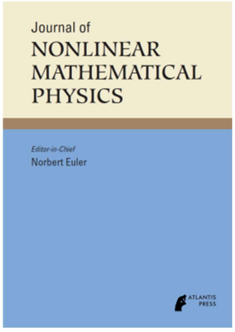

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The Mixed Kuper-Camassa-Holm-Hunter-Saxton Equations

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In this paper, a mixed Kuper-CH-HS equation by a Kupershmidt deformation is introduced and its integrable properties are studied. Moreover, that the equation can be viewed as a constraint Hamiltonian flow on the coadjoint orbit of Neveu-Schwarz superalgebra is shown.

Keywords: Lax Pair, mixed Kuper-CH-HS equation, Neveu-Schwarz superalgebra, Hamiltonian structure.

2000 Mathematics Subject Classification: 37K10, 35Q51, 35Q53

1. Introduction

There are many interesting differential equations in mathematics and physics, such as the Camassa-Holm (CH in brief) equation [1] which is the model for the propagation of shallow water waves of moderate amplitude

$$u_t - u_{txx} = 2u_x u_{xx} + u u_{xxx} - 3uu_x,$$

the Hunter-Saxton (HS in brief) equation [9] which is used as a progressive equation of liquid crystal rotator

$$-u_{txx} = 2u_x u_{xx} + u u_{xxx},$$

and the μHS equation [16] which is closely related to the HS equation

$$-u_{txx} = -2\mu(u)u_x + 2u_x u_{xx} + u u_{xxx},$$

with

$$\mu(u) = \int_{S^1} u dx.$$

It is worth mentioning that the above three equations can be expressed as

$$m_t = -2mu_x - um_x, \tag{1.1}$$

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where

$$m = u_{xx} - c\mu(u) - 4ku.$$

The CH-HS equation (1.1) is just CH equation, HS equation and μHS equation when the values of c and k are given respectively

$$\begin{cases} c = 0,\, k = \frac{1}{4}; \\ c = 0,\, k = 0; \\ c = 1,\, k = 0. \end{cases}$$

The CH-HS equation (1.1) admits a Lax pair, a bi-Hamiltonian structure

$$m_t = P_1 \frac{\delta H_2}{\delta m} = P_2 \frac{\delta H_1}{\delta m},$$

where the P_1 and P_2

$$\begin{aligned} P_1 &= \partial^3 - 4k\partial, \\ P_2 &= m\partial + \partial m, \end{aligned}$$

are two compatible operators of the CH-HS equation (1.1) and

$$\begin{aligned} H_1 &= -\frac{1}{2}\int mudx, \\ H_2 &= \int (-\frac{1}{2}u^3 + \frac{1}{6}uu_x^2 + \frac{1}{3}u^2u_{xx})dx; \end{aligned}$$

are the first two conserved quantities of the CH equation, and

$$\begin{aligned} H_1 &= \frac{1}{2}\int mudx, \\ H_2 &= -\int (\frac{1}{6}uu_x^2 + \frac{1}{3}u^2u_{xx})dx; \end{aligned}$$

are the first two conserved quantities of the HS equation, and

$$\begin{aligned} H_1 &= \frac{1}{2}\int mudx, \\ H_2 &= \int (-\frac{1}{6}uu_x^2 - \frac{1}{3}u^2u_{xx} + \frac{2}{3}u^2\mu(u)u)dx. \end{aligned}$$

are the first two conserved quantities of the μHS equation. And CH-HS equation (1.1) is formally integrable through the inverse scattering method and can be regarded as geodesic equations on the diffeomorphism group of the circle (or of the line) for the right-invariant H^1 metric, see [1–5, 11, 12] and references therein.

The Kuper-CH equation [6] and Kuper-μHS equation [23] we firstly proposed and further researched in [23, 24] as Euler equation related to the Neveu-Schwarz superalgebra, especially, when taking H^1-metric and $\mu\dot{H}^1$-metric, two new super-integrable systems—Kuper-CH system and Kuper-μHS system with Lax pair and local super-biHamiltonian structures, which are fermionic versions of the CH equation and μHS equation in $(1|1)$ superspace are given. The Super-HS equation ([17, 23]) is supersymmetric extensions of HS equation, super-bi-Hamiltonian. The Kuper-CH

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equation, Super-HS equation and the Kuper-μHS equation can also be rewritten as a unified form, which is called Kuper-CH-HS euqation here.

For an arbitrary integrable equation

$$u_t = P_1 \frac{\delta H_n}{\delta u}$$

with two compatible Hamiltonian operators P_1 and P_2, Kupershmidt [15] proposed a nonholonomic deformation as follows

$$\begin{cases} u_t = P_1 \frac{\delta H_n}{\delta u} - P_1 f, \\ P_2 f = 0. \end{cases} \tag{1.2}$$

Zhou [21] proposed the concept of mixed hierarchy of soliton equations based on Lenard scheme and defined the nonholonomic deformation as Kupershmidt deformation. Naturally we want to consider the fermionic cases of the CH-HS equation (1.1) and the mixed Kuper-CH-HS equation and theirs propertities.

In this paper, motivated by the work about sKdV6 [22], we will study the mixed kuper-CH equation and its integrable properties, and study the relation to the corresponding Neveu-Schwarz superalgebra.

2. The Kuper-CH-HS equation and the mixed Kuper-CH-HS equation

The Kuper-CH-HS equation can be rewritten as

$$\begin{cases} m_t = 2k_1 m u_x + k_1 m_x u + \frac{1}{2} k_1 \alpha_x \eta_x + \frac{3}{2} k_1 \alpha \eta_{xx}, \\ \alpha_t = \frac{3}{2} k_1 u_x \alpha + k_1 u \alpha_x + \frac{1}{2} k_1 m \eta_x. \end{cases} \tag{2.1}$$

which is a fermionic extension of the CH-HS equation(1.1), $m = u_{xx} - c\mu(u) - 4ku$ is a bosonic function and $\alpha = \eta_{xx} - k\eta$ is a fermionic function.

$$\begin{cases} c = 0,\, k = \frac{1}{4},\, k_1 = -1, & \text{Kuper-CH equation;} \\ c = 0,\, k = 0,\; k_1 = 1, & \text{Super-HS equation;} \\ c = 1,\, k = 0,\; k_1 = 1, & \text{Kuper-}\mu\text{HS equation.} \end{cases}$$

The Kuper-CH-HS equation(2.1) has the spectral problem

$$\Phi_x = U\Phi, \quad U = \begin{pmatrix} 0 & 1 & 0 \\ k + \frac{1}{2}\lambda m & 0 & \frac{1}{2}\lambda\alpha \\ \frac{1}{2}\lambda\alpha & 0 & 0 \end{pmatrix}. \tag{2.2}$$

The Kuper-CH-HS equation(2.1) has super-bi-Hamiltonian structures, which can be rewritten as

$$\begin{pmatrix} m \\ \alpha \end{pmatrix}_t = K \begin{pmatrix} \frac{\delta H_2}{\delta m} \\ \frac{\delta H_2}{\delta \alpha} \end{pmatrix} = J \begin{pmatrix} \frac{\delta H_1}{\delta m} \\ \frac{\delta H_1}{\delta \alpha} \end{pmatrix},$$

the K and J

$$K = \begin{pmatrix} \partial^3 - 4k\partial & 0 \\ 0 & \partial^2 - k \end{pmatrix},$$

$$J=\begin{pmatrix} m\partial+\partial m & \alpha\partial+\frac{1}{2}\partial\alpha \\ \partial\alpha+\frac{1}{2}\alpha\partial & \frac{m}{2} \end{pmatrix}, \tag{2.3}$$

are two compatible operators of Kuper-CH-HS equation and

$$\begin{aligned} H_1 &= \frac{k_1}{2}\int(mu+\alpha\eta_x)dx, \\ H_2 &= -\frac{k_1}{3}\int(-6ku^3+\frac{1}{2}uu_x^2+u^2u_{xx}-2c\mu(u)u^2-\frac{3k}{2}u\eta\eta_x \\ &\quad -\frac{1}{2}u\eta_x\eta_{xx}+\frac{1}{2}u\eta\alpha_x+\frac{3}{2}u_x\eta\alpha+\frac{m}{2}u\eta\eta_x)dx. \end{aligned}$$

are the first two conserved quantities H_1, H_2 of the Kuper-CH-HS equation (2.1).

Motivated by the Kupershmidt deformation (1.2), we propose a mixed Kuper-CH-HS equation as a nonholonomic deformation of the Kuper-CH-HS equation(2.1).

Definition 2.1. The mixed Kuper-CH-HS equation is defined as

$$\begin{aligned} \begin{pmatrix} m \\ \alpha \end{pmatrix}_t &= K\begin{pmatrix} \frac{\delta H_2}{\delta m} \\ \frac{\delta H_2}{\delta \alpha} \end{pmatrix} - K\begin{pmatrix} f \\ \phi \end{pmatrix}, \\ J\begin{pmatrix} f \\ \phi \end{pmatrix} &= 0 \end{aligned} \tag{2.4}$$

which is equivalent to

$$\begin{aligned} m_t &= 2k_1mu_x+k_1m_xu+\frac{1}{2}k_1\alpha_x\eta_x+\frac{3}{2}k_1\alpha\eta_{xx}-4kf_x+f_{xxx}, \\ \alpha_t &= \frac{3}{2}k_1u_x\alpha+k_1u\alpha_x+\frac{1}{2}k_1m\eta_x-k\phi+\phi_{xx}, \\ & 2mf_x+m_xf+\frac{3}{2}\alpha\phi_x+\frac{1}{2}\alpha_x\phi=0, \\ & \frac{3}{2}\alpha f_x+\alpha_xf+\frac{1}{2}m\phi=0. \end{aligned} \tag{2.5}$$

where $m=u_{xx}-c\mu(u)-4ku$ and f are bosonic functions and $\alpha=\eta_{xx}-k\eta$ and ϕ are fermionic functions.

Corresponding we can get

$$\begin{cases} c=0,\ k=\frac{1}{4},\ k_1=-1, & \text{mixed Kuper-CH equation;} \\ c=0,\ k=0,\ k_1=1, & \text{mixed Super-HS equation;} \\ c=1,\ k=0,\ k_1=1, & \text{mixed Kuper-}\mu\text{HS equation.} \end{cases}$$

It's easy to prove that

$$\begin{aligned} \frac{dH_n}{dt} &= \nabla H_nK\nabla H_2-\nabla H_nK\begin{pmatrix} f \\ \phi \end{pmatrix} \\ &= -\nabla H_{n-1}J\begin{pmatrix} f \\ \phi \end{pmatrix} \end{aligned}$$

$$= 0.$$

where functional gradient

$$\nabla = \left(\frac{\delta}{\delta m}, \frac{\delta}{\delta \alpha} \right)^T.$$

So we have following proposition

Proposition 2.2. *The mixed Kuper-CH-HS equation* (2.5) *has infinite many conserved quantities.*

3. Lax Pair of mixed Kuper-CH Equation

Zero curvature representation and Lax pairs are two kinds of Commutator representations. A systematic approach for constructing zero curvature representation has been well developed by several papers, see [18, 20] and references therein for details. In this section we adopt direct method to construct the Lax pair of the mixed Kuper-CH-HS equation (2.5). From the spectral problem of Kuper-CH-HS equation (2.1), We have got its the Lax pair [24]

$$\begin{cases} \Phi_x = U\Phi \\ \quad = \begin{pmatrix} 0 & 1 & 0 \\ k + \frac{1}{2}\lambda m & 0 & \frac{1}{2}\lambda\alpha \\ \frac{1}{2}\lambda\alpha & 0 & 0 \end{pmatrix} \Phi, \\ \Phi_t = V_0\Phi \\ \quad = \frac{k_1}{2} \begin{pmatrix} -u_x & 2u + \frac{2}{\lambda} & -\eta_x \\ -\frac{2}{\lambda} - 2ku - c\mu(u) - (mu - \frac{1}{2}\eta_x\alpha)\lambda & u_x & \alpha u\lambda - k\eta \\ \alpha u\lambda + k\alpha & \eta_x & 0 \end{pmatrix} \Phi. \end{cases} \tag{3.1}$$

Motivated by the (3.1), we assume that the Lax pair of the mixed Kuper-CH-HS equation has the following form

$$\begin{cases} \Phi_x = U\Phi, \\ \Phi_t = V\Phi \end{cases} \tag{3.2}$$

with

$$V = V_0 + \lambda V_1$$

and

$$V_1 = \begin{pmatrix} a & b & \xi \\ c_1 + c_2\lambda & -a & \beta \\ \beta & -\xi & 0 \end{pmatrix},$$

where a, b, c_1, c_2 are bosonic fields, ξ, β are fermionic fields. From the compatibility condition

$$U_t - V_x + [U, V] = 0, \tag{3.3}$$

considering its componentwise elements, we have

$$a_x = c_1 - kb + \lambda(c_2 - \frac{1}{2}mb - \frac{1}{2}\xi\alpha),$$
$$a = -\frac{1}{2}b_x,$$

$$\beta = \frac{1}{2}\lambda b\alpha + \xi_x,$$
$$m_t = 2k_1 m u_x + k_1 m_x u + \frac{1}{2}k_1\alpha_x\eta_x + \frac{3}{2}k_1\alpha\eta_{xx} + 2c_{1x} - 4ka + 2\lambda(c_{2x} - ma - \alpha\beta),$$
$$\alpha_t = \frac{3}{2}k_1 u_x\alpha + k_1 u\alpha_x + \frac{1}{2}k_1 m\eta_x + 2\beta_x - 2k\xi - \lambda(m\xi + a\alpha). \tag{3.4}$$

Furthermore, from (3.4) and the first two terms in (2.5), the following two relations are given

$$-4kf_x + f_{xxx} - 2c_{1x} + 4ka = 2\lambda(c_{2x} - ma - \alpha\beta),$$
$$-k\phi + \phi_{xx} - 2\beta_x + 2k\xi = -\lambda(m\xi + a\alpha). \tag{3.5}$$

By choosing $b = -f$ and $\xi = \frac{1}{2}\phi$, we can get

$$a = \frac{1}{2}f_x,$$
$$c_1 = \frac{1}{2}f_{xx} - kf,$$
$$c_2 = \frac{1}{4}\phi\alpha - \frac{1}{2}mf,$$
$$\beta = -\frac{1}{2}\lambda f\alpha + \frac{1}{2}\phi_x.$$

meanwhile the system (3.5) reduces to

$$4mf_x + 2m_x f + 3\alpha\phi_x + \alpha_x\phi = 0,$$
$$3\alpha f_x + 2\alpha_x f + m\phi = 0. \tag{3.6}$$

which are exactly the last terms of the mixed Kuper-CH-HS equation (2.5). So we have

Proposition 3.1. *The mixed Kuper-CH-HS equation* (2.5) *has the Lax pair* (3.2)*, where U and*

$$V = V_0 + \lambda V_1$$

are defined above.

4. Geodesic Flow

The CH equation can be described as the geodesic flow on the Bott-Virasoro group for the right-invariant H^1-metric on the group of diffeomorphisms [3, 11, 13]. The HS equation can be regarded as geodesic equations on the quotient space $\mathrm{Diff}(S^1)/S^1$ of the group $\mathrm{Diff}(S^1)$ of orientation-preserving diffeomorphisms of the unit circle S^1 modulo the subgroup of rigid rotations [12]. Furthermore, the μHS equation can also be regarded as a remember of this frame. Zuo [22] described Euler equations associated to the generalized Neveu-Schwarz algebra and got many super-bi-Hamiltonian structure or supersymmetric equation, such as Kuper-CH equation, Kuper-μHS equation, super-CH equation, super-HS equation and Kuper-2CH equation etc. In this section, we want to descibe the relation between Neveu-Schwarz algebra and the mixed Kuper-CH equation (2.5).

The Neveu-Schwarz superalgebra [19, 23] is an algebra

$$\mathscr{G} = \mathrm{Vect}(S^1) \oplus \mathrm{C}^\infty(S^1) \oplus \mathbb{R}$$

with the bilinear operation

$$[\hat{f}, \hat{g}] = (A, B, C),$$

where

$$A = (fg' - f'g - \frac{1}{2}\phi\chi)\frac{d}{dx},$$

$$B = (f\chi' - \frac{1}{2}f'\chi - g\phi' + \frac{1}{2}g'\phi)dx^{-\frac{1}{2}},$$
$$C = \int_{S^1}(f'g'' - 4\phi'\chi')dx,$$

with

$$\hat{f} = (f(x,t)\frac{d}{dx}, \phi(x,t)dx^{-\frac{1}{2}}, a),$$
$$\hat{g} = (g(x,t)\frac{d}{dx}, \chi(x,t)dx^{-\frac{1}{2}}, b),$$
$$f' = \frac{\partial f}{\partial x}.$$

Let us denote

$$\mathscr{G}^* = \mathrm{C}^\infty(S^1)\oplus \mathrm{C}^\infty(S^1)\oplus \mathbb{R}$$

to be the dual space of $\mathscr{G}$, under the following pair

$$\langle \hat{m}, \hat{f}\rangle^* = \int_{S^1}(mf + \alpha\phi)dx + \varsigma a,$$

where

$$\hat{m} = (m(x,t)dx^2, \alpha(x,t)dx^{\frac{3}{2}}, \varsigma) \in \mathscr{G}^*.$$

By the definition,

$$\langle ad^*_{\hat{f}}(\hat{m}), \hat{g}\rangle^* = -\langle \hat{m}, [\hat{f}, \hat{g}]\rangle^*,$$

using integration by parts

$$\begin{aligned}
&\langle ad^*_{\hat{f}}(\hat{m}), \hat{g}\rangle^* = -\langle \hat{m}, [\hat{f}, \hat{g}]\rangle^* \\
&= \int_{S^1}(2mf' + m'f - \varsigma f''' + \frac{3}{2}\alpha\phi' + \frac{1}{2}\alpha'\phi)g dx \\
&\quad + \int_{S^1}(\frac{m}{2}\phi - \varsigma\phi'' + \frac{3}{2}f'\alpha + f\alpha')\chi dx \\
&= \left\langle \left((2mf' + m'f - \varsigma f''' + \frac{3}{2}\alpha\phi' + \frac{1}{2}\alpha'\phi)dx^2, (\frac{m}{2}\phi - \varsigma\phi'' + \frac{3}{2}f'\alpha + f\alpha')dx^{\frac{3}{2}}, 0\right), \hat{g}\right\rangle^*.
\end{aligned}$$

Observe that the stabilizer space of the coadjoint action of the Neveu-Schwarz superalgebra $\mathscr{G}$ on the hyperplane $\varsigma = 0$ of $\mathscr{G}^*$ is given by

$$2mf' + m'f - \varsigma f''' + \frac{3}{2}\alpha\phi' + \frac{1}{2}\alpha'\phi = 0,$$
$$\frac{m}{2}\phi - \varsigma\phi'' + \frac{3}{2}f'\alpha + f\alpha' = 0.$$

which are exactly the latter two equations in (2.5). Thus we have

Proposition 4.1. *The mixed Kuper-CH-HS equation* (2.5) *is the constraint Hamiltonian flow on the Neveu-Schwarz coadjoint orbit, that is to say*

$$\begin{pmatrix} m \\ \alpha \end{pmatrix}_t = ad^*_{(\frac{\delta H}{\delta m}, \frac{\delta H}{\delta \alpha})}\begin{pmatrix} m \\ \alpha \end{pmatrix} - K\begin{pmatrix} f \\ \phi \end{pmatrix},$$

with

$$ad^*_{\hat{f}}(\hat{m}) = 0,$$

$$H = \frac{k_1}{2}\int (mu + \alpha\eta_x)dx,$$

where

$$\hat{m} = (m(x,t)dx^2, \alpha(x,t)dx^{\frac{3}{2}}, 0) \in \mathscr{G}^*,$$
$$\hat{f} = (f(x,t)\frac{d}{dx}, \phi(x,t)dx^{-\frac{1}{2}}, a) \in \mathscr{G}.$$

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