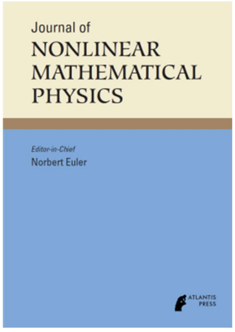


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# The number of independent Traces and Supertraces on the Symplectic Reflection Algebra $H_{1,\eta}(\Gamma \wr S_N)$

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Symplectic reflection algebra $H_{1,\eta}(G)$ has a $T(G)$-dimensional space of traces whereas, when considered as a superalgebra with a natural parity, it has an $S(G)$-dimensional space of supertraces. The values of $T(G)$ and $S(G)$ depend on the symplectic reflection group $G$ and do not depend on the parameter $\eta$.

In this paper, the values $T(G)$ and $S(G)$ are explicitly calculated for the groups $G = \Gamma \wr S_N$, where $\Gamma$ is a finite subgroup of $\mathrm{Sp}(2,\mathbb{C})$.

## 1. Introduction

Let $V := \mathbb{C}^{2N}$, let $G \subset Sp(2N,\mathbb{C})$ be a finite group generated by symplectic reflections. In [11], it was shown that Symplectic Reflection Algebra $H_{1,\eta}(G)$ has $T(G)$ independent traces, where $T(G)$ is the number of conjugacy classes of elements without eigenvalue 1 belonging to the group $G \subset Sp(2N) \subset \mathrm{End}(V)$, and that the algebra $H_{1,\eta}(G)$, considered as a superalgebra with a natural parity, has $S(G)$ independent supertraces, where $S(G)$ is the number of conjugacy classes of elements without eigenvalue $-1$ belonging to $G \subset Sp(2N) \subset \mathrm{End}(V)$. Hereafter, speaking about spectrum, eigenvalues and eigenvectors, the rank of an element of the group algebra $\mathbb{C}[G]$ of the group $G$, etc., we have in mind the representation of the group algebra $\mathbb{C}[G]$ in the space $V$. Besides, we denote all the units in groups, algebras, etc., by 1, and $c \cdot 1$ by $c$ for any number $c$.

Apart from a few cases, there are two families of groups generated by symplectic reflections, see [7] and also [9], [2], [5]:

Family 1): $G$ is a complex reflection group acting on $\mathfrak{H} \oplus \mathfrak{H}^*$, where $\mathfrak{H}$ is the space of reflection representation. In this case, $G$ is a direct product of several groups from the following set of Coxeter groups

$$A_n\,(n \geq 1),\; B_n = C_n\,(n \geq 2),\; D_n\,(n \geq 3),\; E_6,\; E_7,\; E_8,\; F_4,\; G_2,\; H_3,\; H_4,\; I_2(n)\,(n \geq 5, n \neq 6). \quad (1.1)$$

Family 2): $G = \Gamma \wr S_N$, which means here $G = \Gamma^N \rtimes S_N$ acting on $(\mathbb{C}^2)^N$, where $\Gamma$ is a finite subgroup of $Sp(2,\mathbb{C})$.

For groups $G$ from the set (1.1), the list of values $T(G)$ and $S(G)$ is given in [10].

In this work, we give the values of $T(G)$ and $S(G)$ for the 2nd family. Namely, we found the generating functions

$$t(\Gamma,x) := \sum_{N=0}^{\infty} T(\Gamma \wr S_N)x^N, \tag{1.2}$$

$$s(\Gamma,x) := \sum_{N=0}^{\infty} S(\Gamma \wr S_N)x^N \tag{1.3}$$

for each finite subgroup $\Gamma \subset Sp(2,\mathbb{C})$, see Theorem 5.1.

All needed definitions are given in the Section 2; the structure, conjugacy classes and characteristic polynomials of the groups $\Gamma \wr S_N$ are described in Section 3.

To include the case $N = 0$ in consideration in formulas (1.2)–(1.3), it is natural to set $\Gamma \wr S_0 := \{E\}$ and, since $\dim V = 0$, to set $H_{1,\eta}(\Gamma \wr S_0) := \mathbb{C}[\{E\}]$, where $\{E\}$ is the group containing only one element $E$.

Applying the definitions given in Section 2 to the algebra $H_{1,\eta}(\Gamma \wr S_0)$ we deduce that

a) if the algebra $H_{1,\eta}(\Gamma \wr S_0) = \mathbb{C}[\{E\}]$ is considered as superalgebra, it has only a trivial parity $\pi \equiv 0$;
b) the algebra $H_{1,\eta}(\Gamma \wr S_0) = \mathbb{C}[\{E\}]$ has 1-dimensional space of traces and 1-dimensional space of supertraces; these spaces coincide;
c) it is natural to set $T(\Gamma \wr S_0) = S(\Gamma \wr S_0) = 1$;
d) the algebra $H_{1,\eta}(\Gamma \wr S_0) = \mathbb{C}[\{E\}]$ contains two Klein operators (i.e., elements satisfying conditions (2.1)–(2.3)), namely, 1 and $-1$.

## 2. Preliminaries

### 2.1. *Traces*

Let $\mathscr{A}$ be an associative superalgebra with parity $\pi$. All expressions of linear algebra are given for homogenous elements only and are supposed to be extended to inhomogeneous elements via linearity.

A linear complex-valued function *str* on $\mathscr{A}$ is called a *supertrace*, if

$$\mathrm{str}(fg) = (-1)^{\pi(f)\pi(g)}\mathrm{str}(gf) \quad \text{for all } f,g \in \mathscr{A}.$$

A linear complex-valued function tr on $\mathscr{A}$ is called a *trace*, if

$$\mathrm{tr}(fg) = \mathrm{tr}(gf) \quad \text{for all } f,g \in \mathscr{A}.$$

The element $K \in \mathscr{A}$ is called a *Klein operator*, if

$$\pi(K) = 0, \tag{2.1}$$

$$K^2 = 1, \tag{2.2}$$

$$Kf = (-1)^{\pi(f)} fK \qquad \text{for all } f \in \mathscr{A}. \tag{2.3}$$

Any Klein operator, if exists, establishes an isomorphism between the space of traces on $\mathscr{A}$ and the space of supertraces on $\mathscr{A}$.

Namely, if $f \mapsto \mathrm{tr}(f)$ is a trace, then $f \mapsto \mathrm{tr}(fK^{1+\pi(f)})$ is a supertrace, and if $f \mapsto \mathrm{str}(f)$ is a supertrace, then $f \mapsto \mathrm{str}(fK^{1+\pi(f)})$ is a trace.

### 2.2. *Symplectic reflection group*

Let $V = \mathbb{C}^{2N}$ be endowed with a non-degenerate anti-symmetric $\mathrm{Sp}(2N)$-invariant bilinear form $\omega(\cdot,\cdot)$, let the vectors $e_i \in V$, where $i = 1 \dots, 2N$, constitute a basis in $V$.

The matrix $(\omega_{ij}) := \omega(e_i, e_j)$ is anti-symmetric and non-degenerate.

Let $x^i$ be the coordinates of $x \in V$, i.e., $x = e_i x^i$. Then $\omega(x, y) = \omega_{ij} x^i y^j$ for any $x, y \in V$. The indices $i$ are lowered and raised by means of the forms $(\omega_{ij})$ and $(\omega^{ij})$, where $\omega_{ij}\omega^{kj} = \delta_i^k$.

**Definition 2.1.** The element $R \in \mathrm{Sp}(2N) \subset \mathrm{End}\, V$ is called a *symplectic reflection*, if $\mathrm{rank}(R-1) = 2$.

**Definition 2.2.** Any finite subgroup $G$ of $\mathrm{Sp}(2N)$ generated by a set of symplectic reflections is called a *symplectic reflection group*.

In what follows, $G$ stands for a symplectic reflection group, and $\mathscr{R}$ stands for the set of all symplectic reflections in $G$.

Let $R \in \mathscr{R}$. Set

$$V_R := \mathrm{Im}(R-1), \tag{2.4}$$

$$Z_R := \mathrm{Ker}(R-1). \tag{2.5}$$

Clearly, $V_R$ and $Z_R$ are symplectically perpendicular, i.e., $\omega(V_R, Z_R) = 0$, and $V = V_R \oplus Z_R$.

So, let $x = x_{V_R} + x_{Z_R}$ for any $x \in V$, where $x_{V_R} \in V_R$ and $x_{Z_R} \in Z_R$. Set

$$\omega_R(x,y) := \omega(x_{V_R}, y_{V_R}). \tag{2.6}$$

### 2.3. *Symplectic reflection algebra (following* [3]*)*

Let $\mathbb{C}[G]$ be the *group algebra* of $G$, i.e., the set of all linear combinations $\sum_{g\in G} \alpha_g \bar{g}$, where $\alpha_g \in \mathbb{C}$.

If we were rigorists, we would write $\bar{g}$ to distinguish $g$ considered as an element of $G \subset \mathrm{End}(V)$ from the same element $\bar{g} \in \mathbb{C}[G]$ considered as an element of the group algebra. The addition in $\mathbb{C}[G]$ is defined as follows:

$$\sum_{g\in G} \alpha_g \bar{g} + \sum_{g\in G} \beta_g \bar{g} = \sum_{g\in G} (\alpha_g + \beta_g)\bar{g}$$

and the multiplication is defined by setting $\overline{g_1}\,\overline{g_2} = \overline{g_1 g_2}$. In what follows, however, we abuse notation and omit the bar sign over elements of the group algebra.

Let $\eta$ be a function on $\mathscr{R}$, i.e., a set of constants $\eta_R$ with $R \in \mathscr{R}$ such that $\eta_{R_1} = \eta_{R_2}$, if $R_1$ and $R_2$ belong to one conjugacy class of $G$.

**Definition 2.3.** The algebra $H_{t,\eta}(G)$, where $t \in \mathbb{C}$, is an associative algebra with unit 1; it is the algebra $\mathbb{C}[V]$ of (noncommutative) polynomials in the elements of $V$ with coefficients in the group algebra $\mathbb{C}[G]$ subject to the relations

$$gx = g(x)g \text{ for any } g \in G \text{ and } x \in V, \text{ where } g(x) = e_i g^i_j x^j \text{ for } x = e_i x^i, \tag{2.7}$$

$$[x,y] = t\omega(x,y) + \sum_{R\in\mathscr{R}} \eta_R \omega_R(x,y) R \text{ for any } x, y \in V. \tag{2.8}$$

The algebra $H_{t,\eta}(G)$ is called a *symplectic reflection algebra*, see [3].

The commutation relations (2.8) suggest to define the *parity* $\pi$ by setting:

$$\pi(x) = 1,\ \pi(g) = 0 \ \text{ for any } x \in V, \text{ and } g \in G, \tag{2.9}$$

enabling one to consider $H_{t,\eta}(G)$ as an associative *superalgebra*.

We consider the case $t \neq 0$ only, for any such $t$ it is equivalent to the case $t = 1$.

Let $\mathscr{A}$ and $\mathscr{B}$ be superalgebras such that $\mathscr{A}$ is a $\mathscr{B}$-module. We say that the superalgebra $\mathscr{A} * \mathscr{B}$ is a *crossed product* of $\mathscr{A}$ and $\mathscr{B}$, if $\mathscr{A} * \mathscr{B} = \mathscr{A} \otimes \mathscr{B}$ as a superspace and

$$(a_1 \otimes b_1) * (a_2 \otimes b_2) = a_1 b_1(a_2) \otimes b_1 b_2,$$

see [14]. The element $b_1(a_2)$ may include a sign factor imposed by the Sign Rule, see [1], p. 45.

The (super)algebra $H_{1,\eta}(G)$ is a deform of the crossed product of the Weyl algebra $W_N$ and the group algebra of a finite subgroup $G \subset \mathrm{Sp}(2N)$ generated by symplectic reflections.

### 2.4. *The number of independent traces and supertraces on the symplectic reflection algebras*

**Theorem 2.1 ([11]).** *Let the symplectic reflection group $G \subset \mathrm{End}(V)$ have $T_G$ conjugacy classes without eigenvalue* 1 *and $S_G$ conjugacy classes without eigenvalue* $-1$.

*Then the algebra $H_{1,\eta}(G)$ has $T(G) = T_G$ independent traces whereas $H_{1,\eta}(G)$ considered as a superalgebra, see* (2.9)*, has $S(G) = S_G$ independent supertraces.*

**Proposition 2.1.** *Let $G_1 \subset \mathrm{End}(V_1)$, $G_2 \subset \mathrm{End}(V_2)$ and $G = G_1 \times G_2 \subset \mathrm{End}(V_1 \oplus V_2)$ be symplectic reflection groups. Then $T(G) = T(G_1)T(G_2)$ and $S(G) = S(G_1)S(G_2)$.*

Proof follows from evident relations $T_G = T_{G_1} T_{G_2}$, $S_G = S_{G_1} S_{G_2}$ and Theorem 2.1.

**Proposition 2.2.** *If there exists a $K \in G$ such that $K|_V = -1$, then $K$ is a Klein operator.*

## 3. The group $\Gamma \wr S_N$

### 3.1. *Finite subgroups of* $\mathrm{Sp}(2, \mathbb{C})$

The complete list of the finite subgroups $\Gamma \subset \mathrm{Sp}(2, \mathbb{C})$ is as follows, see, e.g., [15]:

| $\Gamma$ | Order | Presence of $-1$ | The number of conjugacy classes $C(\Gamma)$ |
|---|---|---|---|
| Cyclic group $\mathbf{Z}_n := \mathbb{Z}/n\mathbb{Z}$ | $n$ | yes, if $n$ is even; no, if $n$ is odd | $n$ |
| Binary dihedral group $\mathscr{D}_n$ | $4n$ | yes | $n+3$ |
| Binary tetrahedral group $\mathscr{T}$ | 24 | yes | 7 |
| Binary octahedral group $\mathscr{O}$ | 48 | yes | 8 |
| Binary icosahedral group $\mathscr{I}$ | 120 | yes | 9 |

It is easy to see that each of these groups, except $\mathbf{Z}_{2k+1}$, has $C(\Gamma) - 1$ conjugacy classes without $+1$ in the spectrum and has $C(\Gamma) - 1$ conjugacy classes without $-1$ in the spectrum. The group $\mathbf{Z}_{2k+1}$ has $C(\mathbf{Z}_{2k+1}) - 1$ conjugacy classes without $+1$ in the spectrum and it has $C(\mathbf{Z}_{2k+1})$ conjugacy classes without $-1$ in the spectrum.

### 3.2. *Symplectic reflections in* $\Gamma\wr S_N$ (*following* [4])

Let $V=\mathbb{C}^{2N}$ and let the symplectic form $\omega$ have the shape

$$\omega := \begin{pmatrix} \varpi & & & \\ & \varpi & & \\ & & \ddots & \\ & & & \varpi \end{pmatrix}, \text{ where } \varpi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.1}$$

The elements of the group $\Gamma\wr S_N$ have the form of $N\times N$ block matrix with $2\times 2$ blocks. Consider the following elements of $\Gamma\wr S_N$

$$(D_{g,i})_{kl} := \begin{cases} g, & \text{if } k=l=i, \\ 1, & \text{if } k=l\neq i, \\ 0, & \text{otherwise,} \end{cases} \tag{3.2}$$

$$(K_{ij})_{kl} := \begin{cases} \delta_{kl}, & \text{if } k,l\neq i,\ k,l\neq j, \\ \delta_{ki}\delta_{lj}+\delta_{kj}\delta_{li}, & \text{otherwise,} \end{cases} \tag{3.3}$$

$$S_{g,ij} := D_{g,i}D_{g^{-1},j}K_{ij}, \tag{3.4}$$

where $i,j=1,\dots,N$, $i\neq j$, $1\neq g\in\Gamma$. It is clear that $K_{ij}=K_{ji}$ and $S_{g,ij}=S_{g^{-1},ji}$.

The complete set of symplectic reflections in $\Gamma\wr S_N$ consists of $D_{g,i}$, $K_{ij}$ and $S_{g,ij}$, where $1\leqslant i<j\leqslant N$ and $1\neq g\in\Gamma$. This set generates the group $\Gamma\wr S_N$.

The symplectic reflections $K_{ij}$ and $S_{g,ij}$ lie in one conjugacy class for all $i\neq j$ and $g\neq 1$; the elements $D_{g,i}$ ($g\neq 1$) and $D_{h,j}$ ($h\neq 1$) lie in one conjugacy class, if $g$ and $h$ are conjugate in $\Gamma$. So, the algebra $H_{1,\eta}(\Gamma\wr S_N)$ depends on $C(\Gamma)$ parameters $\eta$, if $N\geqslant 2$, and on $C(\Gamma)-1$ parameters, if $N=1$. Here $C(\Gamma)$ is the number of conjugacy classes in $\Gamma$ including the class $\{1\}$.

### 3.3. *Conjugacy classes* (*following* [13])

Further, the elements of the group $\Gamma\wr S_N$ can be represented in the form $D\sigma$ where $D\in\Gamma^N$ is a diagonal $N\times N$ block matrix, each block being a $2\times 2$-matrix, and $\sigma$ is $N\times N$ block matrix of permutation each block being a $2\times 2$-matrix.

The product has the form:

$$(D_1\sigma_1)(D_2\sigma_2)=D_3\sigma_3$$

where $\sigma_3=\sigma_1\sigma_2$ and $D_3=D_1\sigma_1D_2\sigma_1^{-1}$.

Fix an element $g_0 = D_0\sigma_0$. Since the permutation $\sigma_0$ is a product of cycles, there exists a permutation $\sigma'$ such that

$$\sigma'\sigma_0(\sigma')^{-1} = \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_s \end{pmatrix}, \text{ where } c_k \text{ are the cycles of length } L_k,\ \sum_k L_k = N, \tag{3.5}$$

$$c_k = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.6}$$

The element $\sigma' D_0\sigma_0(\sigma')^{-1}$ has the form

$$\sigma' D_0\sigma_0(\sigma')^{-1} = \begin{pmatrix} D_1c_1 & & & \\ & D_2c_2 & & \\ & & \ddots & \\ & & & D_sc_s \end{pmatrix},$$

where $D_k$ is an $L_k \times L_k$ diagonal block matrix, each block being a $2\times 2$-matrix:

$$D_k = \begin{pmatrix} g_1^k & & & \\ & g_2^k & & \\ & & \ddots & \\ & & & g_{L_k}^k \end{pmatrix}, \quad g_i^k \in \Gamma.$$

Next, consider diagonal block matrices $H_k = diag(h_1^k, h_2^k, \ldots, h_{L_k}^k)$ and the elements

$$H_kD_kc_kH_k^{-1} = \begin{pmatrix} h_1^kg_1^k(h_2^k)^{-1} & & & & \\ & h_2^kg_2^k(h_3^k)^{-1} & & & \\ & & h_3^kg_3^k(h_4^k)^{-1} & & \\ & & & \ddots & \\ & & & & h_{L_k}^kg_{L_k}^k(h_1^k)^{-1} \end{pmatrix} c_k.$$

For any element $h_1^k \in \Gamma$, one can choose

$$h_2^k = h_1^kg_1^k,\ h_3^k = h_2^kg_2^k,\ \ldots,\ h_{L_k}^k = h_{L_k-1}^kg_{L_k-1}^k$$

such that

$$H_kD_kc_kH_k^{-1} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & h_1^kg_1^kg_2^k\cdots g_{L_k}^k(h_1^k)^{-1} \end{pmatrix} c_k.$$

So, each conjugacy class of $\Gamma\wr S_N$ is described by the set of cycles in the decomposition (3.5), (3.6) of $\sigma_0$, where each cycle is marked by some conjugacy class of $\Gamma$.

The cycle of length $r$ marked by the conjugacy class $\alpha$ of $\Gamma$ with representative $g_\alpha \in \Gamma$ has the shape:

$$A_{\alpha,r} := \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \dots & \\ & & & & g_\alpha \end{pmatrix} \cdot c^r = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ g_\alpha & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad (3.7)$$

where $\alpha = 1,\dots,C(\Gamma)$ and $r = 1,2,\dots$; the matrix $A_{\alpha,r}$ and the cycle $c^r$ are the $r\times r$ block matrices, each block being a $2\times 2$ matrix.

So, each element $g \in \Gamma\wr S_N$ is conjugate to the element of the shape

$$\begin{pmatrix} A_{\alpha_1,r_1} & & & \\ & A_{\alpha_2,r_2} & & \\ & & \dots & \\ & & & A_{\alpha_s,r_s} \end{pmatrix} \qquad (3.8)$$

It is convenient to describe the conjugacy class of $\Gamma\wr S_N$ with representative (3.8) by the set of nonnegative integers $p_r^\alpha$, where $r = 1, 2, 3, \dots$, and $\alpha = 1, \dots, C(\Gamma)$, such that

$$\sum_{\alpha,r} r p_r^\alpha = N. \qquad (3.9)$$

The value $p_r^\alpha$ for some conjugacy class is the number of cycles $A_{\alpha,r}$ of length $r$ in the decomposition (3.8) marked by the conjugacy class $\alpha$ of the group $\Gamma$.

Note that in [13] the notation $m_r(\alpha)$ is used instead of $p_r^\alpha$ we use in this paper.
The restriction (3.9) can be omitted and can serve as definition of $N$ for each set of the numbers $p_r^\alpha$.

The number of conjugacy classes in $\Gamma\wr S_N$ is equal to

$$C(\Gamma\wr S_N) = \sum_{p_r^\alpha : \sum_{\alpha,r} r p_r^\alpha = N} 1.$$

The generating function $c(\Gamma,x)$ of the number of conjugacy classes is defined as

$$c(\Gamma,x) := \sum_{N=0}^{\infty} C(\Gamma\wr S_N) x^N$$

and is equal to

$$c(\Gamma,x) = \sum_{p_r^\alpha} x^{\sum_{\alpha,r} r p_r^\alpha} = \sum_{p_r^\alpha=0}^{\infty} \prod_{r=1}^{\infty} \prod_{\alpha=1}^{C(\Gamma)} (x^r)^{p_r^\alpha} = \prod_{r=1}^{\infty} \prod_{\alpha=1}^{C(\Gamma)} \frac{1}{1-x^r} = (\Psi(x))^{C(\Gamma)},$$

where $\Psi(x)$ is the Euler function

$$\Psi(x) := \prod_{r=1}^{\infty} \frac{1}{1-x^r}$$

### 3.4. *Characteristic polynomials of conjugacy classes*

Before seeking the generating functions $t(\Gamma,x)$ and $s(\Gamma,x)$, let us find the characteristic polynomial of the conjugacy class $g$ of $\Gamma \wr S_N$ identified by the set $p_r^\alpha$.

Let $P_M(\lambda) := \det(M-\lambda)$ be the characteristic polynomial of the matrix $M$. Then it is easy to see that

$$P_{A_{\alpha,r}}(\lambda) = \det(A_{\alpha,r}-\lambda) = \det(g_\alpha - \lambda^r) = P_{g_\alpha}(\lambda^r),$$

where the marked cycle $A_{\alpha,r}$ is defined by (3.7).

Let $g \in \Gamma \wr S_N$ be defined by Eq. (3.8).

Now, it is easy to show that

$$\begin{aligned} P_g(\lambda) = \det(g-\lambda) &= \prod_{i=1}^{s} \det(A_{\alpha_i,r_i}-\lambda) \\ &= \prod_{\alpha,r:\, p_r^\alpha \geqslant 1} \det(A_{\alpha,r}-\lambda)^{p_r^\alpha} \\ &= \prod_{\alpha,r:\, p_r^\alpha \geqslant 1} (\det(g_\alpha - \lambda^r))^{p_r^\alpha}, \end{aligned} \tag{3.10}$$

if $g$ is a representative of the congugacy class in $\Gamma \wr S_N$ corresponding to the set $p_r^\alpha$.

**Definition 3.1.** We call a conjugacy class *t-admissible*, if its representative $g \in \Gamma \wr S_N$ is such that $P_g(1) \neq 0$.

**Definition 3.2.** We call a conjugacy class *s-admissible*, if its representative $g \in \Gamma \wr S_N$ is such that $P_g(-1) \neq 0$.

**Definition 3.3.** We call a marked cycle $A_{\alpha,r}$, see Eq. (3.7), *t-admissible*, if $P_{A_{\alpha,r}}(1) \neq 0$.

**Definition 3.4.** We call a marked cycle $A_{\alpha,r}$, see Eq. (3.7), *s-admissible*, if $P_{A_{\alpha,r}}(-1) \neq 0$.

Equation (3.10) implies the following statements:

**Proposition 3.1.** *The conjugacy class of $\Gamma \wr S_N$ identified by the set $p_r^\alpha$ is t-admissible, if and only if the marked cycle $A_{\alpha,r}$ is t-admissible for any pair $\alpha$, $r$ such that $p_r^\alpha \neq 0$,*

**Proposition 3.2.** *The conjugacy class of $\Gamma \wr S_N$ identified by the set $p_r^\alpha$ is s-admissible, if and only if the marked cycle $A_{\alpha,r}$ is s-admissible for any pair $\alpha$, $r$ such that $p_r^\alpha \neq 0$,*

Recall that $g_\alpha \in \Gamma \subset Sp(2,\mathbb{C})$, where $\Gamma$ is a finite group. So $\det g_\alpha = 1$ and the Jordan normal form of $g_\alpha$ is diagonal. This implies that if $g_\alpha$ has $+1$ in its spectrum, then $g_\alpha = 1$ and if $g_\alpha$ has $-1$ in its spectrum, then $g_\alpha = -1$. These facts together with Eq. (3.10) imply, in their turn, the following two propositions:

**Proposition 3.3.** *The conjugacy class of $\Gamma \wr S_N$ identified by the set $p_r^\alpha$ is t-admissible if and only if $g_\alpha \neq 1$ for all $\alpha$, $r$ with $p_r^\alpha \neq 0$.*

**Proposition 3.4.** *The conjugacy class of $\Gamma \wr S_N$ identified by the set $p_r^\alpha$ is s-admissible if and only if for any pair $\alpha$, $r$ such that $p_r^\alpha \neq 0$, at least one of the next three conditions holds:*

a) *$g_\alpha \neq -1$ and $g_\alpha \neq 1$,*

b) *$r$ is even and $g_\alpha = -1$,*
c) *$r$ is odd and $g_\alpha = 1$.*

Note that the three sets of pairs $(r, \alpha)$ defined by the cases a), b), c) in Proposition 3.4 have empty pair-wise intersections.

**Definition 3.5.** Let $t_r(\Gamma)$ for $r = 1, 2 \dots$ be equal to the number of different $\alpha$ such that $A_{\alpha,r}$ is $t$-admissible.

Evidently,

$$t_r(\Gamma) = C(\Gamma) - 1. \tag{3.11}$$

**Definition 3.6.** Let $s_r(\Gamma)$ for $r = 1, 2 \dots$ be equal to the number of different $\alpha$ such that $A_{\alpha,r}$ is $s$-admissible.

Evidently, if $\Gamma \ni -1$, then

$$s_r(\Gamma) = C(\Gamma) - 1. \tag{3.12}$$

and if $\Gamma \not\ni -1$, then

$$s_r(\Gamma) = \begin{cases} C(\Gamma) - 1, & \text{if } r \text{ is even,} \\ C(\Gamma), & \text{if } r \text{ is odd.} \end{cases} \tag{3.13}$$

## 4. Combinatorial problem

Consider the following combinatorial problem (analogous problems are considered in [8]).

Suppose we have an unlimited supply of 1-gram colored weights for each of $n_1$ different colors, an unlimited supply of 2-gram colored weights for each of $n_2$ different colors, an unlimited supply of 3-gram colored weights for each of $n_3$ different colors, and so on. Let $a^N_{n_1,\dots,n_k,\dots}$ be the number of opportunities to choose weights from our set of total mass $N$ grams.

The problem is to find generating function

$$F_{n_1,\dots,n_k,\dots}(x) := \sum_{N=0}^{\infty} a^N_{n_1,\dots,n_k,\dots} x^N.$$

This problem is exactly the problem we discussed earlier. Namely, now we say "$r$-gram weight" instead of cycle of length $r$, and "the number of different colors $n_r$" instead of the number $t_r$ (3.11) or $s_r$ (3.12)–(3.13) of different $\alpha$.

**Proposition 4.1.**

$$F_{n_1+m_1,n_2+m_2,\dots,n_k+m_k,\dots}(x) = F_{n_1,n_2,\dots,n_k,\dots}(x) \cdot F_{m_1,m_2,\dots,m_k,\dots}(x).$$

*Proof.* To prove this proposition, it suffices to note that

$$a^N_{n_1+m_1,n_2+m_2,\dots,n_k+m_k,\dots} = \sum_{M=0}^{N} a^M_{n_1,n_2,\dots,n_k,\dots} \cdot a^{N-M}_{m_1,m_2,\dots,m_k,\dots}.$$

□

Introduce the functions

$$f_i := F_{n^i_1,n^i_2,\dots,n^i_k,\dots}, \qquad \text{where } n^i_k = \delta^i_k.$$

Then

$$f_i(x) = 1 + x^i + x^{2i} + x^{3i} + \dots = \frac{1}{1-x^i}.$$

The next theorem follows from Proposition 4.1

**Theorem 4.1.**

$$F_{n_1,n_2,\dots,n_k,\dots} = \prod_{i=1}^{\infty} (f_i)^{n_i}.$$

The function $F_{1,1,1,\dots} = \Psi(x)$ is the well-known Euler function, the generating function of the number of partitions of $N$ into the sum of positive integers.

## 5. Generating functions $t(\Gamma)$ and $s(\Gamma)$

**Theorem 5.1.** *Set*

$$\Psi(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i} \qquad \textit{(Euler function)}, \tag{5.1}$$

$$\Phi(x) := \prod_{k=0}^{\infty} \frac{1}{1-x^{2k+1}}. \tag{5.2}$$

*Let $T(\Gamma\wr S_N)$ be the dimension of the space of traces on $H_{1,\eta}(\Gamma\wr S_N)$ and let $S(\Gamma\wr S_N)$ be the dimension of the space of supertraces on $H_{1,\eta}(\Gamma\wr S_N)$ considered as a superalgebra.*

*Let*

$$t(\Gamma,x) := \sum_{N=0}^{\infty} T(\Gamma\wr S_N)x^N \quad \textit{and} \quad s(\Gamma,x) := \sum_{N=0}^{\infty} S(\Gamma\wr S_N)x^N.$$

*Then*

$$t(\Gamma,x) = (\Psi(x))^{C(\Gamma)-1},$$
$$s(\Gamma,x) = (\Psi(x))^{C(\Gamma)-1}, \quad \textit{if } \Gamma \neq \mathbf{Z}_{2k+1},$$
$$s(\Gamma,x) = (\Psi(x))^{C(\Gamma)-1}\Phi(x), \ \textit{if } \Gamma = \mathbf{Z}_{2k+1}.$$

*Proof.* To prove Theorem 5.1, we apply Theorem 4.1 to the numbers (3.11)–(3.13) of admissible conjugacy classes. It is clear that

$$t(\Gamma) = F_{t_1(\Gamma),t_2(\Gamma),t_3(\Gamma),\dots} = \Psi^{C(\Gamma)-1},$$

$$s(\Gamma) = F_{s_1(\Gamma),s_2(\Gamma),s_3(\Gamma),\dots} = \begin{cases} \Psi^{C(\Gamma)-1}, & \text{if } \Gamma \ni -1, \\ \Psi^{C(\Gamma)-1}\Phi, & \text{if } \Gamma \not\ni -1. \end{cases}$$

□

Observe that $\Phi(x) = \sum_{i=0}^{\infty} O_N x_N$, where $O_N$ is the number of partitions of $N$ into the sum of odd positive integers, and $O_N$ coinsides with the number of independent supertraces on $H_{1\eta}(S_N)$, see [12].

### 5.1. *Inequality theorem*

**Theorem 5.2.** *Let $G = \Gamma \wr S_N$. For each positive integer N, the following statements hold:*

$$S(G) > 0,$$
$$S(G) \geqslant T(G),$$
$$S(G) = T(G) \text{ if and only if } H_{1,\eta}(G) \text{ contains a Klein operator.}$$

Literally the same statements were proved for the groups $G$ from Family 1) in [10], and hence these statements hold for the direct product of any finite number of groups from Family 1) and Family 2) defined on page 1.

*Proof.* Let $\Gamma \neq \mathbf{Z}_{2k+1}$. Since each finite group $\Gamma \in Sp(2N, \mathbb{C})$, except $\Gamma = \mathbf{Z}_{2k+1}$, contains $-1$, the group $\Gamma \wr S_N$ contains Klein operator $K = \prod_{i=1}^{N} D_{-1,i}$.

There is no Klein operator in $H_{1,\eta}(\mathbf{Z}_{2k+1} \wr S_N)$ since for this algebra, $S(\mathbf{Z}_{2k+1} \wr S_N) > T(\mathbf{Z}_{2k+1} \wr S_N)$, as it follows from Theorem 5.1. □

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