Magnetic curves in Sol₃

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Magnetic curves with respect to the canonical contact structure of the space Sol_3 are investigated.

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Introduction

In electromagnetic theory, magnetic curves represent the trajectories of charged particles moving in Euclidean 3-space $\mathbb{E}^3$ under a static magnetic field $\vec{B}$. Newton's second law of motions under the Lorentz force derived from a static magnetic field implies the law of Lorentz force. More precisely, a particle of mass m and charge q on position $\vec{r}(t)$ in a static magnetic field $\vec{B}$ moves with the velocity $\vec{v}(t)$ satisfying the Lorentz equation

$$m\frac{d\vec{v}}{dt}(t) = q\,\vec{v}(t) \times \vec{B}_{\vec{r}(t)}.$$

As is well known, in $\mathbb{E}^3$ the motion of the particle is described by a circular helix around $\vec{B}$. Particularly, magnetic trajectories of the particle can be circles (and hence periodic curves).

The notion of a static magnetic field can be generalized to arbitrary Riemannian manifolds (see [2, 25]). Let (M, g, F) be a Riemannian manifold with a closed 2-form F. Then F is referred to as a *magnetic field* on M. A curve $\gamma(t)$ is called a *magnetic curve* if it satisfies the Lorentz equation:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = q\,\varphi\,\dot{\gamma}.$$

Here q is a constant (called the *charge*), ∇ is the Levi-Civita connection and φ is an endomorphism field metrically related to the magnetic field F via g. Hence geodesics in Riemannian manifolds are mathematical models of motions of particles without the Lorentz force or charge 0.

On the other hand, according to Thurston, there are eight model spaces in 3-dimensional homogeneous geometries.

- space forms: Euclidean 3-space $\mathbb{E}^3$, 3-sphere $\mathbb{S}^3$, hyperbolic 3-space $\mathbb{H}^3$,
- product spaces: $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$,
- the Heisenberg group Nil$_3$, the universal covering $\widetilde{\mathrm{SL}_2\mathbb{R}}$ of $\mathrm{SL}_2\mathbb{R}$,
- the Sol$_3$ space.

Among these eight model spaces, $\mathbb{S}^3$, Nil$_3$, $\widetilde{\mathrm{SL}_2\mathbb{R}}$ and Sol$_3$ admits a contact structure compatible to the metric (see [24]). The compatible contact structure naturally induces a magnetic field F (called a *contact magnetic field*) on these four model spaces.

The study of magnetic curves in arbitrary Riemannian manifolds was developed in early 1990's, even though related works can be found earlier (see [10, 25]). The notation used here is very similar to notation used in [7] and [8].

In 2009 Cabrerizo et al. have looked for periodic orbits of the contact magnetic field on the unit 3-sphere $\mathbb{S}^3$ in [7]. In addition, Druţă-Romaniuc et al. classified magnetic trajectories in Nil$_3$ and $\widetilde{\mathrm{SL}_2\mathbb{R}}$ with respect to a contact magnetic field [8]. Magnetic trajectories on the space Sol$_3$ with respect to a contact magnetic field are not studied, yet.

The purpose of this paper is to study magnetic trajectories in the model space Sol$_3$ of solvegeometry with respect to a contact magnetic field.

1. Magnetic curves

Let (M, g) be a Riemannian manifold. We equip a *closed* 2-form F on M. Thus we get an endomorphism field φ by

$$g(\varphi X, Y) = F(X, Y). \tag{1.1}$$

We regard F as a (mathematical model of) magnetic field (see [2, 25]). And the endomorphism field φ is referred to as the *Lorentz force* derived from F.

Then a *magnetic trajectory* γ (also called a *magnetic curve*) is defined as a solution to

$$\nabla_{\gamma'} \gamma' = q\,\varphi\,\gamma'. \tag{1.2}$$

Here q is a real constant called the *charge* of the magnetic trajectory $\gamma(t)$ under the magnetic field F.

It is well-known that magnetic trajectories have constant speed. When a magnetic curve $\gamma(s)$ is arc length parametrized, it is called a *normal magnetic curve*.

One can see that the differential equation of magnetic trajectory is a generalization of geodesic equation. In fact if $\varphi = 0$, i.e. $F = 0$ or $q = 0$, the differential equation coincides with geodesic equation.

On a Riemannian manifold (M, g, F) equipped with an *exact* magnetic field $F = dA$, one can consider the variational problem for regular curves $\gamma(t)$ with respect to the following *Landau-Hall functional*:

$$\mathrm{LH}(\gamma) = \int_0^\ell \frac{1}{2} g(\gamma'(t), \gamma'(t))\, dt - q \int_0^\ell A(\gamma'(t))\, dt.$$

Here q is a real constant.

Let p and p' be distinct points. Denote by $C^\infty[a,b]$ the space of smooth curves in M defined on a closed interval $[a,b]$ satisfying the boundary condition

$$\gamma(a) = p, \ \ \gamma(b) = p'.$$

Take a variation γ_ε through γ (i.e., $\gamma_0(s) = \gamma(s)$) satisfying the boundary condition

$$\gamma_\varepsilon(a) = p, \ \ \gamma_\varepsilon(b) = p'.$$

Then the first variation formula of the Landau-Hall functional is given by (see e.g. [11]):

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathrm{LH}(\gamma_\varepsilon) = -\int_0^\ell g(\nabla_{\gamma'}\gamma' - q\,\varphi\,\gamma', V(s))ds,$$

where V is the variational vector field

$$V(s) = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \gamma_\varepsilon(s).$$

Thus the Euler-Lagrange equation of this variational problem is exactly the magnetic equation (1.2).

Note that the magnetic equation (1.2) makes sense even if F is not exact.

Remark 1.1. Magnetic curves with respect to non-standard magnetic fields on Euclidean 3-space $\mathbb{E}^3$ are used in computer aided geometric design (see [29] and [30]).

2. Contact structures

Let M be a 3-dimensional manifold. A 1-form η is said to be a *contact form* if it satisfies $d\eta \wedge \eta \neq 0$. A 3-dimensional manifold M together with a contact form η is called a *contact* 3-*manifold*. Luts and Martinet proved that every compact orientable 3-manifold carries a contact form (see [19, 20, 27]).

On a contact 3-manifold (M, η), there exists a unique vector filed ξ such that $\eta(\xi) = 1$ and $\iota_\xi d\eta = 0$. The vector field ξ is called the *Reeb vector field* of (M, η). In analytical mechanics, ξ is traditionally called the *characteristic vector field* of (M, η).

Moreover, every contact 3-manifold (M, η) admits an endomorphism field φ and a Riemannian metric g such that (see [3]):

$$\varphi^2 = -\mathrm{I} + \eta \otimes \xi, \ \ \eta \circ \varphi = 0, \ \varphi\xi = 0,$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \ X, Y \in \mathfrak{X}(M), \tag{2.1}$$

and

$$d\eta(X,Y) = g(X, \varphi Y), \ X, Y \in \mathfrak{X}(M). \tag{2.2}$$

Here $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M. The exterior derivative $d\eta$ of η is defined by

$$d\eta(X,Y) = \frac{1}{2}\Big(X\eta(Y) - Y\eta(X) - \eta([X,Y])\Big),$$

for any $X, Y \in \mathfrak{X}(M)$.

The structure (φ, ξ, η, g) is called an *almost contact metric structure* associated to the contact form η. The resulting space $(M, \varphi, \xi, \eta, g)$ is called a *contact metric 3-manifold.* Note that the volume element of a contact metric 3-manifold M is $-\eta \wedge d\eta/2$.

Remark 2.1. On a contact 3-manifold (M, η) equipped with an arbitrary chosen Riemannian metric g, one can take a magnetic field $F = k\, d\eta$ (k is a nonzero constant) and consider magnetic curves with respect to F and g. It seems to be natural to demand that the metric g satisfies some "compatibility condition" (see e.g. (2.1)) with respect to F. In this paper we restrict our attention to Riemannian metrics satisfying the condition:

$$g(X, \varphi Y) = kd\eta(X, Y), \quad X, Y \in \mathfrak{X}(M).$$

Remark 2.2. Perrone in [24] classified homogeneous contact metric 3-manifolds. According to [24], among the simply connected model spaces of Thurston geometry, the following spaces admit a homogeneous contact form compatible to the metric:

$$\mathbb{S}^3, \quad \mathrm{Nil}_3, \quad \widetilde{\mathrm{SL}_2\mathbb{R}}, \quad \mathrm{Sol}_3.$$

For more information on contact forms on compact 3-manifolds with universal cover $\widetilde{\mathrm{SL}_2\mathbb{R}}$ and Sol$_3$ refer to [22].

3. Invariant contact structure on Sol$_3$

3.1. *Model of the* Sol$_3$ *space*

In this subsection we recall relevant facts on Sol$_3$ given in [5, 6, 9, 13–18].

The model space Sol$_3$ of solvegometry in the sense of Thurston (see [26]) is the Cartesian 3-space $\mathbb{R}^3(x, y, z)$ equipped with a homogeneous Riemannian metric

$$g = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2. \tag{3.1}$$

The Sol$_3$ space is a Lie group G with respect to the multiplication law:

$$(x, y, z) * (a, b, c) = (x + e^{-z}a, y + e^{z}b, z + c).$$

The unit element is $(0, 0, 0)$ and the inverse element of (x, y, z) is $(-e^{z}x, -e^{-z}y, -z)$. The left translated vector fields associated to the orthonormal basis $\hat{E}_1 = (1, 0, 0)$, $\hat{E}_2 = (0, 1, 0)$, $\hat{E}_3 = (0, 0, 1)$ are

$$\hat{e}_1 = e^{-z}\frac{\partial}{\partial x}, \quad \hat{e}_2 = e^{z}\frac{\partial}{\partial y}, \quad \hat{e}_3 = \frac{\partial}{\partial z}. \tag{3.2}$$

The space Sol$_3$ can be realized as the closed subgroup

$$\left\{ \begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\}$$

of SL$_3\mathbb{R}$. The corresponding Lie algebra $\mathfrak{sol}_3$ is

$$\left\{ \begin{pmatrix} -w & 0 & u \\ 0 & w & v \\ 0 & 0 & 0 \end{pmatrix} \middle| u,v,w \in \mathbb{R} \right\}.$$

The orthonormal basis $\{\hat{E}_1, \hat{E}_2, \hat{E}_3\}$ of $\mathfrak{sol}_3$ is identified with

$$\hat{E}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \hat{E}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ \hat{E}_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The dual coframe field $\vartheta = (\theta^1, \theta^2, \theta^3)$ of $\mathscr{E} = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ is

$$\theta^1 = e^z dx, \ \theta^2 = e^{-z} dy, \ \theta^3 = dz. \tag{3.3}$$

The connection 1-forms $\{\omega_j{}^i\}$ defined by $d\theta^i + \sum_{k=1}^3 \omega_j{}^i \wedge \theta^j = 0$ relative to ϑ are

$$(\omega_j{}^i) = \begin{pmatrix} 0 & 0 & \theta^1 \\ 0 & 0 & -\theta^2 \\ -\theta^1 & \theta^2 & 0 \end{pmatrix}.$$

The curvature 2-forms $\{\Omega_j{}^i\}$ defined by $\Omega_j{}^i = d\omega_j{}^i + \sum_{k=1}^3 \omega_k{}^i \wedge \omega_j{}^k$ relative to ϑ are

$$(\Omega_j{}^i) = \begin{pmatrix} 0 & \theta^1 \wedge \theta^2 & -\theta^1 \wedge \theta^3 \\ -\theta^1 \wedge \theta^2 & 0 & -\theta^2 \wedge \theta^3 \\ \theta^1 \wedge \theta^3 & \theta^2 \wedge \theta^3 & 0 \end{pmatrix}$$

In covariant derivative fashion, the Levi-Civita connection ∇ of Sol$_3$ is described as follows

$$\begin{array}{lll} \nabla_{\hat{e}_1}\hat{e}_1 = -\hat{e}_3, & \nabla_{\hat{e}_1}\hat{e}_2 = 0, & \nabla_{\hat{e}_1}\hat{e}_3 = \hat{e}_1, \\ \nabla_{\hat{e}_2}\hat{e}_1 = 0, & \nabla_{\hat{e}_2}\hat{e}_2 = \hat{e}_3, & \nabla_{\hat{e}_2}\hat{e}_3 = -\hat{e}_2, \\ \nabla_{\hat{e}_3}\hat{e}_1 = 0, & \nabla_{\hat{e}_3}\hat{e}_2 = 0, & \nabla_{\hat{e}_3}\hat{e}_3 = 0. \end{array} \tag{3.4}$$

The Riemannian curvature R is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, \ X,Y,Z \in \mathfrak{X}(M).$$

The Riemannian curvature R is expressed in components R_{kij}^ℓ by

$$R(\hat{e}_i, \hat{e}_j)\hat{e}_k = \sum_{\ell=1}^3 R_{kij}^\ell \hat{e}_\ell$$

is computed as

$$R_{212}^1 = 1, \ R_{313}^1 = -1, \ R_{323}^2 = -1.$$

The Ricci tensor field Ric is defined by

$$\mathrm{Ric}(X,Y) = \mathrm{tr}(Z \longmapsto R(Z,X)Y).$$

The components $R_{ij} = \mathrm{Ric}(\hat{e}_i, \hat{e}_j) = \sum_{l=1}^{3} R^l_{ilj}$ is given by

$$R_{11} = 0, \ R_{22} = 0, \ R_{33} = -2.$$

The scalar curvature $\rho := \mathrm{tr}\,\mathrm{Ric} = \sum_{i=1}^{3} R_{ii}$ is -2.

3.2. *Invariant contact structure on* Sol$_3$

In this subsection, we introduce a left invariant contact structure on Sol$_3$.

For more about a left invariant contact structures see [4, 14, 24].

On the solvable Lie group Sol$_3$, we may take the following left invariant orthonormal frame field:

$$
\begin{aligned}
e_1 &:= \frac{1}{\sqrt{2}}(\hat{e}_1 - \hat{e}_2), \\
e_2 &:= \hat{e}_3, \\
e_3 &:= \frac{1}{\sqrt{2}}(\hat{e}_1 + \hat{e}_2).
\end{aligned}
\tag{3.5}
$$

Here the orthonormal frame field $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ is defined by (3.2).

We choose $\xi := e_3$ and denote by η the metrical dual 1-form of ξ. Namely η is given by

$$\eta = \frac{1}{\sqrt{2}} \left(e^z dx + e^{-z} dy \right).$$

Then η is a left invariant contact form on Sol$_3$. Next we define an endomorphism field φ by

$$\varphi e_1 = -e_2, \ \varphi e_2 = e_1, \ \varphi e_3 = 0. \tag{3.6}$$

Then φ and ξ are also left invariant on Sol$_3$. Direct calculations show that

$$d\eta(X,Y) = \frac{1}{2}g(\varphi X, Y), \ \ X, Y \in \mathfrak{X}(\mathrm{Sol}_3). \tag{3.7}$$

Remark 3.1. Precisely speaking, to adapt to contact metric geometry, we need to perform the following normalization procedure:

$$\varphi \longmapsto -\varphi, \ \xi \longmapsto 2\xi, \ \eta \longmapsto \frac{1}{2}\eta, \ g \longmapsto \frac{1}{4}g.$$

Then the resulting quintet $(\mathrm{Sol}_3, -\varphi, 2\xi, \eta/2, g/4)$ is a *contact metric manifold* (in the sense of [3]) as explained in Section 2.

However, in the study of magnetic curves, this normalization is not essential. So we do not use this normalization hereafter (cf. Remark 2.1).

According to (3.4) and (3.5), the Levi-Civita connection ∇ of Sol$_3$ is rewritten as

$$
\begin{aligned}
\nabla_{e_1}e_1 &= 0, & \nabla_{e_1}e_2 &= e_3, & \nabla_{e_1}e_3 &= -e_2, \\
\nabla_{e_2}e_1 &= 0, & \nabla_{e_2}e_2 &= 0, & \nabla_{e_2}e_3 &= 0, \\
\nabla_{e_3}e_1 &= -e_2, & \nabla_{e_3}e_2 &= e_1, & \nabla_{e_3}e_3 &= 0.
\end{aligned}
\tag{3.8}
$$

The commutation relations are

$$
[e_1,e_2] = e_3, \quad [e_2,e_3] = -e_1, \quad [e_3,e_1] = 0.
$$

Thus $\{e_1,e_2,e_3\}$ is a *unimodular basis* of $\mathfrak{sol}_3$ [21].

The sectional curvature K is determined by

$$
K(e_1 \wedge e_2) = -1, \quad K(e_2 \wedge e_3) = -1, \quad K(e_1 \wedge e_3) = 1.
$$

4. Magnetic curves in Sol$_3$

4.1. *Contact magnetic fields*

Let $M = (M,\varphi,\xi,\eta,g)$ be a contact metric 3-manifold. Then for a constant k, $F = k\,d\eta$ is a magnetic field on M. The magnetic field $F = k\,d\eta$ is called the *contact magnetic field* on a contact metric 3-manifold M. Magnetic trajectories with respect to contact magnetic fields are called *contact magnetic trajectories*.

Contact magnetic trajectories on the 3-sphere are investigated in [7]. Munteanu and Nistor studied periodicity of contact magnetic fields on 3-tori [23].

In the case of Sol$_3$ equipped with the structure (φ,ξ,η,g) defined in Section 3.2, we take the contact magnetic field F given by (see (3.7)).

$$
F(X,Y) = 2\,d\eta(X,Y), \quad X,Y \in \mathfrak{X}(\mathrm{Sol}_3).
\tag{4.1}
$$

Then the corresponding Lorentz force coincides with φ.

The magnetic curve equation on Sol$_3$ with respect to $F = 2d\eta$ with charge q is

$$
\nabla_{\gamma'}\gamma' = q\,\varphi\,\gamma'.
\tag{4.2}
$$

Note that contact magnetic equation (4.2) is the Euler-Lagrange equation of the Landau-Hall functional

$$
\mathrm{LH}(\gamma) = \int_0^{\ell} \frac{1}{2} g(\gamma'(s),\gamma'(s))\,ds - 2q \int_0^{\ell} \eta(\gamma'(s))\,ds.
$$

4.2. *Magnetic curve equation*

First task is to deduce the magnetic curve equation (4.2) for a regular curve $\gamma(s) = (x(s),y(s),z(s))$ in Sol$_3$. We have

$$
\gamma'(s) = x'(s)\frac{\partial}{\partial x} + y'(s)\frac{\partial}{\partial y} + z'(s)\frac{\partial}{\partial z},
$$

and from (3.2) and (3.5) it follows

$$
\gamma'(s) = x'(s)\,\frac{e^z}{\sqrt{2}}(e_1 + e_3) - y'(s)\,\frac{e^{-z}}{\sqrt{2}}(e_1 - e_3) + z'(s)\,e_2
$$

and hence

$$\gamma'(s) = \frac{1}{\sqrt{2}} \left(e^z x'(s) - e^{-z} y'(s) \right) e_1 + z'(s) e_2 + \frac{1}{\sqrt{2}} \left(e^z x'(s) + e^{-z} y'(s) \right) e_3. \tag{4.3}$$

Next we compute the covariant derivative $\nabla_{\gamma'} \gamma'$.

$$\begin{aligned}
\nabla_{\gamma'} \gamma' = &\frac{1}{\sqrt{2}} \left(e^z x'' - e^{-z} y'' + 2z' \left(e^z x' + e^{-z} y' \right) \right) e_1 \\
&+ \left(z'' + e^{-2z}(y')^2 - e^{2z}(x')^2 \right) e_2 \\
&+ \frac{1}{\sqrt{2}} \left(e^z x'' + e^{-z} y'' + 2z' \left(e^z x' - e^{-z} y' \right) \right) e_3.
\end{aligned}$$

Taking in account relations (4.3) and (3.6) we have

$$\varphi \gamma' = z' e_1 - \frac{1}{\sqrt{2}} \left(e^z x' - e^{-z} y' \right) e_2.$$

Hence from the magnetic curve equation (4.2) we obtain system of differential equations

$$\begin{aligned}
e^z x'' - e^{-z} y'' + 2z' \left(e^z x' + e^{-z} y' \right) &= \sqrt{2}\, q\, z', \\
z'' + e^{-2z}(y')^2 - e^{2z}(x')^2 &= \frac{-1}{\sqrt{2}}\, q \left(e^z x' - e^{-z} y' \right), \\
e^z x'' + e^{-z} y'' + 2z' \left(e^z x' - e^{-z} y' \right) &= 0,
\end{aligned} \tag{4.4}$$

Remark 4.1. Notice that the system of differential equations (4.4) for $q = 0$ coincides with the system of differential equations (4.4) in [6] which determines geodesics in Sol$_3$ (*cf.* [5, 28]).

Without loss of generality, we can restrict our attention to magnetic trajectories under the initial conditions:

$$x(0) = 0, \quad y(0) = 0, \quad z(0) = 0, \quad x'(0) = a, \quad y'(0) = b, \quad \text{and} \quad z'(0) = c,$$

since Sol$_3$ is a homogeneous Riemannian space.

After the summing of the first and the third equation of the system (4.4) we obtain

$$x'' + 2x'z' = \frac{\sqrt{2}}{2}\, q\, e^{-z}\, z'. \tag{4.5}$$

Solving this ODE in the first step we get

$$x'(s) = \left(a - \frac{q}{\sqrt{2}} \right) e^{-2z(s)} + \frac{q}{\sqrt{2}} e^{-z(s)} \tag{4.6}$$

and finally

$$x(s) = \left(a - \frac{q}{\sqrt{2}} \right) \int_0^s e^{-2z(\tau)} d\tau + \frac{q}{\sqrt{2}} \int_0^s e^{-z(\tau)} d\tau. \tag{4.7}$$

Analogously for y-coordinate, after subtracting the first from the third equation of the system (4.4) we obtain following equation

$$y'' - 2y'z' = -\frac{\sqrt{2}}{2}\, q\, e^z\, z'. \tag{4.8}$$

Hence

$$y'(s) = \left(b - \frac{q}{\sqrt{2}}\right)e^{2z(s)} + \frac{q}{\sqrt{2}}e^{z(s)} \tag{4.9}$$

and

$$y(s) = \left(b - \frac{q}{\sqrt{2}}\right)\int_0^s e^{2z(\tau)}d\tau + \frac{q}{\sqrt{2}}\int_0^s e^{z(\tau)}d\tau. \tag{4.10}$$

Substituting (4.6) and (4.9) in the second equation of the system (4.4) we get

$$z'' + \left(b - \frac{q}{\sqrt{2}}\right)^2 e^{2z} - \left(a - \frac{q}{\sqrt{2}}\right)^2 e^{-2z} + \frac{q}{\sqrt{2}}\left(\left(b - \frac{q}{\sqrt{2}}\right)e^z - \left(a - \frac{q}{\sqrt{2}}\right)e^{-z}\right) = 0 \tag{4.11}$$

Now we assume that s is an arc length parameter of γ. If we multiply this equation by $2z'(\neq 0)$, after integrating and using $z'(0) = c = \sqrt{1 - a^2 - b^2}$, we obtain

$$(z')^2 + \left[\left(a - \frac{q}{\sqrt{2}}\right)e^{-z} + \frac{q}{\sqrt{2}}\right]^2 + \left[\left(b - \frac{q}{\sqrt{2}}\right)e^z + \frac{q}{\sqrt{2}}\right]^2 = 1 \tag{4.12}$$

After separation of variables, the solution of this equation is given by the following elliptic integral

$$\frac{dz}{\pm\sqrt{1 - \left[\left(a - \frac{q}{\sqrt{2}}\right)e^{-z} + \frac{q}{\sqrt{2}}\right]^2 - \left[\left(b - \frac{q}{\sqrt{2}}\right)e^z + \frac{q}{\sqrt{2}}\right]^2}} = ds \tag{4.13}$$

Hence, the following theorem is proved.

Theorem 4.1. *The normal magnetic curves of the space* Sol_3 *with respect to the contact magnetic field* $F = 2d\eta$ *with charge* $q \neq 0$ *is given by the following equations :*

$$x(s) = \left(a - \frac{q}{\sqrt{2}}\right)\int_0^s e^{-2z(\tau)}d\tau + \frac{q}{\sqrt{2}}\int_0^s e^{-z(\tau)}d\tau,$$

$$y(s) = \left(b - \frac{q}{\sqrt{2}}\right)\int_0^s e^{2z(\tau)}d\tau + \frac{q}{\sqrt{2}}\int_0^s e^{z(\tau)}d\tau,$$

$$ds = \frac{dz}{\pm\sqrt{1 - \left[\left(a - \frac{q}{\sqrt{2}}\right)e^{-z} + \frac{q}{\sqrt{2}}\right]^2 - \left[\left(b - \frac{q}{\sqrt{2}}\right)e^z + \frac{q}{\sqrt{2}}\right]^2}},$$

where a, b, $c \in \mathbb{R}$ *and* $a^2 + b^2 + c^2 = 1$.

In the sequel we consider particular cases of magnetic curves in Sol_3.

Example 1

First we examine case $z' = 0$. Then (4.12) implies $z = 0$ and from (4.7) and (4.10) it follows

$$\gamma(s) = (as, bs, 0) \tag{4.14}$$

where a, $b \in \mathbb{R}$. It is a (geodesic) line in the plane $z = 0$.

Example 2

If we assume that $a = b = \frac{q}{\sqrt{2}}$, then from (4.12), (4.7) and (4.10) it follows

$$\gamma(s) = \left(\frac{-q \cdot e^{-(\sqrt{1-q^2} \cdot s)}}{\sqrt{2(1-q^2)}}, \; \frac{q \cdot e^{(\sqrt{1-q^2} \cdot s)}}{\sqrt{2(1-q^2)}}, \; \sqrt{1-q^2} \cdot s \right). \tag{4.15}$$

Particularly, for $q = 0$ we have the z-axis, which is a geodesic line in Sol₃ space.

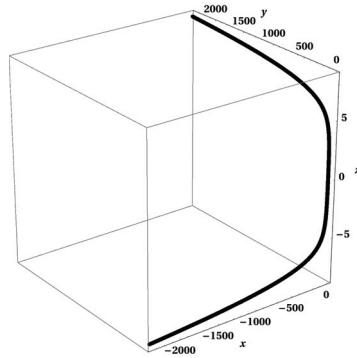Figure 1 shows the magnetic curve for $a = b = \frac{q}{\sqrt{2}}$, $q = \frac{1}{2}$, $s \in [-10, 10]$.

Fig. 1. $\gamma(s) = \left(\frac{-1}{\sqrt{6}} e^{-\frac{\sqrt{3}}{2} s}, \; \frac{1}{\sqrt{6}} e^{\frac{\sqrt{3}}{2} s}, \; \frac{\sqrt{3}}{2} s \right)$

Example 3

If we assume $a = \frac{q}{\sqrt{2}}$, then from (4.11) we have

$$z'' + \left(b - \frac{q}{\sqrt{2}} \right)^2 e^{2z} + \frac{q}{\sqrt{2}} \left(b - \frac{q}{\sqrt{2}} \right) e^z = 0 \tag{4.16}$$

The solution of this equation is

$$z(s) = -\ln \left[\left(b - \frac{q}{\sqrt{2}} \right) \left(\sqrt{1 + \frac{q^2}{2}} \cosh s + \frac{q}{\sqrt{2}} \right) \right].$$

Further, from (4.7) and (4.10) it follows

$$x(s) = \frac{q \cdot (2b - \sqrt{2}q)}{4} \left(q \cdot s + \sqrt{2 + q^2} \cdot \sinh s \right),$$

$$y(s) = \frac{2}{(2b - \sqrt{2}q)} \frac{\sqrt{2 + q^2} \cdot \sinh s}{\left(q + \sqrt{2 + q^2} \cdot \cosh s \right)}.$$

Particularly, for $q = 0$ we obtain geodesic line in yz-plane.

Figure 2 shows the magnetic curve for $b = 1, q = \frac{1}{2}, s \in [-10, 10]$.
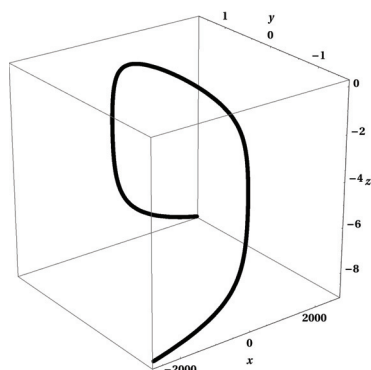
Fig. 2. $\gamma(s) = \left(\frac{4-\sqrt{2}}{32}(s+3\sinh s), \frac{6(4+\sqrt{2})\sinh s}{7(1+3\cosh s)}, \ln\left(\frac{8}{(2\sqrt{2}-1)(1+3\cosh s)} \right) \right)$

Example 4

If we assume $b = \frac{q}{\sqrt{2}}$, then from (4.11) we have

$$z'' - \left(a - \frac{q}{\sqrt{2}} \right)^2 e^{-2z} - \frac{q}{\sqrt{2}} \left(a - \frac{q}{\sqrt{2}} \right) e^{-z} = 0. \tag{4.17}$$

The solution of this equation is

$$z(s) = \ln \left[\left(a - \frac{q}{\sqrt{2}} \right) \left(\sqrt{1 + \frac{q^2}{2}} \cosh s + \frac{q}{\sqrt{2}} \right) \right].$$

Further, from (4.7) and (4.10) it follows

$$x(s) = \frac{2}{(2a - \sqrt{2}q)} \frac{\sqrt{2+q^2} \cdot \sinh s}{\left(q + \sqrt{2+q^2} \cdot \cosh s \right)},$$

$$y(s) = \frac{q \cdot (2a - \sqrt{2}q)}{4} \left(q \cdot s + \sqrt{2+q^2} \cdot \sinh s \right).$$

Particularly, for $q = 0$ we obtain geodesic line in *xz*-plane.

Figure 3 shows the magnetic curve for $a = 1$, $q = \frac{1}{2}$, $s \in [-10, 10]$.

Remark 4.2 (Magnetic Jacobi fields). Adachi [1] and Gouda [11] obtained the second variational formula of the Landau-Hall functional:

$$\frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} \mathrm{LH}(\gamma_\varepsilon) = - \int_0^\ell g(\mathscr{J}_{q,F}(V), V(s)) ds,$$

where $\mathscr{J}_{q,F}$ is an operator acting on the space $\Gamma(\gamma^* TM)$ of all vector fields along γ defined by

$$\mathscr{J}_{q,F}(W) = \nabla_{\gamma'}\nabla_{\gamma'}W + R(W, \gamma')\gamma' - q\varphi(\nabla_{\gamma'}W) - q(\nabla_W \varphi)\gamma'.$$

A vector field $W(s)$ along γ is said to be a *magnetic Jacobi field* if it satisfies $\mathscr{J}_{q,F}(W) = 0$. Detailed study on magnetic Jacobi fields gives us insight on how small variations in the initial conditions
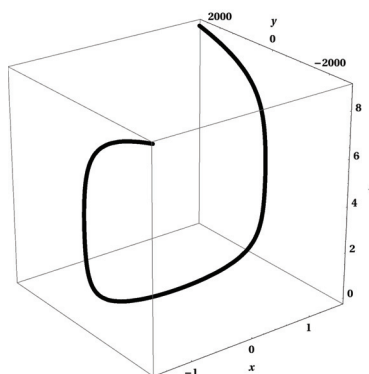
Fig. 3. $\gamma(s) = \left(\frac{6(4+\sqrt{2})\sinh s}{7(1+3\cosh s)}, \frac{4-\sqrt{2}}{32}(s+3\sinh s), \ln\left(\frac{1}{8}(2\sqrt{2}-1)(1+3\cosh s)\right) \right)$

affect the evolution of magnetic curves. In this direction, Adachi obtained the comparison theorem for magnetic curves on Kähler manifold whose Lorentz force is a complex structure [1]. Gouda [11, 12] studied magnetic Jacobi fields on Riemannian 2-manifolds equipped with compatible Kähler structure. The parallelism of the complex structure (the Lorentz force) is crucial in these works.

In case $(M,g) = \text{Sol}_3$, the sectional curvature function can have both signs. In addition the Lorentz force φ is non-parallel. Thus the behavior of magnetic Jacobi fields along contact magnetic curves in Sol$_3$ appears complicated. This will be addressed in future work.

Acknowledgments

References

[1] T. Adachi, A comparison theorem on magnetic Jacobi fields, *Proc. Edinburgh Math. Soc.* **40** (1997), 293–308.

[2] V. I. Arnold, Some remarks on flows of line elements and frames, *Sov. Math. Dokl.* **2** (1961), 562–564.

[3] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Math. **203** (Birkhäuser, Basel, 2002).

[4] D. E. Blair, D. Perrone, Conformally Anosov flows in contact metric geometry, *Balkan J. Geom. Appl.* **3** (1998) 2, 33–46.

[5] A. Bölcskei, B. Szilágyi, Visualization of curves and spheres in Sol geometry, *KoG* **10** (2006) 27–32.

[6] A. Bölcskei, B. Szilágyi, Frenet formulas and geodesics in Sol geometry, *Beit. Alg. Geom.* **48** (2007) 2, 411–421.

[7] J. L. Cabrerizo, M. Fernández, and J. S. Gómez, The contact magnetic flow in 3D Sasakian manifolds, *J. Phys. A: Math. Theor.* **42** (2009) 19, 195201, 10 pages.

[8] S. L. Druţă-Romaniuc, J. Inoguchi, M. I. Munteanu, A. I. Nistor, Magnetic curves in Sasakian manifolds, *J. Nonlinear Math. Phys.* **22** (2015) 3, 428–447.

[9] Z. Erjavec, On a certain class of Weingarten surfaces in Sol space, *Int. J. Appl. Math.* **28** (2015) 5, 507-514.

[10] V. L. Ginzburg, A charge in a magnetic field: Arnold's problems 1981-9, 1982-24, 1984-4, 1994-14, 1994-35, 1996-17,1996-18, in *Arnold's problems* (V.I. Arnold ed.) (Springer-Verlag and Phasis, 2004) 395–401.

[11] N. Gouda, Magnetic flows of Anosov type, *Tohoku Math. J.* **49** (1997), 165–183.

[12] N. Gouda, The theorem of E. Hopf under uniform magnetic fields, *J. Math. Soc. Japan* **50** (1998) 3, 767–779.

[13] J. Inoguchi, Minimal surfaces in 3-dimensional solvable Lie groups, *Chinese Ann. Math. B.* **24** (2003), 73–84.

[14] J. Inoguchi, On homogeneous contact 3-manifolds, *Bull. Fac. Edu. Utsunomiya Univ.Sect.* 2. **59** (2009), 1–12.

[15] J. Inoguchi, S. Lee, A Weierstrass type representation for minimal surfaces in Sol, *Proc. Amer. Math. Soc.* **146** (2008) 2209–2216.

[16] R. López, M. I. Munteanu, On the geometry of constant angle surfaces in *Sol*$_3$, *Kyushu J. Math.* **65** (2011) 237–249.

[17] R. López, M. I. Munteanu, Surfaces with constant mean curvature in Sol geometry, *Differential. Geom. Appl.* **29** (2011) 238–245.

[18] R. López, M. I. Munteanu, Minimal translation surfaces in Sol$_3$, *J. Math. Soc. Japan* **64** (2012) 3, 985–1003.

[19] R. Lutz, Sur quelques propriétés des formes différentielles en dimension trois, Thèse, Strasbourg, 1971.

[20] J. Martinet, Formes de contact sur les variétés de dimension 3, *Proc. Liverpool Singularity Sympos.* II, Lecture Notes in Math. **209** (1971), 142–163.

[21] J. Milnor, Curvatures of left invariant metrics on Lie groups, *Adv. Math.* **21** (1976) 293–329.

[22] Y. Mitsumatsu, Anosov flows and non-Stein symplectic manifolds, *Ann. Inst. Fourier* **45** (1995) 5, 1407–1421.

[23] M. I. Munteanu, A. I. Nistor, On some closed magnetic curves on a 3-torus, *Math. Phys. Anal. Geom.* **20** (2017), art. no. 8, 13 pages.

[24] D. Perrone, Homogeneous contact Riemannian three-manifolds, *Illinois J. Math.* **13** (1997), 243–256.

[25] T. Sunada, Magnetic flows on a Riemann surface, in *Proc. KAIST Mathematics Workshop: Analysis and Geometry*, (KAIST, Taejeon, Korea, 1993), pp. 93–108.

[26] W. M. Thurston, *Three-dimensional Geometry and Topology* I, Princeton Math. Series., vol. 35 (S. Levy ed.), 1997.

[27] W. M. Thurston, H. E. Winkelnkemper, On the existence of contact forms, *Proc. Amer. Math. Soc.* **52** (1975), 1, 345–347.

[28] M. Troyanov, L'horizon de SOL, *Expo. Math.* **16** (1998) 441–479.

[29] M. S. Wo, R. U. Gobithaasan, K. T. Miura, Log-aesthetic magnetic curves and their application for CAD systems, *Math. Probl. Eng.* **2014** Article ID 504610 (2014) 16 pages.

[30] L. Xu, D. Mould, Magnetic curves: Curvature-controlled aesthetic curves using magnetic fields, *Computational Aesthetics in Graphics, Visualization and Imaging*, (O. Deussen and P. Hall eds.) (Victoria, British Columbia, Canada, 2009) 1–8.