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Morphisms Cohomology and Deformations of Hom-algebras

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The purpose of this paper is to define cohomology complexes and study deformation theory of Hom-associative algebra morphisms and Hom-Lie algebra morphisms. We discuss infinitesimal deformations, equivalent deformations and obstructions. Moreover, we provide various examples.

Keywords: Hom-associative algebra morphism; Hom-Lie algebra morphism; cohomology; deformation.

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Introduction

The first instance of Hom-type algebras appeared in physics literature when looking for q -deformations, which consists of replacing the usual derivation by a σ -derivation, of some algebras of vector fields, [1, 3, 13]. The main examples dealt with Witt and Virasoro algebras and used Jackson derivation, defined for a polynomial P by $D_q(P(t)) = \frac{P(qt) - P(t)}{qt - t}$. It turns out that the obtained algebras no longer satisfy Jacobi identity, but a modified version involving a homomorphism. These algebras were called *Hom-Lie algebras* and studied first, from mathematical viewpoint, by Hartwig, Larsson and Silvestrov in [11, 12]. *Hom-associative algebras* play the role of associative algebras in the Hom-Lie setting. They were introduced by the last author and Silvestrov in [14], where it is shown that the commutator bracket defined by the multiplication in a Hom-associative algebra leads naturally to a Hom-Lie algebra. The adjoint functor was considered by D. Yau [19].

The deformation theory was developed by Gerstenhaber for rings and algebras using formal power series in [7]. It is closely related to Hochschild cohomology. Then, it was extended to Lie algebras, using Chevalley-Eilenberg cohomology, by Nijenhuis and Richardson [17]. Deformation

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theory of associative algebra morphisms have been studied by Gerstenhaber and Schack in a series of papers [8–10], while deformations of Lie algebra morphisms have been considered by Nijenhuis and Richardson in [17], and more recently by Frégier [4], see also [5, 6]. Cohomology and deformations of Hom-associative algebras and Hom-Lie algebras were studied in [2, 16].

The purpose of this paper is to provide first a Hom-type Hochschild cohomology of Hom-associative algebra morphisms and a Hom-type Chevalley-Eilenberg cohomology of Hom-Lie algebra morphisms, generalizing the cohomology theory associated to deformations of Lie algebra morphisms given by Frégier for Lie algebras in [4] and that introduced by Gerstenhaber and Schack in [9] for associative algebra morphisms. To this end, we need to generalize the algebra valued cohomology theory in [2] to any bimodule. Moreover, we discuss a deformation theory of Hom-associative algebra morphisms and Hom-Lie algebra morphisms. Furthermore, various examples are presented.

1. Preliminaries

We assume that \mathbb{K} is an algebraically closed field of characteristic 0, even if most of the results are valid for any field. A *Hom-algebra* is a triple consisting of a \mathbb{K} -vector space, or a module, together with a bilinear map (multiplication) and a linear map. These Hom-algebras aim to generalize classical algebraic structures, their main feature is that the identities defining the structures are twisted by homomorphisms. In the sequel, we will write \otimes for $\otimes_{\mathbb{K}}$, $\mathcal{A}^{\otimes n} := \mathcal{A} \otimes \cdots \otimes \mathcal{A}$ and $\mathcal{A}^{\times n} := \mathcal{A} \times \cdots \times \mathcal{A}$.

Definition 1.1. A *Hom-associative algebra* is a triple $(\mathcal{A}, \mu, \alpha)$ consisting of a \mathbb{K} -vector space \mathcal{A} , a bilinear map $\mu : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ and a linear map $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ satisfying

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)), \text{ for all } x, y, z \in \mathcal{A} \quad (\text{Hom-associativity}).$$

A *Hom-Lie algebra* is a triple $(\mathcal{L}, [\cdot, \cdot], \alpha)$ consisting of a \mathbb{K} -vector space \mathcal{L} , a skew-symmetric bilinear map $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ and a linear map $\alpha : \mathcal{L} \rightarrow \mathcal{L}$ satisfying

$$\bigcirc_{x,y,z} [\alpha(x), [y, z]] = 0 \text{ for all } x, y, z \in \mathcal{L} \quad (\text{Hom-Jacobi identity}),$$

where $\bigcirc_{x,y,z}$ denotes summation over the cyclic permutation on x, y, z .

A Hom-associative algebra or a Hom-Lie algebra is called *multiplicative* if α is an algebra morphism.

Definition 1.2. Let $(\mathcal{A}, \mu, \alpha)$ and $(\mathcal{A}', \mu', \alpha')$ (resp. $(\mathcal{L}, [\cdot, \cdot], \alpha)$ and $(\mathcal{L}', [\cdot, \cdot]', \alpha')$) be two Hom-associative (resp. Hom-Lie) algebras. A linear map $\phi : \mathcal{A} \rightarrow \mathcal{A}'$ (resp. $\phi : \mathcal{L} \rightarrow \mathcal{L}'$) is called a *Hom-associative* (resp. *Hom-Lie*) *algebra morphism* if

$$\mu' \circ (\phi \otimes \phi) = \phi \circ \mu \text{ (resp. } [\cdot, \cdot]' \circ (\phi \otimes \phi) = \phi \circ [\cdot, \cdot]) \text{ and } \phi \circ \alpha = \alpha' \circ \phi.$$

The following theorem provides an easy way to deform a usual associative algebra morphism (resp. Lie algebra morphism) to a Hom-associative algebra morphism (resp. Hom-Lie algebra morphism).

Theorem 1.3. Let $A = (A, \mu)$ and $B = (B, \nu)$ be two associative algebras (resp. Lie algebras) and $\phi : A \rightarrow B$ be an algebra morphism. Consider two algebra morphisms $\alpha : A \rightarrow A$ and $\beta : B \rightarrow B$ such

that $\phi \circ \alpha = \beta \circ \phi$. Then $A_\alpha = (A, \mu_\alpha = \alpha \circ \mu, \alpha)$ and $B_\beta = (B, \nu_\beta = \beta \circ \nu, \beta)$ are Hom-associative algebras (resp. Hom-Lie algebras) and $\phi : A_\alpha \rightarrow B_\beta$ is a Hom-associative algebra morphism.

The first assertion was proved in [20] and we have $\phi \circ \alpha \circ \mu = \beta \circ \phi \circ \mu = \beta \circ \nu \circ (\phi \otimes \phi)$.

Now, we discuss concepts of module and representation for Hom-associative algebras and Hom-Lie algebras.

Definition 1.4. Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-associative algebra. A (left) \mathcal{A} -module is a triple (M, f, γ) , where M is a \mathbb{K} -vector space, $f : M \rightarrow M$ and $\gamma : \mathcal{A} \otimes M \rightarrow M$ are \mathbb{K} -linear maps such that $\gamma \circ (\mu \otimes f) = \gamma \circ (\alpha \otimes \gamma)$.

Proposition 1.5 (Left adjoint \mathcal{A} -module). Let $(\mathcal{A}, \mu, \alpha)$ and $(\mathcal{A}', \mu', \alpha')$ be two Hom-associative algebras and $\phi : \mathcal{A} \rightarrow \mathcal{A}'$ be a Hom-associative algebra morphism. The triple (M, f, γ) , where $M = \mathcal{A}'$, $\gamma = \rho_l = \mu'(\phi \otimes id)$ and $f = \alpha'$, is an \mathcal{A} -module called adjoint representation of \mathcal{A} induced by the Hom-associative algebra morphism ϕ .

Proof. Straightforward. □

Remark 1.6. We similarly define the right adjoint module by considering the triple $(\mathcal{A}', \rho_r = \mu'(id \otimes \phi), \alpha')$. A bimodule structure is given by left and right actions with maps ρ_r and ρ_l that satisfy the following additional condition $\rho_r(\rho_l(x, y), \alpha(z)) = \rho_l(\alpha(x), \rho_r(y, z))$. Left and right modules are special cases of bimodules, one may set $\rho_r = 0$ (resp. $\rho_l = 0$).

Let $(\mathcal{L}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra and $\beta \in \mathfrak{gl}(V)$ be an arbitrary linear self map on V , where V is an arbitrary vector space. We denote a left action of \mathcal{L} on V by the bracket $[\cdot, \cdot]_V : \mathcal{L} \times V \rightarrow V$ such that $(x, v) \rightarrow [x, v]_V$.

Definition 1.7. A triple $(V, [\cdot, \cdot]_V, \beta)$ is called a left \mathcal{L} -module with respect to $\beta \in \mathfrak{gl}(V)$, if for any $x, y \in \mathcal{L}$ and $v \in V$, we have $[\alpha(x), \beta(v)]_V = \beta([x, v]_V)$ and $[[x, y], \beta(v)]_V = [\alpha(x), [y, v]_V]_V - [\alpha(y), [x, v]_V]_V$.

We say that $(V, [\cdot, \cdot]_V, \beta)$ is a representation of \mathcal{L} .

Proposition 1.8. Let $(\mathcal{L}, [\cdot, \cdot], \alpha)$ and $(\mathcal{L}', [\cdot, \cdot]', \alpha')$ be two Hom-Lie algebras and $\phi : \mathcal{L} \rightarrow \mathcal{L}'$ be a Hom-Lie algebra morphism. Let (Π, β, ρ_l) be a triple, where $\Pi = \mathcal{L}'$, $\beta = \alpha'$, $\rho_l = [\cdot, \cdot]_V = [\phi, id]'$. Then, Π is a left adjoint \mathcal{L} -module via ϕ .

2. Cohomology of Hom-associative algebra morphisms

2.1. Cohomology of Hom-associative algebras with values in an adjoint \mathcal{A} -bimodule

We construct a cochain complex $C_{\alpha, \alpha'}^*(\mathcal{A}, M)$ that defines a Hom-type Hochschild cohomology for multiplicative Hom-associative algebras in an adjoint \mathcal{A} -bimodule M . Let $(\mathcal{A}, \mu, \alpha)$ and $(\mathcal{A}', \mu', \alpha')$ be two Hom-associative algebras over \mathbb{K} and $\phi : \mathcal{A} \rightarrow \mathcal{A}'$ be a Hom-associative algebra morphism. Let $M = (\mathcal{A}', \rho_l, \rho_r)$ be an \mathcal{A} -bimodule, where ρ_l and ρ_r are defined in Proposition 1.5 (resp. Remark 1.6). Regard \mathcal{A}' as an \mathcal{A} -bimodule via the adjoint representation of \mathcal{A} induced by ϕ .

The set of n -cochains on \mathcal{A} with values in an \mathcal{A} -bimodule M , is defined to be the set of n -linear maps which are compatible with α and α' . We denote by $C^n(\mathcal{A}, M)$ the set of n -linear maps from

\mathcal{A} to M and let $C_{\alpha, \alpha'}^0(\mathcal{A}, M) := M$ and for $n > 0$ we set

$$C_{\alpha, \alpha'}^n(\mathcal{A}, M) := \{f \in C^n(\mathcal{A}, M) : \alpha' \circ f = f \circ \alpha^{\otimes n}\}.$$

Definition 2.1. For $n \geq 1$, a n -coboundary operator associated to the triple (\mathcal{A}, M, ϕ) is the linear map $\delta_{\text{Hom}}^n : C_{\alpha, \alpha'}^n(\mathcal{A}, M) \rightarrow C_{\alpha, \alpha'}^{n+1}(\mathcal{A}, M)$ defined by

$$\begin{aligned} \delta_{\text{Hom}}^n \varphi(x_0, x_1, \dots, x_n) &= \mu'(\phi(\alpha^{n-1}(x_0))), \varphi(x_1, x_2, \dots, x_n) \\ &+ \sum_{k=1}^n (-1)^k \varphi(\alpha(x_0), \alpha(x_1), \dots, \alpha(x_{k-2}), \mu(x_{k-1}, x_k), \alpha(x_{k+1}), \dots, \alpha(x_n)) \\ &+ (-1)^{n+1} \mu'(\varphi(x_0, \dots, x_{n-1}), \phi(\alpha^{n-1}(x_n))). \end{aligned} \tag{2.1}$$

Theorem 2.2. The pair $(C_{\alpha, \alpha'}^*(\mathcal{A}, M), \delta_{\text{Hom}}^*)$ defines a cohomology complex of Hom-associative algebras with values in the \mathcal{A} -bimodule M .

Remark 2.3. In the general case, we let $\mathcal{A}, \mathcal{A}'$ be two Hom-associative algebras and $M = (\mathcal{A}', \rho_l, \rho_r)$ be a \mathcal{A} -bimodule, where ρ_l and ρ_r are left \mathcal{A} -module and right \mathcal{A} -module respectively. For $\varphi \in C_{\alpha, \alpha'}^n(\mathcal{A}, M)$, we set

$$\begin{aligned} \delta_{\text{Hom}}^n \varphi(x_0, x_1, \dots, x_n) &= \rho_l(\alpha^{n-1}(x_0), \varphi(x_1, x_2, \dots, x_n)) \\ &+ \sum_{k=1}^n (-1)^k \varphi(\alpha(x_0), \alpha(x_1), \dots, \alpha(x_{k-2}), \mu(x_{k-1}, x_k), \alpha(x_{k+1}), \dots, \alpha(x_n)) \\ &+ (-1)^{n+1} \rho_r(\varphi(x_0, \dots, x_{n-1}), \alpha^{n-1}(x_n)). \end{aligned}$$

In the particular case where $M = \mathcal{A}$ and $\rho_l, \rho_r = \mu$, the Hom-associative algebra is an \mathcal{A} -bimodule over itself. We recover the coboundary operator defined in [2]. One considers the previous definition with $\rho_l = \rho_r = \mu$ and denote $C_{\alpha, \alpha}^n(\mathcal{A}, M)$ by $C_{\text{Hom}}^n(\mathcal{A}, M)$.

The space of n -cocycles is $Z_{\text{Hom}}^n(\mathcal{A}, M) := \{\varphi \in C_{\alpha, \alpha'}^n(\mathcal{A}, M) : \delta_{\text{Hom}}^n \varphi = 0\}$, and the space of n -coboundaries is $B_{\text{Hom}}^n(\mathcal{A}, M) := \{\psi = \delta_{\text{Hom}}^{n-1} \varphi : \varphi \in C_{\alpha, \alpha'}^{n-1}(\mathcal{A}, M)\}$. The n^{th} Hochschild cohomology group of the Hom-associative algebra \mathcal{A} with values in an adjoint \mathcal{A} -bimodule is the quotient $H_{\text{Hom}}^n(\mathcal{A}, M) := \frac{Z_{\text{Hom}}^n(\mathcal{A}, M)}{B_{\text{Hom}}^n(\mathcal{A}, M)}$.

2.2. Cohomology Complex of Hom-associative algebra morphisms

The original cohomology theory associated to deformation of associative algebra morphisms was introduced by M. Gerstenhaber in [9]. In this section, we will discuss this theory for Hom-associative algebra morphisms. Let \mathcal{A}, \mathcal{B} be two Hom-associative algebras and $\phi : \mathcal{A} \rightarrow \mathcal{B}$ be a Hom-associative algebra morphism. Regard \mathcal{B} as a representation of \mathcal{A} via ϕ wherever appropriate. Define the module of n -cochains of ϕ by

$$C_{\text{Hom}}^n(\phi, \phi) = C_{\text{Hom}}^n(\mathcal{A}, \mathcal{A}) \times C_{\text{Hom}}^n(\mathcal{B}, \mathcal{B}) \times C_{\alpha, \alpha'}^{n-1}(\mathcal{A}, \mathcal{B}).$$

The coboundary operator $\delta^n : C_{\text{Hom}}^n(\phi, \phi) \rightarrow C_{\text{Hom}}^{n+1}(\phi, \phi)$ is defined by

$$\delta^n(\varphi_1, \varphi_2, \varphi_3) = (\delta_{\text{Hom}}^n \varphi_1, \delta_{\text{Hom}}^n \varphi_2, \phi \circ \varphi_1 - \varphi_2 \circ \phi^{\otimes n} - \delta_{\text{Hom}}^{n-1} \varphi_3),$$

where $\delta_{\text{Hom}}^n \varphi_1$ and $\delta_{\text{Hom}}^n \varphi_2$ are defined in Remark 2.3 and $\delta_{\text{Hom}}^n \varphi_3$ is defined by (2.1).

Theorem 2.4. We have $\delta^{n+1} \circ \delta^n = 0$. Hence $(C_{\text{Hom}}^*(\phi, \phi), \delta^*)$ is a cochain complex.

Proof. The most-right component of $(\delta^{n+1} \circ \delta^n)(\varphi_1, \varphi_2, \varphi_3)$ is $\phi \circ (\delta_{\text{Hom}}^n \varphi_1) - (\delta_{\text{Hom}}^n \varphi_2) \circ \phi - \delta_{\text{Hom}}^n(\phi \circ \varphi_1 - \varphi_2 \circ \phi^{\otimes n} - \delta_{\text{Hom}}^{n-1} \varphi_3) = \phi \circ (\delta_{\text{Hom}}^n \varphi_1) - (\delta_{\text{Hom}}^n \varphi_2) \circ \phi^{\otimes n+1} - \delta_{\text{Hom}}^n(\phi \circ \varphi_1) - \delta_{\text{Hom}}^n(\varphi_2 \circ \phi^{\otimes n})$. To finish the proof, one checks that $\phi \circ (\delta_{\text{Hom}}^n \varphi_1) = \delta_{\text{Hom}}^n(\phi \circ \varphi_1)$ and $(\delta_{\text{Hom}}^n \varphi_2) \circ \phi^{\otimes n+1} = \delta_{\text{Hom}}^n(\varphi_2 \circ \phi^{\otimes n})$. Indeed, $\phi \circ \varphi_1$ is defined as follows: $(\phi \circ \varphi_1)(x_0, \dots, x_n) = \phi \circ (\varphi_1(x_0, \dots, x_n))$ and $\varphi_2 \circ \phi^{\otimes n}$ as $\varphi_2 \circ \phi(x_0, \dots, x_n) = \varphi_2 \circ (\phi(x_1), \dots, \phi(x_n))$. \square

Remark 2.5. If $H_{\text{Hom}}^n(\mathcal{A}, \mathcal{A})$, $H_{\text{Hom}}^n(\mathcal{B}, \mathcal{B})$ and $H_{\alpha, \alpha'}^{n-1}(\mathcal{A}, \mathcal{B})$ are all trivial, then so is $H_{\text{Hom}}^n(\phi, \phi)$. The proof is similar to that of Proposition 3.3 in [18].

We define the n^{th} Hochschild cohomology group of a Hom-associative algebra morphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$ to be

$$H_{\text{Hom}}^n(\phi, \phi) := H_{\text{Hom}}^n(\mathcal{A}, \mathcal{A}) \times H_{\text{Hom}}^n(\mathcal{B}, \mathcal{B}) \times H_{\alpha, \alpha'}^{n-1}(\mathcal{A}, \mathcal{B}).$$

The corresponding cohomology modules of the cochain complex $(C_{\text{Hom}}^*(\phi, \phi), \delta^*)$ are denoted by $H_{\text{Hom}}^n(\phi, \phi) := H_{\text{Hom}}^n(C_{\text{Hom}}^*(\phi, \phi), \delta)$.

Example 2.6. We consider a 3-dimensional Hom-associative algebra \mathcal{A} defined in [15], with respect to a basis $\{e_1, e_2, e_3\}$, by the multiplication $\mu_{\mathcal{A}}$ and the linear map $\alpha_{\mathcal{A}}$ such that

$$\begin{aligned} \mu_{\mathcal{A}}(e_1, e_1) &= ae_1, & \mu_{\mathcal{A}}(e_1, e_2) &= \mu_{\mathcal{A}}(e_2, e_1) = ae_2, & \mu_{\mathcal{A}}(e_1, e_3) &= \mu_{\mathcal{A}}(e_3, e_1) = be_3, \\ \mu_{\mathcal{A}}(e_2, e_2) &= ae_2, & \mu_{\mathcal{A}}(e_2, e_3) &= be_3, & \mu_{\mathcal{A}}(e_3, e_2) &= \mu_{\mathcal{A}}(e_3, e_3) = 0, \\ \alpha_{\mathcal{A}}(e_1) &= ae_1, & \alpha_{\mathcal{A}}(e_2) &= ae_2, & \alpha_{\mathcal{A}}(e_3) &= be_3, \end{aligned}$$

where a, b are parameters.

We consider also a 2-dimensional Hom-associative algebra \mathcal{B} defined, with respect to a basis $\{f_1, f_2\}$, by the multiplication $\mu_{\mathcal{B}}$ and the linear map $\alpha_{\mathcal{B}}$ such that

$$\mu_{\mathcal{B}}(f_1, f_1) = f_1, \quad \mu_{\mathcal{B}}(f_i, f_j) = f_2 \text{ for } (i, j) \neq (1, 1), \quad \alpha_{\mathcal{B}}(f_1) = \beta f_1 - \beta f_2, \quad \alpha_{\mathcal{B}}(f_2) = 0,$$

where β is a parameter.

Let $\phi : \mathcal{A} \rightarrow \mathcal{B}$ be a Hom-associative algebra morphism. It is defined, when $a = \beta = 1$, as $\phi(e_1) = f_1 - f_2$, $\phi(e_2) = f_1 - f_2$, $\phi(e_3) = 0$.

In the following, we compute the second cohomology spaces $H_{\text{Hom}}^2(\mathcal{A}, \mathcal{A})$ and $H_{\text{Hom}}^2(\mathcal{B}, \mathcal{B})$. The 2-cocycles $\psi : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ are of the form

$$\begin{aligned} \psi(e_1, e_1) &= p_1 e_1 + \left(\frac{p_2}{b} - p_1\right) e_2, & \psi(e_2, e_1) &= \frac{p_2}{b} e_2, & \psi(e_3, e_1) &= b p_1 e_3, \\ \psi(e_1, e_2) &= \frac{p_2}{b} e_2, & \psi(e_2, e_2) &= p_3 e_1 + p_4 e_2, & \psi(e_3, e_2) &= -b p_3 e_3, \\ \psi(e_1, e_3) &= p_2 e_3, & \psi(e_2, e_3) &= b(p_3 + p_4) e_3, & \psi(e_3, e_3) &= 0. \end{aligned}$$

where p_1, p_2, p_3, p_4 are parameters.

It turns out that they are all coboundaries. Moreover, we get

$$\begin{aligned} H_{\text{Hom}}^2(\mathcal{A}, \mathcal{A}) &= \{0\}, \\ H_{\text{Hom}}^2(\mathcal{B}, \mathcal{B}) &= \{\psi \mid \psi(f_1, f_1) = cf_1 + df_2, \psi(f_i, f_j) = pf_2, p \neq (c+d), (i, j) \neq (1, 1)\} \\ H_{\text{Hom}}^1(\mathcal{A}, \mathcal{B}) &= \{\phi_1 \mid \phi_1(e_1) = (p_1 - c)f_1 + (c - p_1)f_2, \\ &\quad \phi_1(e_2) = (p_2 - c)f_1 + (c - p_2)f_2; \phi_1(e_3) = 0\}. \end{aligned}$$

3. Cohomology Complex of Hom-Lie algebra morphisms

In this section, we deal with a cohomology of Hom-Lie algebra morphisms. We construct first a cochain complex $C_{\text{HL}}^*(\mathcal{L}, \Pi)$ that defines a Chevalley-Eilenberg cohomology for multiplicative Hom-Lie algebras with values in a left \mathcal{L} -module Π , then a cohomology complex of Hom-Lie algebra morphisms.

3.1. Cohomology complex of multiplicative Hom-Lie algebras with values in a left \mathcal{L} -module

Let \mathcal{L} and \mathcal{L}' be two Hom-Lie algebras and $\phi : \mathcal{L} \rightarrow \mathcal{L}'$ be a Hom-Lie morphism. Regard \mathcal{L}' as a representation Π of \mathcal{L} via ϕ defined by (1.8). The set $C_{\text{HL}}^n(\mathcal{L}, \Pi)$ of n -cochains on \mathcal{L} with values in Π is the set of skewsymmetric \mathbb{K} -linear maps from $\mathcal{L}^{\times n}$ to Π . We set $\tilde{C}_{\alpha, \alpha'}^n(\mathcal{L}, \Pi) := \{f \in C_{\text{HL}}^n(\mathcal{L}, \Pi) : \alpha' \circ f = f \circ \alpha^{\otimes n}\}$. For $n = 0$, we have $\tilde{C}_{\alpha, \alpha'}^0(\mathcal{L}, \Pi) = \Pi$.

Definition 3.1. Let $(\mathcal{L}, [\cdot, \cdot], \alpha)$ and $(\mathcal{L}', [\cdot, \cdot]', \alpha')$ be two Hom-Lie algebras. Let $\phi : \mathcal{L} \rightarrow \mathcal{L}'$ be a Hom-Lie algebra morphism. Regard \mathcal{L}' as a representation of \mathcal{L} via ϕ wherever appropriate. For $n \geq 1$, the n -coboundary operator associated to the triple (\mathcal{L}, Π, ϕ) is the linear map $\delta_{\text{HL}}^n : C_{\text{HL}}^n(\mathcal{L}, \Pi) \rightarrow C_{\text{HL}}^{n+1}(\mathcal{L}, \Pi)$ defined by

$$\begin{aligned} \delta_{\text{HL}}^n \varphi(x_0, \dots, x_n) &= \sum_{i=0}^n (-1)^i [\phi(\alpha^{n-1}(x_i)), \varphi(x_0, \dots, \hat{x}_i, \dots, x_n)]' \\ &\quad + \sum_{0 \leq i < j \leq n} (-1)^{i+j} \varphi([x_i, x_j], \alpha(x_0), \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, \alpha(x_n)). \end{aligned} \quad (3.1)$$

Theorem 3.2. The pair $(\tilde{C}_{\alpha, \alpha'}^*(\mathcal{L}, \Pi), \delta_{\text{HL}})$ defines a cochain complex. The corresponding cohomology denoted by $H_{\text{HL}}^*(\mathcal{L}, \Pi)$, is called the cohomology of the Hom-Lie algebra \mathcal{L} with coefficients in the representation Π .

Proof. The proof of $\delta_{\text{HL}}^{n+1} \circ \delta_{\text{HL}}^n = 0$ is straightforward and lengthy. One may view it in the arxiv version of this paper, see arXiv:1710.07599. \square

Remark 3.3. In the case where $\Pi = \mathcal{L}$ and $[\cdot, \cdot] = [\cdot, \cdot]'$, we recover the coboundary operator defined in [2]. The space of n -cochains is denoted by $C_{\text{HL}}^n(\mathcal{L}, \mathcal{L})$.

The space of n -cocycles is $Z_{\text{HL}}^n(\mathcal{L}, \Pi) := \{\varphi \in \tilde{C}_{\alpha, \alpha'}^n(\mathcal{L}, \Pi) : \delta_{\text{HL}}^n \varphi = 0\}$, and the space of n -coboundaries is $B_{\text{HL}}^n(\mathcal{L}, \Pi) := \{\psi = \delta_{\text{HL}}^{n-1} \varphi : \varphi \in \tilde{C}_{\alpha, \alpha'}^{n-1}(\mathcal{L}, \Pi)\}$. The n^{th} cohomology group of the Hom-Lie algebra \mathcal{L} with coefficients in Π is the quotient $H_{\text{HL}}^n(\mathcal{L}, \Pi) := \frac{Z_{\text{HL}}^n(\mathcal{L}, \Pi)}{B_{\text{HL}}^n(\mathcal{L}, \Pi)}$.

3.2. Cohomology complex of Hom-Lie algebra morphisms

In this section, we generalize the cohomology theory of Lie algebra morphisms developed by Fréгий in [4] to Hom-Lie algebras. We adopt the same notations. Consider the product \diamond defined, for $\lambda \in C_{\text{Hom}}^n(\mathcal{L}', \mathcal{L}')$ and $\phi \in \text{Hom}(\mathcal{L}, \mathcal{L}')$, as $\lambda \diamond \phi \in C_{\text{HL}}^n(\mathcal{L}, \mathcal{L}')$ such that $\lambda \diamond \phi(x_1, \dots, x_n) := \lambda(\phi(x_1), \dots, \phi(x_n))$, for any $x_1, \dots, x_n \in \mathcal{L}$.

Let $\phi : \mathcal{L} \rightarrow \mathcal{L}'$ be a Hom-Lie algebra morphism. Regard \mathcal{L}' as a representation of \mathcal{L} via ϕ wherever appropriate. Define the *module of n-cochains of ϕ* by

$$C_{\text{HL}}^n(\phi, \phi) := C_{\text{HL}}^n(\mathcal{L}, \mathcal{L}) \times C_{\text{HL}}^n(\mathcal{L}', \mathcal{L}') \times \tilde{C}_{\alpha, \alpha'}^{n-1}(\mathcal{L}, \mathcal{L}').$$

The coboundary operator $\delta_{(\phi, \phi)}^n : C_{\text{HL}}^n(\phi, \phi) \rightarrow C_{\text{HL}}^{n+1}(\phi, \phi)$ is defined by

$$\delta_{(\phi, \phi)}^n(\varphi_1, \varphi_2, \varphi_3) = (\delta_{\text{HL}}^n \varphi_1, \delta_{\text{HL}}^n \varphi_2, \delta_{\text{HL}}^{n-1} \varphi_3 + (-1)^{n-1}(\phi \circ \varphi_1 - \varphi_2 \diamond \phi)),$$

where $\delta_{\text{HL}}^n \varphi_1$ and $\delta_{\text{HL}}^n \varphi_2$ are defined in Remark 3.3 and $\delta_{\text{HL}}^n \varphi_3$ by formula (3.1).

Theorem 3.4. We have $\delta_{(\phi, \phi)}^{n+1} \circ \delta_{(\phi, \phi)}^n = 0$. Hence $(C_{\text{HL}}^*(\phi, \phi), \delta^*)$ is a cohomology complex.

Proof. The proof is similar to the proof of Theorem 2.4. □

Remark 3.5. The corresponding cohomology modules of the cochain complex $(C_{\text{HL}}^*(\phi, \phi), \delta^*)$ are denoted by $H_{\text{HL}}^n(\phi, \phi) := H_{\text{HL}}^n(C_{\text{HL}}^*(\phi, \phi), \delta^*)$. If $H_{\text{HL}}^n(\mathcal{L}, \mathcal{L})$, $H_{\text{HL}}^n(\mathcal{L}', \mathcal{L}')$ and $H_{\text{HL}}^{n-1}(\mathcal{L}, \mathcal{L}')$ are all trivial then so is $H_{\text{HL}}^n(\phi, \phi)$. The proof is similar to that of Proposition 3.3 in [18].

4. Deformations of Hom-associative algebra morphisms

In this section, we study 1-parameter formal deformations of Hom-associative algebra morphisms and Hom-Lie algebra morphisms using the approach introduced by Gerstenhaber [7]. Recall that the main idea is to change the scalar field \mathbb{K} to a formal power series ring $\mathbb{K}[[t]]$, in one indeterminate t , and the main results provide us with cohomological interpretations. Let $A[[t]]$ be the set of formal power series whose coefficients are elements of A .

4.1. Deformation of Hom-associative algebra morphisms

First, we recall the definition of a formal deformation of a Hom-associative algebra, then discuss deformations of Hom-associative algebra morphisms.

Definition 4.1. A 1-parameter formal deformation of a Hom-associative algebra $(\mathcal{A}, \mu_0, \alpha)$ is a Hom-associative $\mathbb{K}[[t]]$ -algebra $(\mathcal{A}[[t]], \mu_t, \alpha)$, where $\mu_t = \sum_{i \geq 0} \mu_i t^i$, which is a $\mathbb{K}[[t]]$ -bilinear map satisfying the condition $\mu_t \circ (\mu_t \otimes \alpha) = \mu_t \circ (\alpha \otimes \mu_t)$.

The deformation is said to be of order N if $\mu_t = \sum_{i \geq 0}^N \mu_i t^i$ and infinitesimal if $N = 1$.

Let $(\mathcal{A}, \mu_{\mathcal{A}}, \alpha)$ and $(\mathcal{B}, \mu_{\mathcal{B}}, \beta)$ be two Hom-associative algebras and $\phi : \mathcal{A} \rightarrow \mathcal{B}$ be a Hom-associative algebra morphism. A deformation of ϕ is given by a triple $\Theta_t = (\mu_{\mathcal{A}, t}, \mu_{\mathcal{B}, t}, \phi_t)$ where

- $\mu_{\mathcal{A}, t} = \sum_{n \geq 0} \mu_{\mathcal{A}, n} t^n$ is a deformation of \mathcal{A} ,
- $\mu_{\mathcal{B}, t} = \sum_{n \geq 0} \mu_{\mathcal{B}, n} t^n$ is a deformation of \mathcal{B} ,

- $\phi_t : \mathcal{A}[[t]] \rightarrow \mathcal{B}[[t]]$ is a Hom-associative algebra morphism of the form $\phi_t = \sum_{n \geq 0} \phi_n t^n$, where each $\phi_n : \mathcal{A} \rightarrow \mathcal{B}$ is a \mathbb{K} -linear map and $\phi_0 = \phi$.

Proposition 4.2. *The linear coefficient $\Theta_1 = (\mu_{\mathcal{A},1}, \mu_{\mathcal{B},1}, \phi_1)$, called the infinitesimal part of the deformation Θ_t of ϕ , is a 2-cocycle in $C^2_{\text{Hom}}(\phi, \phi)$.*

Proof. Straightforward. □

4.2. Equivalent deformations and Rigidity

Let $\mu_{\mathcal{A},t}$ and $\tilde{\mu}_{\mathcal{A},t}$ be two formal deformations of a Hom-associative algebra \mathcal{A} . A formal automorphism $\psi_t : \mathcal{A}[[t]] \rightarrow \mathcal{A}[[t]]$ is a power series $\psi_t = \sum_{n \geq 0} \psi_n t^n$ in which each $\psi_n \in \text{End}(\mathcal{A})$ and $\psi_0 = \text{Id}_{\mathcal{A}}$ such that $\psi_t(\mu_{\mathcal{A},t}(x, y)) = \tilde{\mu}_{\mathcal{A},t}(\psi_t(x), \psi_t(y))$ and $\psi_t \circ \alpha = \tilde{\alpha} \circ \psi_t$ for all $x, y \in \mathcal{A}$. Two deformations $\mu_{\mathcal{A},t}$ and $\tilde{\mu}_{\mathcal{A},t}$ are said to be *equivalent* if and only if there exists a formal automorphism which transforms $\mu_{\mathcal{A},t}$ to $\tilde{\mu}_{\mathcal{A},t}$.

Definition 4.3. Let $\Theta_t = (\mu_{\mathcal{A},t}, \mu_{\mathcal{B},t}, \phi_t)$ and $\tilde{\Theta}_t = (\tilde{\mu}_{\mathcal{A},t}, \tilde{\mu}_{\mathcal{B},t}, \tilde{\phi}_t)$ be two deformations of a Hom-associative algebra morphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$. A formal automorphism $\phi_t : \Theta_t \rightarrow \tilde{\Theta}_t$ is a pair $(\psi_{\mathcal{A},t}, \psi_{\mathcal{B},t})$, where $\psi_{\mathcal{A},t} : \mathcal{A}[[t]] \rightarrow \mathcal{A}[[t]]$ and $\psi_{\mathcal{B},t} : \mathcal{B}[[t]] \rightarrow \mathcal{B}[[t]]$ are formal automorphisms such that $\tilde{\phi}_t = \psi_{\mathcal{B},t} \circ \phi_t \circ \psi_{\mathcal{A},t}^{-1}$. Two deformations Θ_t and $\tilde{\Theta}_t$ are said to be *equivalent* if and only if there exists a formal automorphism such that $\Theta_t \rightarrow \tilde{\Theta}_t$.

Given a deformation Θ_t and a pair of power series

$$\psi_t = (\psi_{\mathcal{A},t} = \sum_n \psi_{\mathcal{A},n} t^n, \psi_{\mathcal{B},t} = \sum_n \psi_{\mathcal{B},n} t^n),$$

one can define a deformation $\tilde{\Theta}_t$ which is automatically equivalent to Θ_t .

In [2], it was shown that if $\mu_{\mathcal{A},t}$ and $\tilde{\mu}_{\mathcal{A},t}$ are, thanks to the automorphism $\phi_{\mathcal{A},t} : \mathcal{A}[[t]] \rightarrow \mathcal{A}[[t]]$, equivalent deformations of \mathcal{A} , then the infinitesimals of $\mu_{\mathcal{A},t}$ and $\tilde{\mu}_{\mathcal{A},t}$ belong to the same cohomology class.

Theorem 4.4. *The infinitesimal part of a deformation Θ_t of ϕ is a 2-cocycle in $C^2_{\text{Hom}}(\phi, \phi)$ whose cohomology class is determined by the equivalence class of the first term of Θ_t .*

Proof. In view of Proposition 4.2, it remains to show that if $\psi_t : \Theta_t \rightarrow \tilde{\Theta}_t$ is a formal automorphism, then 2-cocycles Θ_1 and $\tilde{\Theta}_1$ differ by a 2-coboundary. Write $\psi_t = (\psi_{\mathcal{A},t}, \psi_{\mathcal{B},t})$ and $\tilde{\Theta}_t = (\tilde{\mu}_{\mathcal{A},t}, \tilde{\mu}_{\mathcal{B},t}, \tilde{\phi}_t)$. According to [2], we have $\delta^1 \psi_{*,1} = \mu_{*,1} - \tilde{\mu}_{*,1} \in C^2_{\text{Hom}}(*, *)$ for $*$ denoting either \mathcal{A} or \mathcal{B} . To finish the proof, we develop both sides of $\tilde{\phi}_t = \psi_{\mathcal{B},t} \phi_t \psi_{\mathcal{A},t}^{-1}$ and collecting the coefficients of t^n yield for $n = 1$ the equality $\phi_1 - \tilde{\phi}_1 = \phi \psi_{\mathcal{A},1} - \psi_{\mathcal{B},1} \phi$. It follows that a 1-cochain $\alpha = (\psi_{\mathcal{A},1}, \psi_{\mathcal{B},1}, 0) \in C^1_{\text{Hom}}(\phi, \phi)$ satisfies $\delta^1_{\phi, \phi} \alpha = \Theta_1 - \tilde{\Theta}_1$. □

Definition 4.5. Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-associative algebra and $\Theta_1 = (\mu_{\mathcal{A},1}, \mu_{\mathcal{B},1}, \phi_1)$ be an element of $Z^2_{\text{Hom}}(\phi, \phi)$. The 2-cocycle Θ_1 is said to be *integrable* if there exists a family $(\mu_{\mathcal{A},t} = \sum_n \mu_{\mathcal{A},n}, \mu_{\mathcal{B},t} = \sum_n \mu_{\mathcal{B},n}, \phi_t = \sum_n \phi_n)$ defining a formal deformation Θ_t of ϕ .

The integrability of Θ_1 depends only on its cohomology class.

Theorem 4.6. Let $(\mathcal{A}, \mu_{\mathcal{A}}, \alpha)$ and $(\mathcal{B}, \mu_{\mathcal{B}}, \beta)$ be two Hom-associative algebras and $\Theta_t = (\mu_{\mathcal{A},t}, \mu_{\mathcal{B},t}, \phi_t)$ be a deformation of a Hom-associative algebra morphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$. Then, there exists an equivalent deformation $\tilde{\Theta}_t = (\tilde{\mu}_{\mathcal{A},t}, \tilde{\mu}_{\mathcal{B},t}, \tilde{\phi}_t)$ such that $\tilde{\Theta}_1 \in Z_{\text{Hom}}^2(\phi, \phi)$ and $\tilde{\Theta}_1 \notin B_{\text{Hom}}^2(\phi, \phi)$. Hence, if $H_{\text{Hom}}^2(\phi, \phi) = 0$, then every formal deformation is equivalent to a trivial deformation.

Proof. Define a pair of power series $\psi_t = (\Psi_{\mathcal{A},t}, \Psi_{\mathcal{B},t})$. According to Definition 4.3, we define an equivalent deformation $\tilde{\Theta}_t = (\tilde{\mu}_{\mathcal{A},t}, \tilde{\mu}_{\mathcal{B},t}, \tilde{\phi}_t) = \sum \tilde{\Theta}_n t^n$.

We have $\mu_{*,1} \in Z_{\text{Hom}}^2(*, *)$ and also $\mu_{*,1} - \tilde{\mu}_{*,1} \in Z_{\text{Hom}}^2(*, *)$ for $* \in \{\mathcal{A}, \mathcal{B}\}$. Moreover, since $\phi_1 \in Z_{\text{Hom}}^1(\mathcal{A}, \mathcal{B})$, we have $\phi_1 - \tilde{\phi}_1 \in Z_{\text{Hom}}^1(\mathcal{A}, \mathcal{B})$. If $\Theta_1 \in B_{\text{Hom}}^2(\phi, \phi)$, then $\Theta_1 - \tilde{\Theta}_1 = \delta_{\phi, \phi}^1 \varphi$ for some $\varphi \in C_{\text{Hom}}^1(\phi, \phi)$. □

Remark 4.7. Let $\Theta_t = \sum_{i \geq 0} \Theta_i t^i$ be a deformation of a Hom-associative algebra morphism, in which $\Theta_i = 0$ for $i = 1, \dots, n$ and Θ_{n+1} is a coboundary in $C_{\text{Hom}}^2(\phi, \phi)$, then there exists a deformation $\tilde{\Theta}_t$ equivalent to Θ_t and a formal automorphism $\psi_t : \Theta_t \rightarrow \tilde{\Theta}_t$ such that $\tilde{\Theta}_i = 0$ for $i = 1, \dots, n + 1$.

A morphism of Hom-associative algebras for which every formal deformation is equivalent to a trivial deformation $(\mu_{\mathcal{A},0}, \mu_{\mathcal{B},0}, \psi)$ is said to be *analytically rigid*. The vanishing of the second cohomology group, $H_{\text{Hom}}^2(\phi, \phi) = 0$, gives a sufficient criterion for rigidity.

4.2.1. Obstructions

A deformation of order N of ϕ is a triple $\Theta_t = (\mu_{\mathcal{A},t}, \mu_{\mathcal{B},t}, \phi_t)$ satisfying $\phi_t(\mu_{\mathcal{A},t}(a, b)) = \mu_{\mathcal{B},t}(\phi_t(a), \phi_t(b))$ or equivalently

$$\sum_{i=0}^n \phi_i(\mu_{\mathcal{A},n-i}(a, b)) = \sum_{i+j+k=n} \mu_{\mathcal{B},i}(\phi_j(a), \phi_k(b)) \text{ for } n \leq N.$$

Given a deformation Θ_t of order N , it can be extended to order $N + 1$ if and only if there exists a 2-cochain $\Theta_{N+1} = (\mu_{\mathcal{A},N+1}, \mu_{\mathcal{B},N+1}, \phi_{N+1}) \in C_{\text{Hom}}^2(\phi, \phi)$ such that $\bar{\Theta}_t = \Theta_t + t^{N+1} \Theta_{N+1}$ is a deformation of order $N + 1$.

The primary obstruction of a deformation $\mu_{\mathcal{A},t} = \sum_{i=0}^{N+1} \mu_i t^i$ is

$$- \sum_{\substack{p+q=N+1, \\ p>0, q>0}} \mu_p \circ \alpha \mu_q.$$

An analogous obstruction to an infinitesimal deformation of a morphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is obtained by calculating the second order term in the deformation equation of ϕ . Then

$$\begin{aligned} & \mu_{\mathcal{B},2}(\phi(a), \phi(b)) + \mu_{\mathcal{B},1}(\phi_1(a), \phi(b)) + \mu_{\mathcal{B},1}(\phi(a), \phi_1(b)) \\ & + \mu_{\mathcal{B},0}(\phi_1(a), \phi_1(b)) + \mu_{\mathcal{B},0}(\phi_2(a), \phi(b)) + \mu_{\mathcal{B},0}(\phi(b), \phi_2(b)) \\ & = \phi_2 \mu_{\mathcal{A},0}(a, b) + \phi_1 \mu_{1,\mathcal{A}}(a, b) + \phi \mu_{\mathcal{A},2}(a, b). \end{aligned}$$

Therefore, we get

$$(\mu_{\mathcal{A},1} \bar{\circ} \mu_{\mathcal{A},1}, \mu_{\mathcal{B},1} \bar{\circ} \mu_{\mathcal{B},1}, \mu_{1,\mathcal{B}} \bar{\circ} \phi_1 - \phi_1 \circ \mu_{\mathcal{A},1} + \phi_1 \smile \phi_1) = \delta^2(\mu_{\mathcal{A},2}, \mu_{\mathcal{B},2}, \phi_2),$$

where for a p -cochain ϕ_1 (resp. q -cochain ϕ_2), we have

$$\begin{aligned} \phi_1 \bar{\circ} \phi_2(x_1, \dots, x_{p+q-1}) = \\ \sum_{i=1}^q (-1)^{i(p-1)} \phi_1(\phi(x_1), \dots, \phi(x_{i-1}), \phi_2(x_i, \dots, x_{i+q-1}), \phi(x_{i+q}), \dots, \phi(x_{p+q-1})). \end{aligned}$$

Following [2], we express the obstructions for the algebras using the Gerstenhaber bracket. For $\varphi \in C_{\text{Hom}}^p(\mathcal{A}, \mathcal{A})$ and $\psi \in C_{\text{Hom}}^q(\mathcal{A}, \mathcal{A})$, where $p, q \geq 0$, we define $j_{\varphi}^{\alpha} \psi \in C_{\text{Hom}}^{p+q+1}(\mathcal{A}, \mathcal{A})$ to be the composition product given by the operator

$$\begin{aligned} j_{\varphi}^{\alpha}(\psi)(x_0, \dots, x_{p+q}) = \\ \sum_{k=0}^q (-1)^{pk} \psi(\alpha^p(x_0), \dots, \alpha^p(x_{k-1}), \varphi(x_k, \dots, x_{k+p}), \alpha^p(x_{p+k+1}), \dots, \alpha^p(x_{p+q})). \end{aligned}$$

and set $[\varphi, \psi]_{\alpha}^{\Delta} := j_{\psi}^{\alpha}(\varphi) - (-1)^{ab} j_{\varphi}^{\alpha}(\psi)$ and let

$$\begin{aligned} \mathcal{O}b_{\mathcal{A}} &= \sum_{\substack{p+q=N+1 \\ p>0, q>0}} \frac{1}{2} [\mu_{\mathcal{A},p}, \mu_{\mathcal{A},q}]_{\alpha}^{\Delta}, \\ \mathcal{O}b_{\mathcal{B}} &= \sum_{\substack{p+q=N+1 \\ p>0, q>0}} \frac{1}{2} [\mu_{\mathcal{B},p}, \mu_{\mathcal{B},q}]_{\alpha}^{\Delta} \end{aligned}$$

for the obstruction of a deformation of any Hom-associative algebra \mathcal{A} . Set

$$\mathcal{O}b_{\phi} := \sum_{\substack{p+q=N+1 \\ p>0, q>0}} \mu_{\mathcal{B},p} \bar{\circ} \phi_q - \sum_{\substack{p+q=N+1 \\ p>0, q>0}} \phi_p \circ \mu_{\mathcal{A},q} + \sum_{\substack{p+q=N+1 \\ p>0, q>0}} \phi_p \smile \phi_q + \sum_{\substack{p+q=N+1 \\ p>0, q>0, k>0}} \mu_{\mathcal{B},p} \circ (\phi_q, \phi_k)$$

for the obstruction of the extension of the Hom-associative algebra morphism ϕ .

Theorem 4.8. *Let $(\mathcal{A}, \mu_{\mathcal{A},0}, \alpha_{\mathcal{A}})$ and $(\mathcal{B}, \mu_{\mathcal{B},0}, \alpha_{\mathcal{B}})$ be two Hom-associative algebras. Let $\phi : \mathcal{A} \rightarrow \mathcal{B}$ be a Hom-associative algebra morphism and $\Theta_t = (\mu_{\mathcal{A},t}, \mu_{\mathcal{B},t}, \phi_t)$ be an order k 1-parameter formal deformation of ϕ . Then $\mathcal{O}b = (\mathcal{O}b_{\mathcal{A}}, \mathcal{O}b_{\mathcal{B}}, \mathcal{O}b_{\phi}) \in Z_{\text{Hom}}^3(\phi, \phi)$ and Θ_t extends to a deformation of order $k+1$ if and only if $\mathcal{O}b_{\phi}$ is a coboundary.*

Proof. The proof is straightforward but lengthy. One may view it in the arXiv version arXiv:1710.07599. □

Corollary 4.9. *If $H_{\text{Hom}}^3(\phi, \phi) = 0$, then every infinitesimal deformation can be extended to a formal deformation of larger order.*

4.3. Deformations of Hom-Lie algebra morphisms

In this section, we discuss deformations of Hom-Lie algebra morphisms. We obtain similar results as in previous section. For proofs, see arXiv:1710.07599.

Definition 4.10. Let $(\mathcal{L}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra. A 1-parameter formal Hom-Lie deformation of \mathcal{L} is given by a $\mathbb{K}[[t]]$ -bilinear map $[\cdot, \cdot]_t : \mathcal{L}[[t]] \times \mathcal{L}[[t]] \rightarrow \mathcal{L}[[t]]$ of the form $[\cdot, \cdot]_t = \sum_{i \geq 0} t^i [\cdot, \cdot]_i$, where each $[\cdot, \cdot]_i$ is a skewsymmetric bilinear map $[\cdot, \cdot]_i : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ (extended to $\mathbb{K}[[t]]$ -bilinear map), such that $[\cdot, \cdot] = [\cdot, \cdot]_0$ and satisfying

$$\circ_{x,y,z} [\alpha(x), [y, z]_t]_t = 0 \quad (\text{Hom-Jacobi identity}).$$

Let $\phi : \mathcal{L} \rightarrow \mathcal{L}'$ be a Hom-Lie algebra morphism. A deformation of ϕ is a triple $\Theta_t = ([\cdot, \cdot]_t; [\cdot, \cdot]'_t; \phi_t)$ in which:

- $[\cdot, \cdot]_t = \sum_{i \geq 0} t^i [\cdot, \cdot]_i$ is a deformation of \mathcal{L} ,
- $[\cdot, \cdot]'_t = \sum_{i \geq 0} t^i [\cdot, \cdot]'_i$ is a deformation of \mathcal{L}' ,
- $\phi_t : \mathcal{L}[[t]] \rightarrow \mathcal{L}'[[t]]$ is a Hom-Lie algebra morphism of the form $\phi_t = \sum_{n \geq 0} \phi_n t^n$, where each $\phi_n : \mathcal{L} \rightarrow \mathcal{L}'$ is a \mathbb{K} -linear map and $\phi_0 = \phi$.

Proposition 4.11. The linear coefficient, $\Theta_1 = ([\cdot, \cdot]_1, [\cdot, \cdot]'_1, \phi_1)$, which is called the infinitesimal part of the deformation Θ_t , is a 2-cocycle in $C^2_{HL}(\phi, \phi)$.

Let $(\mathcal{L}, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie algebra. Let $\mathcal{L}_t = (\mathcal{L}, [\cdot, \cdot]_t, \alpha)$ and $\mathcal{L}'_t = (\mathcal{L}, [\cdot, \cdot]'_t, \alpha)$ be two deformations of \mathcal{L} , where $[\cdot, \cdot]_t = \sum_{i \geq 0} t^i [\cdot, \cdot]_i$ and $[\cdot, \cdot]'_t = \sum_{i \geq 0} t^i [\cdot, \cdot]'_i$ with $[\cdot, \cdot]_0 = [\cdot, \cdot]'_0 = [\cdot, \cdot]$. We say that \mathcal{L}_t and \mathcal{L}'_t are equivalent if there exists a formal automorphism $\psi_t : \mathcal{L}[[t]] \rightarrow \mathcal{L}[[t]]$, that may be written in the form $\psi_t = \sum_{i \geq 0} \psi_i t^i$, where $\psi_i \in \text{End}(\mathcal{L})$ and $\psi_0 = \text{id}$, such that $\psi_t([x, y]_t) = [\psi_t(x), \psi_t(y)]'_t$.

A deformation \mathcal{L}_t is said to be trivial if and only if \mathcal{L}_t is equivalent to \mathcal{L} .

Definition 4.12. Let $(\mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$, $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \beta)$ be two Hom-Lie algebras and $\phi : \mathcal{L} \rightarrow \mathcal{G}$ be a Hom-Lie algebra morphism. Let $\Theta_t = ([\cdot, \cdot]_{\mathcal{L},t}, [\cdot, \cdot]_{\mathcal{G},t}, \phi_t)$ and $\tilde{\Theta}_t = ([\cdot, \cdot]'_{\mathcal{L},t}, [\cdot, \cdot]'_{\mathcal{G},t}, \tilde{\phi}_t)$ be two deformations of a Hom-Lie algebra morphism ϕ .

A formal automorphism $\psi_t : \Theta_t \rightarrow \tilde{\Theta}_t$ is a pair $(\psi_{\mathcal{L},t}, \psi_{\mathcal{G},t})$, where $\psi_{\mathcal{L},t} : \mathcal{L}[[t]] \rightarrow \mathcal{L}[[t]]$ and $\psi_{\mathcal{G},t} : \mathcal{G}[[t]] \rightarrow \mathcal{G}[[t]]$ are formal automorphisms, such that $\tilde{\phi}_t = \psi_{\mathcal{L},t} \circ \phi_t \circ \psi_{\mathcal{G},t}^{-1}$. Two deformations Θ_t and $\tilde{\Theta}_t$ are equivalent if and only if there exists a formal automorphism that transforms Θ_t in to $\tilde{\Theta}_t$.

Remark 4.13. Given a deformation Θ_t and a pair of power series $\psi_t = (\psi_{\mathcal{L},t} = \sum_n \psi_{\mathcal{L},n} t^n, \psi_{\mathcal{G},t} = \sum_n \psi_{\mathcal{G},n} t^n)$, one can define a deformation $\tilde{\Theta}_t$. The deformation $\tilde{\Theta}_t$ is automatically equivalent to Θ_t .

Theorem 4.14. The infinitesimal of a deformation Θ_t of ϕ is a 2-cocycle in $C^2_{HL}(\phi, \phi)$ whose cohomology class is determined by the equivalence class of the first term of Θ_t .

Theorem 4.15. Let $(\mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ and $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \beta)$ be two Hom-Lie algebras. Let $\Theta_t = ([\cdot, \cdot]_{\mathcal{L},t}, [\cdot, \cdot]_{\mathcal{G},t}, \phi_t)$ be a deformation of a Hom-Lie algebra morphism ϕ . Then there exists an equivalent deformation $\tilde{\Theta}_t = ([\cdot, \cdot]_{\mathcal{L},t}, [\cdot, \cdot]_{\mathcal{G},t}, \tilde{\phi})$ such that $\tilde{\Theta}_1 \in Z_{HL}^2(\phi, \phi)$ and $\tilde{\Theta}_1 \notin B_{HL}^2(\phi, \phi)$. Hence, if $H_{HL}^2(\phi, \phi) = 0$, then every formal deformation is equivalent to a trivial deformation.

4.3.1. Obstructions

A deformation of order N of ϕ is a triple, $\Theta_t = ([\cdot, \cdot]_{\mathcal{L},t}, [\cdot, \cdot]_{\mathcal{G},t}, \phi_t)$ satisfying $\phi_t([x, y]_{\mathcal{L},t}) = [\phi_t(x), \phi_t(y)]_{\mathcal{G},t}$ or equivalently

$$\sum_{i=0}^N \phi_i([x, y]_{\mathcal{L},N-i}) = \sum_{i+j+k=N} [\phi_i(x), \phi_j(y)]_{\mathcal{G},k}.$$

Given a deformation Θ_t of order N , it extends to a deformation of order $N + 1$ if and only if there exists a 2-cochain $\Theta_{N+1} = ([\cdot, \cdot]_{\mathcal{L},N+1}, [\cdot, \cdot]_{\mathcal{G},N+1}, \phi_{N+1}) \in C_{HL}^2(\phi, \phi)$ such that $\bar{\Theta}_t = \Theta_t + t^{N+1}\Theta_{N+1}$ is a deformation of order $N + 1$. Then $\bar{\Theta}_t$ is said to be an *extension* of Θ_t of order $N + 1$.

Let

$$\mathcal{O}b_{\mathcal{L}} = \frac{1}{2} \sum_{\substack{p+q=N+1 \\ p>0, q>0}} [[\cdot, \cdot]_{\mathcal{L},p}, [\cdot, \cdot]_{\mathcal{L},q}]_{\alpha}^{\wedge}$$

be the obstruction of a deformation of the Hom-Lie algebra \mathcal{L} , where $[\cdot, \cdot]_{\alpha}^{\wedge}$ is the Gerstenhaber bracket defined in [2]. Let

$$\mathcal{O}b_{\phi} = \sum_{i=0}^{N+1} \phi_i([x, y]_{\mathcal{L},N+1-i}) - \sum' [\phi_i(x), \phi_j(y)]_{\mathcal{G},k}$$

be the obstruction of the extension of the Hom-Lie algebra morphism ϕ , where

$$\sum' = \sum_{\substack{i+j=N+1 \\ i,j>0 \\ k=0}} + \sum_{\substack{i+k=N+1 \\ i,k>0 \\ j=0}} + \sum_{\substack{j+k=N+1 \\ j,k>0 \\ i=0}} + \sum_{\substack{i+k+j=N+1 \\ i,k,j>0}}.$$

Theorem 4.16. Let $(\mathcal{L}, [\cdot, \cdot]_{\mathcal{L},0}, \alpha)$ and $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G},0}, \beta)$ be two Hom-Lie algebras and $\phi : \mathcal{L} \rightarrow \mathcal{G}$ be a Hom-Lie algebra morphism. Let $\Theta_t = ([\cdot, \cdot]_{\mathcal{L},t}, [\cdot, \cdot]_{\mathcal{G},t}, \phi_t)$ be an order k 1-parameter formal deformation of ϕ . Then $\mathcal{O}b = (\mathcal{O}b_{\mathcal{L}}, \mathcal{O}b_{\mathcal{G}}, \mathcal{O}b_{\phi}) \in Z_{HL}^3(\phi, \phi)$. Therefore, the deformation extends to a deformation of order $k + 1$ if and only if $\mathcal{O}b$ is a coboundary.

5. Example

We compute in this section a cohomology of a given Hom-Lie algebra morphism and discuss some deformations. Let $\mathfrak{g}_1 = (\mathfrak{g}_1, [\cdot, \cdot]_1, \alpha_1)$ and $\mathfrak{g}_2 = (\mathfrak{g}_2, [\cdot, \cdot]_2, \alpha_2)$ be two Hom-Lie algebras defined with

respect to the basis $\{e_1, e_2, e_3\}$ (resp. $\{f_1, f_2, f_3\}$) by the following relations

$$\begin{aligned} [e_1, e_2]_1 &= e_3, [e_2, e_3]_1 = 0, [e_1, e_3]_1 = 0, \\ \alpha_1(e_1) &= p_1 e_1, \alpha_1(e_2) = p_2 e_2, \alpha_1(e_3) = p_1 p_2 e_3; \\ [f_1, f_2]_2 &= f_1 + f_3, [f_2, f_3]_2 = f_2, [f_1, f_3]_2 = f_1 + 2f_3, \\ \alpha_2(f_1) &= f_1, \alpha_2(f_2) = 2f_2, \alpha_2(f_3) = 2f_3, \end{aligned}$$

where p_1, p_2 are parameters.

Let $\phi_{1,2} : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ be a Hom-Lie algebra morphism. We have the following cases

- (1) If $p_1 = p_2 = 2$: $\phi_{1,2}^1(e_1) = \lambda_{2,1} f_2 + \lambda_{3,1} f_3, \phi_{1,2}^1(e_2) = \lambda_{2,2} f_2 + \frac{\lambda_{2,2} \lambda_{3,1}}{\lambda_{2,1}} f_3, \phi_{1,2}^1(e_3) = 0$.
- (2) If $p_1 = 2$ and $p_2 = 0$: $\phi_{1,2}^2(e_1) = \lambda_{2,1} f_2 + \lambda_{3,1} f_3, \phi_{1,2}^2(e_2) = 0, \phi_{1,2}^2(e_3) = 0$.

A 2-cochain is given by a triple (ψ, φ, ρ) , where $\psi : \mathfrak{g}_1 \times \mathfrak{g}_1 \rightarrow \mathfrak{g}_1, \varphi : \mathfrak{g}_2 \times \mathfrak{g}_2 \rightarrow \mathfrak{g}_2, \rho : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$. The 2-cochains of the Hom-Lie algebra \mathfrak{g}_1 are defined by ψ_i for $i = 1, \dots, 8$

$$\begin{aligned} \begin{cases} \psi_1(e_1, e_2) = a_1 e_2 \\ \psi_1(e_2, e_3) = 0 \\ \psi_1(e_1, e_3) = 0 \end{cases} & \begin{cases} \psi_2(e_1, e_2) = a_2 e_3 \\ \psi_2(e_2, e_3) = 0 \\ \psi_2(e_1, e_3) = 0 \end{cases} & \begin{cases} \psi_3(e_1, e_2) = 0 \\ \psi_3(e_2, e_3) = a_3 e_3 \\ \psi_3(e_1, e_3) = 0 \end{cases} & \begin{cases} \psi_4(e_1, e_2) = 0 \\ \psi_4(e_2, e_3) = 0 \\ \psi_4(e_1, e_3) = a_4 e_2 \end{cases} \\ \begin{cases} \psi_5(e_1, e_2) = 0 \\ \psi_5(e_2, e_3) = 0 \\ \psi_5(e_1, e_3) = a_5 e_3 \end{cases} & \begin{cases} \psi_6(e_1, e_2) = a_6 e_1 \\ \psi_6(e_2, e_3) = 0 \\ \psi_6(e_1, e_3) = 0 \end{cases} & \begin{cases} \psi_7(e_1, e_2) = 0 \\ \psi_7(e_2, e_3) = a_7 e_1 \\ \psi_7(e_1, e_3) = 0 \end{cases} & \begin{cases} \psi_8(e_1, e_2) = 0 \\ \psi_8(e_2, e_3) = -\frac{p_2}{p_1} a_8 e_2 \\ \psi_8(e_1, e_3) = a_8 e_1 \end{cases} \end{aligned}$$

where a_1, \dots, a_8 are parameters.

We obtain the following results

- (1) If $p_1 = 0$, then $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1) = \langle \psi_2, \psi_3, \psi_5, \psi_6, \psi_7 \rangle$. Hence, $\dim Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1) = 5$.
- (2) If $p_1 = 1$ and $p_2 \notin \{-1, 0, 1\}$, then $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1) = \langle \psi_1, \psi_2, \psi_4, \psi_5 \rangle$.
 - (a) If $p_2 = 1$, then $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$ is generated in addition by $\{\psi_3, \psi_6, \psi_7, \psi_8\}$.
 - (b) If $p_2 = 0$, then $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$ is generated in addition by $\{\psi_3\}$.
 - (c) If $p_2 = -1$, then $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$ is generated in addition by $\{\psi_7\}$.
- (3) If $p_1 = -1$ and $p_2 \notin \{-1, 0, 1\}$, then $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1) = \langle \psi_2, \psi_4 \rangle$.
 - (a) If $p_2 = 1$, then $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$ is generated in addition by $\{\psi_3, \psi_6, \psi_7\}$.
 - (b) If $p_2 = 0$, then $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$ is generated in addition by $\{\psi_1, \psi_3, \psi_5\}$.
 - (c) If $p_2 = -1$, then $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$ is generated in addition by $\{\psi_7, \psi_8\}$.
- (4) If $p_1 \notin \{-1, 0, 1\}$ and $p_2 = \frac{1}{p_1}$, then $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1) = \langle \psi_2, \psi_8 \rangle$.
 - (a) If $p_2 = 1$, then $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$ is generated by $\{\psi_2, \psi_3, \psi_6, \psi_7\}$.
 - (b) If $p_2 = 0$, then $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$ is generated by $\{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}$.
 - (c) If $p_2 = -1$, then $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1)$ is generated by $\{\psi_2, \psi_7\}$.
- (5) If $p_1 \notin \{-1, 0, 1\}$ and $p_2 \notin \{-1, 0, 1\}$ such that $p_2 \neq \frac{1}{p_1}$, then $Z_{HL}^2(\mathfrak{g}_1, \mathfrak{g}_1) = \langle \psi_2 \rangle$.

All cocycles are not coboundaries except ψ_2 .

Now, we consider the Lie algebra \mathfrak{g}_2 . The 2-cocycles are defined to be

$$\varphi(f_1, f_2) = k_1 f_2 + k_2 f_3, \quad \varphi(f_2, f_3) = 0, \quad \varphi(f_1, f_3) = k_3 f_2 - k_1 f_3,$$

where k_1, k_2, k_3 are parameters. Therefore, $\dim H_{\text{HL}}^2(\mathfrak{g}_2, \mathfrak{g}_2) = 1$ and it is generated by

$$\varphi(f_1, f_2) = f_3, \quad \varphi(f_2, f_3) = 0, \quad \varphi(f_1, f_3) = 0.$$

Now, we consider the third component. For the first morphism $\phi_{1,2}^1$, there is only one 1-cocycle corresponding to $\psi = \psi_2$ and φ . It is given by

$$\rho(e_1) = \left(-\frac{a_{3,2}\lambda_{2,1}^2}{\lambda_{2,2}\lambda_{3,1}} + \frac{a_{2,2}\lambda_{2,1}}{\lambda_{2,2}} + \frac{a_{3,1}\lambda_{2,1}}{\lambda_{3,1}} \right) f_2 + a_{3,1}f_3, \quad \rho(e_2) = a_{2,2}f_2 + a_{3,2}f_3, \quad \rho(e_3) = 0.$$

Therefore $H_{\text{HL}}^1(\mathfrak{g}_1, \mathfrak{g}_2)$ is 3-dimensional.

For the second morphism $\phi_{1,2}^2$, we have

$$\rho(e_1) = a_{2,1}f_2 + a_{3,1}f_3, \quad \rho(e_2) = 0, \quad \rho(e_3) = 0.$$

Therefore $H_{\text{HL}}^1(\mathfrak{g}_1, \mathfrak{g}_2)$ is 2-dimensional.

Now, we provide examples of deformations. For \mathfrak{g}_1 , we consider the deformed bracket

$$[e_1, e_2]_1' = e_3; \quad [e_2, e_3]_1' = 0; \quad [e_1, e_3]_1' = wte_2,$$

where w is a parameter, or

$$[e_1, e_2]_1' = e_3 + te_2; \quad [e_2, e_3]_1' = 0; \quad [e_1, e_3]_1' = te_3.$$

For \mathfrak{g}_2 , we consider the deformed bracket defined by

$$[f_1, f_2]_2' = af_1 + (b + tk_2)f_3; \quad [f_2, f_3]_2' = cf_2; \quad [f_2, f_3]_2' = df_1 + 2af_3.$$

Let $\tilde{\phi}$ be a deformation of the second morphism given by

$$\tilde{\phi}(e_1) = (\lambda_{2,1} + a_{2,1}t)f_2 + (\lambda_{3,1} + a_{3,1}t)f_3; \quad \tilde{\phi}(e_2) = 0; \quad \tilde{\phi}(e_3) = 0.$$

Then $([,]_1', [,]_2', \tilde{\phi})$ is an infinitesimal deformation of $\phi_{1,2}^2$.

Let $\tilde{\phi}$ be an infinitesimal deformation of the first morphism, where

$$\begin{aligned} \tilde{\phi}(e_1) &= \left(\lambda_{2,1} + \left(-\frac{a_{3,2}\lambda_{2,1}^2}{\lambda_{2,2}\lambda_{3,1}} + \frac{a_{2,2}\lambda_{2,1}}{\lambda_{2,2}} + \frac{a_{3,1}\lambda_{2,1}}{\lambda_{3,1}} \right) t \right) f_2 + (\lambda_{3,1} + ta_{3,1})f_3; \\ \tilde{\phi}(e_2) &= (\lambda_{2,2} + ta_{2,2})f_2 + \left(\frac{\lambda_{2,2}\lambda_{3,1}}{\lambda_{2,1}} + ta_{3,2} \right) f_3; \\ \tilde{\phi}(e_3) &= 0. \end{aligned}$$

Then $([,]_1, [,]_2', \tilde{\phi})$ is an infinitesimal deformation of $\phi_{1,2}^1$.

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