 + t}}{\sqrt{t_2 + t^2 - t}} \right). \]

(3.3)

Thus

\[ \sqrt{f + t_1^2 + t_1} = (\sqrt{t_2 + t^2 + t})A(x, n, t, t_1), \]

(3.4)

where \( A \) is some function of variables \( x, n, t \) and \( t_1 \). The last equality is equivalent to

\[ f = (2A^2 t - 2At_1)\sqrt{t_2 + t^2 + t}A^2 t_2 + t(2A^2 t - 2At_1). \]

(3.5)

We substitute \( f \) given by (3.5) into equality (3.1), use (3.4) and equality

\[ \sqrt{f + t_1^2 - t_1} = \frac{f(\sqrt{t_2 + t^2 - t})}{At_2}, \]

to get

\[ \frac{1}{\sqrt{t_2 + t^2}} \left( \Lambda_1 t_2^2 + \Lambda_2 t_2 \sqrt{t_2 + t^2} + \Lambda_3 t_3 + \Lambda_4 \sqrt{t_2 + t^2} + + \Lambda_5 t_5^2 \right) = 0, \]

(3.6)

where

\[ \Lambda_i = \alpha_{i1} A + \alpha_{i2} A + \alpha_{i3} A t_1 + \alpha_{i4}, \quad 1 \leq i \leq 5, \]

(3.7)
and

\[ \alpha_{11} = 0, \alpha_{12} = 1, \alpha_{13} = A^2, \alpha_{14} = \frac{A}{t - \epsilon} - \frac{A^3}{t_1 - \epsilon} \],

\[ \alpha_{21} = 0, \alpha_{22} = t - \frac{t_1}{A}, \alpha_{23} = -3t_1A + 3tA^2, \alpha_{24} = \frac{-t_1 + 2tA}{t - \epsilon} + \frac{2t_1A^2 - 3tA^3}{t_1 - \epsilon} + A^2 - A, \]

\[ \alpha_{31} = 1, \alpha_{32} = t^2, \alpha_{33} = 2t_1^2 + 5t^2A^2 - 6t_1tA, \]

\[ \alpha_{34} = \frac{-t_1t + t + 2t \epsilon)A}{t - \epsilon} + \frac{-5t^2A^3 + 4t_1tA^2 - 3t_1A^2}{t_1 - \epsilon} + t_1 + 2tA^2 - t_1A, \]

\[ \alpha_{41} = t - \frac{t_1}{A}, \alpha_{42} = 0, \alpha_{43} = 4t_1^3A^2 - 6t_1t^2A + 2t^2t, \]

\[ \alpha_{44} = \frac{2t_1t^2A + \epsilon t t_1}{t - \epsilon} + \frac{-4t_1^3A^3 + 4t_1tA^2 - t_1^2A}{t_1 - \epsilon} + 2t^2A^2 - t_1tA, \]

\[ \alpha_{51} = 1, \alpha_{52} = 0, \alpha_{53} = 2t_1^2 + 4t^2A^2 - 6t_1tA, \]

\[ \alpha_{54} = \frac{-t_1t + 2t \epsilon + t_1^2A}{t - \epsilon} + \frac{-4t^2A^3 + 4t_1tA^2 - t_1^2A}{t_1 - \epsilon} + t_1 + 2tA^2 - t_1A. \]

We can solve the overdetermined system of linear equations \( A_i = 0, i = 1, 2 \ldots 5 \), with respect to \( A_x, A_y, A_{t_1} \) and obtain

\[
\begin{align*}
A_x &= 0, \\
A_y &= \frac{A}{t - \epsilon} + \frac{A^2}{2(t_1 - tA)} \left( \frac{A \epsilon}{t_1 - \epsilon} - \frac{\epsilon}{t - \epsilon} \right), \\
A_{t_1} &= \frac{A}{t_1 - \epsilon} + \frac{1}{2(t_1 - tA)} \left( \frac{A \epsilon}{t_1 - \epsilon} - \frac{\epsilon}{t - \epsilon} \right). 
\end{align*}
\] (3.8)

By direct calculations one can check that \( A_{t_1} = A_{t_1} \), so the above system has a solution. **Existence of an \( x \)-integral**: We are looking for a function \( F(t, t_1, t_2) \) such that \( D_xF = 0 \) that is

\[ F_x t_x + F_{t_1} t_{1x} + F_{t_2} t_{2x} = 0, \] (3.9)

where \( t \) satisfies equation (1.7) with function \( f \) given by (3.5). We use

\[ t_{1x} = A^2(t, t_1) t_x + 2A(t, t_1) (tA(t, t_1) - t_1) (\sqrt{t_x + t^2 + t}) \]

and

\[ \sqrt{f + t_1^2} = (\sqrt{t_x + t^2 + t}) A - t_1, \]
By substituting these expressions for \( t_1 \) and \( t_2 \), into equality (3.9) and comparing the coefficients of \( \sqrt{t_x + t^2} \), \( t_x \) and \( t^0 \), we obtain the following system of equations

\[
\begin{align*}
F_1 &= 2A(t,t_1)(tA(t,t_1) - t_1)F_1 + 2A(t,t_1)A(t_1,t_2)(tA(t,t_1)A(t_1,t_2) - t_2)F_2 = 0, \\
&\quad + A^2(t,t_1)F_1 \quad + A^2(t,t_1)A^2(t_1,t_2)F_2 = 0, \\
F_1 &= 2tA(t,t_1)(tA(t,t_1) - t_1)F_1 + 2tA(t,t_1)A(t_1,t_2)(tA(t,t_1)A(t_1,t_2) - t_2)F_2 = 0.
\end{align*}
\]

To check for the existence of a solution we transform the above system to its row reduced form

\[
\begin{align*}
F_1 + \frac{A^2(t,t_1)A(t_1,t_2)(t_2 - t_1A(t_1,t_2))}{tA(t,t_1) - t_1}F_2 &= 0, \\
F_1 + \frac{A(t_1,t_2)(t_2 - tA(t,t_1)A(t_1,t_2))}{-tA(t,t_1) + t_1}F_2 &= 0.
\end{align*}
\]

(3.10)

The corresponding vector fields

\[
\begin{align*}
V_1 &= \frac{\partial}{\partial t} + \frac{A^2(t,t_1)A(t_1,t_2)(t_2 - t_1A(t_1,t_2))}{tA(t,t_1) - t_1} \frac{\partial}{\partial t_1}, \\
V_2 &= \frac{\partial}{\partial t_1} + \frac{A(t_1,t_2)(t_2 - tA(t,t_1)A(t_1,t_2))}{-tA(t,t_1) + t_1} \frac{\partial}{\partial t_2},
\end{align*}
\]

commute, that is \( [V_1, V_2] = 0 \), provided \( A \) satisfies system (3.8). Thus by the Jacobi theorem, system (3.10) has a solution. To solve the system define a function \( E(t,t_1,t_2) \) by

\[
E_t = \frac{A^2}{tA - t_1}, \quad E_{t_2} = \frac{1}{A(tA_A - t_2)}, \quad E_{t_1} = \frac{t_2 - tAA_1}{(tA - t_1)(t_1A - t_2)} + \frac{1}{t_1 - \epsilon t},
\]

where \( A = A(t,t_1) \) and \( A_1 = A(t_1,t_2) \).

One can check that \( E_{t} = E_{t,1} \) and \( E_{t_2} = E_{t_2,1} \), so such a function \( E \) exists. Function \( E \) is a first integral of the first equation of system (3.10). We write system (3.10) using new variables

\[
\tilde{t} = t, \quad \tilde{t}_1 = t_1, \quad \tilde{t}_2 = E(t,t_1,t_2)
\]

and obtain

\[
\begin{align*}
\tilde{F}_1 &= 0, \\
\tilde{F}_1 + \frac{\tilde{t}}{\tilde{t}_1 - \epsilon \tilde{t}} \tilde{F}_2 &= 0.
\end{align*}
\]

(3.11)

Therefore one of the \( x \)-integrals is \( F(t,t_1,t_2) = E(t,t_1,t_2)/(t_1 - \epsilon (n + 1)) \) where function \( E \) defined above.
4. Nonexistence of a chain (1.7) admitting the minimal order $n$-integral (1.4)

Let us find a function $f(x, n, t, t_1, t_x)$ such that equation (1.7) has the $n$-integral

$$I = \frac{t_{xx} - \frac{t_1^2}{2} - t_{xx} \frac{1+5\sqrt{5}+4t}{f-x} - 2\sqrt{\frac{t}{x}} - 2t_x \sqrt{t - x} - 10t_x^2 \sqrt{t - x} - 4t_x}{t_{xx} - \frac{2t_x \sqrt{t} + 2t_x}{f-x}}.$$  

We have,

$$t_{1x} = f(x, n, t, t_1, t_x),$$

$$t_{1xx} = f_x + f_1 t_x + f_1 f + f_{t_1} t_x,$$

$$t_{1xxx} = \left((f_{xx} + f_{tt} t_x + f_{tt} f + f_{tt} t_{xx}) + t_1(f_{tt} t_x + f_{tt} f + f_{tt} t_{xx}) + f_{tt} t_{xx}\right)$$

$$+ f(f_{tt} t_x + f_{tt} f + f_{tt} t_{xx}) + f_1(t_x + f_{tt} f + f_{tt} t_{xx})$$

$$+ t_1(t_1 + f_{tt} t_x + f_{tt} f + f_{tt} t_{xx}) + f_{tt} t_{xxx}.$$  

Equality $DI = I$ is equivalent to $J := L(DL)(DI - I) = 0,$ where $L = \sqrt{2}t_x(t - x)(t_{xx}(t - x) - 2t_x(\sqrt{t} + 1)^2).$ We have,

$$J = \Lambda_1 t_{xxx} + \Lambda_2 t_{xx}^3 + \Lambda_3 t_{xx}^2 + \Lambda_4 t_{xx} + \Lambda_5,$$

where $\Lambda_k$, $1 \leq k \leq 5$, are some functions of variables $x, n, t, t_1, t_x.$ In particular,

$$\frac{\Lambda_1}{2(t-x)(t_1-x)t_x} f = 2(t-x)f(1 + \sqrt{f})^2 - 2(t_1-x)t_x f_x (1 + \sqrt{t_x})^2 - (t_1-x) \left((f_{xx} + f_{tt} f + f_{tt} t_{xx}) + f_{tt} t_{xx}\right)$$

$$- t_1 \left((f_{tt} t_x + f_{tt} f + f_{tt} t_{xx}) + f_{tt} t_{xxx}\right) + f_1(t_x + f_{tt} f + f_{tt} t_{xx}).$$

Equality $\Lambda_1 = 0$ implies that $f f_{tt} - t_{tt} f_x^2 + 2 t_{tt} f x f = 0,$ thus

$$\frac{f^2}{f_{tt}} \left(\frac{t_{tt} f_x^2}{f_x}$$

$$\left\{ t_{tt} f_x^2 \right\} = 0.$$
Hence, \( t_x^2 f_x^2 = A^2(x,n,t,t_1) \) for some function \( A \) depending on \( x, n, t, t_1 \) only. Therefore, \( \frac{f_x}{\sqrt{f}} = \frac{A}{\sqrt{t_x}} \)

and hence \( \frac{\partial}{\partial t_x} \{ \sqrt{f} - A\sqrt{t_x} = 0 \} \). We have,

\[ \sqrt{f} = A\sqrt{t_x} + B \]

where \( A = A(x,n,t,t_1) \) and \( B = B(x,n,t,t_1) \). We substitute \( f = A^2 t_x + 2AB\sqrt{t_x} + B^2 \) into \( \Lambda_1 = 0 \) and get

\[ \alpha_1 t_x^2 + \alpha_2 t_x^{3/2} + \alpha_3 t_x + \alpha_4 \sqrt{t_x} + \alpha_5 = 0. \]

We solve the system of equations \( \alpha_k = 0, \ 1 \leq k \leq 5 \), and obtain \( B = 0 \), that is

\[
\begin{align*}
A_x &= \frac{B}{2A} B_t - 3 A^2 B B_t + \frac{2(t_1 - t)B + A\{2(t - t_1) + 6(t - x)B + 3(t - x)B^2\}}{2(t - x) (x - t_1)}, \\
A_t &= \frac{A}{2B} B_t + \frac{3 A^2}{2B} B_t + \frac{A\{2(t_1 - x)A + 2(x - t_1)B - (t - x)^2 + 2\}}{2(t - x) (x - t_1)B}, \\
A_{t_1} &= -\frac{1}{2AB} B_t - \frac{A}{2B} B_{t_1} + \frac{2(x - t_1)B + 2(t - x)A(2 + 3B)}{2(t - x) (x - t_1)B}, \\
B_x &= -B^2 B_t - \frac{B(1 + B)^2}{t_1 - x}.
\end{align*}
\]

We substitute \( f = A^2 t_x + 2AB\sqrt{t_x} + B^2 \) into \( \Lambda_3 = 0 \) and get

\[ \beta_5 t_x^3 + \beta_7 t_x^{5/2} + \beta_3 t_x^2 + \beta_4 t_x^{3/2} + \beta_6 \sqrt{t_x} + \beta_7 = 0. \]

We solve the system of equations \( \beta_k = 0, \ 1 \leq k \leq 7 \), and obtain \( B = 0 \), or

\[
\begin{align*}
A_x &= \frac{3 B}{8A} B_t - \frac{23}{24} A^2 B B_t + \frac{21(t_1 - x)B + A\{16(t - t_1) + 51(t - x)B + 23(t - x)B^2\}}{24(t - x) (x - t_1)}, \\
A_t &= \frac{3 A}{8B} B_t + \frac{3 A^2}{8B} B_t + \frac{A\{7(t_1 - x)A + 8(x - t_1)B - (t - x)^2(7 + 3B)\}}{8(t - x) (x - t_1)B}, \\
A_{t_1} &= -\frac{3}{8AB} B_t - \frac{3 A}{8B} B_{t_1} + \frac{7(x - t_1)B + 7(x - t_1)A(7 + 11B)}{8(t - x) (x - t_1)B}, \\
B_x &= -B^2 B_t - \frac{B(1 + B)^2}{t_1 - x}.
\end{align*}
\]

We equate expressions for \( A_x \) and \( A_t \) from (4.1) and (4.2) and find

\[
\begin{align*}
B_t &= -\frac{A\{2(t_1 - x)B + A((t - t_1) + (t - x)B)\}}{2(t - x) (x - t_1)B}, \\
B_{t_1} &= \frac{t - t_1 + 3(t - x)B + 2(t - x)B^2}{2(t - x) (x - t - 1)B}.
\end{align*}
\]

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Then, it follows from (4.1) that

\[
\begin{align*}
A_x &= \frac{(t_1 - x + (t - x)A)B}{2(t-x)(x-t_1)}, \\
A_t &= \frac{A((t_1 - x)A + (x-t)A^2 + 2(x-t_1)B)}{2(t-x)(x-t_1)B}, \\
A_{tt} &= \frac{x-t_1 + (t-x)A(1+2B)}{2(t-x)(x-t_1)B}, \\
B_x &= \frac{B(t+t_1-2x+(t-x)B)}{2(t-x)(x-t_1)}. 
\end{align*}
\]  

(4.4)

Equality \(A_{tt} - A_{tx} = 0\) becomes

\[
\frac{(t_1 - x)^2 - (t-x)^2 A^3}{(t-x)^2(t_1 - x)^2 B} = 0,
\]

thus

\[
A^3 = \frac{(t_1 - x)^2}{(t-x)^2}.
\]  

(4.5)

Equality \(A_{tt} - A_{tx} = 0\) becomes

\[
\frac{- (t_1 - x)^2 + (t-x)^2 A(1+B)^2}{(t-x)^2(t_1 - x)^2 B} = 0,
\]

thus

\[
A(1+B)^2 = \frac{(t_1 - x)^2}{(t-x)^2}.
\]  

(4.6)

Equality \(A_{tt} - A{tx} = 0\) becomes

\[
\frac{(t_1 - x)^2(A - B)^2 - (t-x)^2 A^3}{(t-x)^2(t_1 - x)^2 B} = 0.
\]

It implies that

\[
\frac{A^3}{(A-B)^2} = \frac{(t_1 - x)^2}{(t-x)^2},
\]  

(4.7)

or \(A = B\), that leads to \(A = B = 0\) and \(f = 0\). It follows from (4.5) and (4.7) that \(A - B = 1\) or \(A = -B = -1\). It follows from (4.5) and (4.6) that \(1+B = A\) or \(1+B = -A\). This gives rise to four possibilities:

1) \(A - B = 1\);
2) \(A = B = 1\) and \(A + B = -1\) which gives \(A = 0\), \(B = -1\) and therefore \(f = 1\);
3) \(A - B = -1\) and \(A = B = 1\) which is an inconsistent system;
4) \(A - B = -1\) and \(A + B = -1\) which gives \(A = -1\), \(B = 0\) and therefore \(f = t_x\).

We have to study case 1) only. In this case we get \(B = A - 1\) and equation \(\sqrt{t_1} = A\sqrt{t_x} + B\) becomes \(\sqrt{t_1} + 1 = A(\sqrt{t_x} + 1)\), that can be written as well as

\[
(\sqrt{t_1} + 1)^3 = A^3(\sqrt{t_x} + 1)^3.
\]  

(4.8)

Due to (4.5), our equation (4.8) becomes

\[
\frac{(\sqrt{t_1} + 1)^3}{(t_1 - x)^2} = \frac{(\sqrt{t_x} + 1)^3}{(t-x)^2}.
\]

The last equation admits an \(n\)-integral \(I = \frac{(\sqrt{t_x} + 1)^3}{(t-x)^2}\) of order one.
Let us consider case $B = 0$. We write $D I - I = 0$ for the chain $t_{1x} = C(x, n, t, t_1) t_x$ and get

$$\Lambda_1 t_{xxx} + \Lambda_2 t_{xx}^2 + \Lambda_3 t_{xx} + \Lambda_4 = 0$$

where $\Lambda_k = \Lambda_k(x, n, t_1, t_x)$, $1 \leq k \leq 4$. Equation $\Lambda_1 = 0$ implies

$$\alpha_1 t_x + \alpha_2 \sqrt{t_x} + \alpha_3 = 0$$

where $\alpha_k = \alpha_k(x, n, t, t_1)$, $1 \leq k \leq 3$. In particular, $\alpha_2 = 4C(- (t_1 - x) + (t - x) \sqrt{C})$. Since $\alpha_2 = 0$, we have $C = (t_1 - x)^2 (t - x)^{-2}$. The chain becomes $t_{1x} = (t_1 - x)^2 (t - x)^{-2} t_x$. It admits the $n$-integral $I = (t - x)^{-2} t_x$ of order one.

Therefore, if equation (1.7) admits $n$-integral (1.4) then (1.4) is not a minimal order integral.

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References


