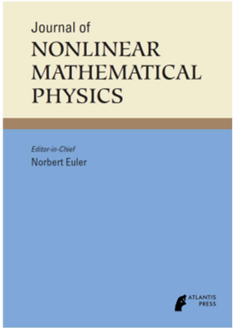


ATLANTIS PRESS

**Journal of Nonlinear Mathematical Physics**

**ISSN (Online): 1776-0852** **ISSN (Print): 1402-9251**
**Journal Home Page: https://www.atlantis-press.com/journals/jnmp**

# Solvable nonlinear discrete-time evolutions and Diophantine findings

Francesco Calogero

**To cite this article**: Francesco Calogero (2018) Solvable nonlinear discrete-time evolutions and Diophantine findings, Journal of Nonlinear Mathematical Physics 25:4, 515–517, DOI: https://doi.org/10.1080/14029251.2018.1503395

**To link to this article**: https://doi.org/10.1080/14029251.2018.1503395

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 25, No. 4 (2018) 515–517

LETTER TO THE EDITOR

# Solvable nonlinear discrete-time evolutions and Diophantine findings

Francesco Calogero
*Physics Department, University of Rome "La Sapienza", Rome, Italy*
*Istituto Nazionale di Fisica Nucleare, Sezione di Roma 1*
*francesco.calogero@uniroma1.it, francesco.calogero@roma1.infn.it*

Received 18 March 2018

Accepted 18 May 2018

Certain nonlinearly-coupled systems of $N$ discrete-time evolution equations are identified, which can be solved by algebraic operations; and some remarkable Diophantine findings are thereby obtained. These results might be useful to test the accuracy of numerical routines yielding the $N$ roots of polynomials of arbitrary degree $N$.

## 1. Introduction and notation

As the reader will easily see, the results of this paper amount to a transfer—from *continuous* to *discrete* time, via the approach introduced in [1]—of the findings reported in [2] and [3] (see also Chapters 3 and 7 of [4]).

Throughout this paper the following notation is used: $N$ and $L$ are two *arbitrary positive integers* ($N \geq 2$, $L \geq 2$), the indices $n$ and $m$ run from 1 to $N$, the *discrete-time* variable $\ell = 0, 1, 2, ...$ takes all *nonnegative integer* values, the $N$ dependent variables $z_n(\ell)$ are generally *complex* numbers and, being generally defined (see below) as the $N$ zeros of a polynomial of degree $N$ in its (complex) argument $z$, they are the elements of an *unordered* set of $N$ elements identified hereafter with the notation $\tilde{z}(\ell)$; likewise in the following the notation $\tilde{f}$ denotes the *unordered* set of $N$ elements $f_n$.

## 2. Results

**Proposition 2.1.** *Consider the system of N second-order discrete-time evolution equations*

$$2\prod_{m=1}^{N}\left[z_n(\ell+2)-z_m(\ell+1)\right]-\prod_{m=1}^{N}\left[z_n(\ell+2)-z_m(\ell)\right]=0\;; \tag{2.1a}$$

*note that this formula provides the unordered set* $\tilde{z}(\ell+2)$*, the elements of which are the N values* $z_n(\ell+2)$*, as the N zeros of the polynomial of degree N in z defined in terms of the two unordered sets* $\tilde{z}(\ell)$ *and* $\tilde{z}(\ell+1)$ *as follows:*

$$P_N\left(z;\tilde{z}(\ell),\tilde{z}(\ell+1)\right)=2\prod_{m=1}^{N}\left[z-z_m(\ell+1)\right]-\prod_{m=1}^{N}\left[z-z_m(\ell)\right]\;. \tag{2.1b}$$

Let this system of second-order discrete-time evolution equations, (2.1a), be complemented by the following assignments of the two *unordered* sets $\tilde{z}(0)$ respectively $\tilde{z}(1)$ of $2N$ *initial* data $z_n(0)$

respectively $z_n(1)$: (i) the $N$ data $z_n(0)$ are assigned *arbitrarily*; (ii) the $N$ data $z_n(1)$ are defined—in terms of the parameter $L$, $(L \neq 1)$, the *unordered* set $\tilde{z}(0)$, and the *unordered* set $\tilde{f}$ the elements of which are $N$ *arbitrarily* assigned (generally *complex*) numbers $f_m$—by the $N$ algebraic equations

$$\prod_{m=1}^{N} [z_n(1) - z_m(0)] + \frac{(-1)^N}{L-1} \prod_{m=1}^{N} [z_n(1) - f_m] = 0 \; ; \tag{2.2a}$$

hence these $N$ data $z_n(1)$ are the $N$ roots of the polynomial $p_N^{(1)}\left(z;\tilde{z}(0),\tilde{f};L\right)$, of degree $N$ in $z$, defined as follows in terms of the two unordered sets $\tilde{z}(0)$ and $\tilde{f}$:

$$p_N^{(1)}\left(z;\tilde{z}(0),\tilde{f};L\right) = \prod_{m=1}^{N} [z - z_m(0)] + \frac{(-1)^N}{L-1} \prod_{m=1}^{N} [z - f_m] \; . \tag{2.2b}$$

The solution $z(\ell)$ of the system of second-order discrete-time evolution equations (2.1a) is then given by the $N$ roots of the following polynomial of degree $N$ in $z$:

$$\psi_N\left(z;\tilde{z}(0),\tilde{f};L;\ell\right) = \left(\frac{L-\ell}{L}\right) \prod_{m=1}^{N} [z - z_m(0)] + \left(\frac{\ell}{L}\right) (-1)^N \prod_{m=1}^{N} (z - f_m) \; . \qquad \square \tag{2.3}$$

Proposition 2.1 is proven in the following Section. In the meantime the reader may immediately verify the validity of the formula (2.3) at $\ell = 0$ and—via (2.2b)—at $\ell = 1$.

**Remark 2.1.** Note that—while in the formulation of this Proposition 2.1 we considered the system of $N$ equations (2.2) as determining the $N$ elements of the unordered set $\tilde{z}(1)$ in terms of the $2N$ elements of the two, *arbitrarily* assigned, unordered sets $\tilde{z}(0)$ and $\tilde{f}$, this system (2.2) of $N$ algebraic equations might as well be considered to define the $N$ elements $f_m$ of the unordered set $\tilde{f}$ in terms of the $2N$ elements of the two—both then *arbitrarily* assigned—unordered sets $\tilde{z}(0)$ and $\tilde{z}(1)$. $\square$

**Corollary 2.1.** *At $\ell = L$ the unordered set $\tilde{z}(L)$ coincides with the unordered set $\tilde{f}$:*

$$\tilde{z}(L) \equiv \tilde{f} \; . \qquad \square \tag{2.4}$$

The validity of this Corollary 2.1 is an immediate consequence of the Proposition 2.1, being obtained by setting $\ell = L$ in (2.3). And it has an obvious Diophantine implication if the $N$, *a priori arbitrary*, numbers $f_m$ are chosen to be *integers* or *rationals*.

## 3. Proof

The starting point of the proof of Proposition 2.1 is the definition (2.3) of the polynomial $\psi_N(z;\ell)$. The consistency of this definition with the assignment of the initial data $\tilde{z}(0)$ and $\tilde{z}(1)$ has already been noted above. What remains to be proven is that the formula

$$\psi_N(z;\ell) = \prod_{n=1}^{N} [z - z_n(\ell)] \tag{3.1}$$

—which, with $\psi_N(z;\ell)$ defined by (2.3), clearly coincides with the statement of Proposition 2.1—implies that the $N$ zeros $z_n(\ell)$ satisfy the evolution equation (2.1a). To this end we note that since by definition (see (2.3)) the dependence of $\psi_N(z;\ell)$ on the discrete-time variable $\ell$ is *linear*, $\psi_N(z;\ell)$ satisfies identically the *linear second-order* difference equation

$$\psi_N(z;\ell+2) - 2\psi_N(z;\ell+1) + \psi_N(z;\ell) = 0 \,. \quad (3.2)$$

For $z = z_n(\ell+2)$, via (3.1), this formula implies (2.1a). Q. E. D.

## 4. Envoy

The result reported in the above Proposition 2.1 is likely to look, at least at first sight, somewhat *remarkable*, especially in view of the arbitrariness of the assignment of the $2N$ numbers $z_n(0)$ and $f_n$ (or, equivalently, $z_n(0)$ and $z_n(1)$; see the above Remark 2.1). But of course, after its validity has been proven, it shall be considered *obvious*—as all valid mathematical results in some sense are. A potential application of this finding is as a tool to test the accuracy of numerical routines to compute the zeros of polynomials of *arbitrary* degree $N$: by comparing, with the simple *explicit* outcome detailed in the above Corollary 2.1, the results yielded by the application of such routines in order to solve numerically—from the initial data detailed in Proposition 2.1, up to $\ell = L$—the discrete-time evolution (2.1); which indeed requires finding the zeros of appropriate polynomials of degree $N$ at every step of this discrete-time evolution. In this context the flexibility implied by the possibility to assign *arbitrarily* the two integers $N$ and $L$ and the $2N$, generally *complex*, numbers $z_n(0)$ and $f_n$ might be quite useful. Specialists in numerical analysis might be interested to explore in detail the vistas implied by such possibilities: note for instance that, for $N = 20$ and $f_m = m$, Corollary 2.1—for any arbitrary assignment of the parameters $L$ and $x_n(0)$—yields the 20 zeros of the perfidious Wilkinson polynomial [5].

An extension of the findings reported in this paper to the case in which the finite positive integer $N$ is replaced by $\infty$ is of course possible, see [2].

A (perhaps less elegant) variant of the approach described in this paper—characterized by the replacement of the system of *second-order* discrete-time evolution equations (2.1a) by systems of *first-order* discrete-time evolution equations—is of course possible, in analogy to the treatments of the continuous-time cases, see [1], [2] and Chapter 3 of [4].

## References


[1] O. Bihun and F. Calogero, "Generations of solvable discrete-time dynamical systems", *J. Math. Phys.* **58**, 052701 (21 pages) (2017); DOI: 10.1063/1.4982959.

[2] F. Calogero, "Finite and infinite systems of nonlinearly coupled ordinary differential equations the solutions of which feature remarkable Diophantine findings", *J. Nonlinear Math. Phys.* **25** (3) (2018) 433–441.

[3] F. Calogero, "Novel differential algorithm to evaluate all the zeros of any generic polynomial", *J. Nonlinear Math. Phys.* **24**, 469–472 (2017). DOI: 10.1080/14029251.2017.1375685.

[4] F. Calogero, *Zeros of Polynomials and Solvable Nonlinear Evolution Equations*, Cambridge University Press, Cambridge, U.K., 2018 (in press).

[5] J.H. Wilkinson, "The perfidious polynomial", in *Studies in Numerical Analysis*, vol. **24**, pp. 1–28, 1984; G.H. Golub (editor), Mathematical Association of America, Washington DC, USA.