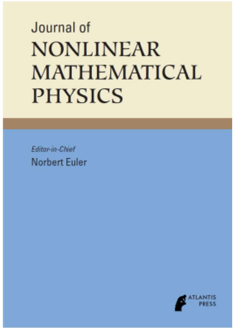

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Bilinear Identities and Hirota's Bilinear Forms for the (γ_n, σ_k)-KP Hierarchy

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In this paper, we discuss how to construct the bilinear identities for the wave functions of the (γ_n, σ_k)-KP hierarchy and its Hirota's bilinear forms. First, based on the corresponding squared eigenfunction symmetry of the KP hierarchy, we prove that the wave functions of the (γ_n, σ_k)-KP hierarchy are equal to the bilinear identities given in Sec.3 by introducing N auxiliary parameters z_i, $i = 1, 2, \ldots, N$. Next, we derived the bilinear equations for the tau-function of the (γ_n, σ_k)-KP hierarchy. Then, we obtain the bilinear equations for the tau-function of the mixed type of KP equation with self-consistent sources (KPESCS), which includes both the first and the second type of KPESCS as special cases by setting $n = 2$ and $k = 3$. Finally, using the relation between the Hirota bilinear derivatives and the usual partial derivatives, we show the procedure of translating the Hirota's bilinear equations into the mixed type of KPESCS.

Keywords: (γ_n, σ_k)-KP hierarchy; bilinear identity; τ-function; Hirota's bilinear form.

2000 Mathematics Subject Classification:35Q51, 37K60

1. Introduction

Sato theory has important applications in the theory of integrable systems. It reveals the infinite dimensional Grassmannian structure of space of tau-functions, where the tau-function are solutions for the Hirota's bilinear form of KP hierarchy. The KP hierarchy can be expressed in terms of pseudo-differential operator and has the bilinear identities [3, 4].

Soliton equations with self-consistent sources (SESCS) are important integrable models in many fields of physics, such as hydrodynamics, state physics, plasma physics. For example, the KdV equation with self-consistent sources describes the interaction of long and short capillary-gravity waves. The nonlinear Schrödinger equation with self-consistent sources represents the nonlinear interaction of an electrostatic high frequency wave with the ion acoustic wave in a two component homogeneous plasma. The KP equation with self-consistent sources describes the interaction of a long wave with a short wave packet propagating on the x-y plane at some angle to each other.

As an infinite dimensional integrable system, it has been generalized to large sets of integrable hierarchies by introducing new flows [7,16]. In [8], Liu and his collaborators construct an extended KP hierarchy by introducing a new vector field ∂_{τ_k}. This new extended KP hierarchy can be reduced to the k-constrained KP hierarchy , the Gelfand-Dickey hierarchy with self-consistent sources, the first type of KP equation with self-consistent sources(KPESCS) and the second type of KPESCS. In [18], Yao and her collaborators propose a new (γ_n, σ_k)-KP hierarchy with two new time series γ_n and σ_k. This new (γ_n, σ_k)-KP hierarchy can be regarded as a generalization of the extended KP hierarchy, which consists of a γ_n-flow, a σ_k-flow as well as a mixed γ_n- and σ_k-evolution equations of the eigenfunctions [8]. The (γ_n, σ_k)-KP hierarchy contains the mixed type of KP equation with self-consistent sources (KPESCS), which can also be reduced to both the first type and the second type of KPESCS as special cases. Also, the constrained flows of the (γ_n, σ_k)-KP hierarchy can be regarded as a generalization of the Gelfand-Dickey hierarchy (GDH), which contains the first,the second as well as the mixed type of GDH with self-consistent sources.

The KP hierarchy can be expressed in bilinear form using Hirota's bilinear operators [6]. In this formalism, solutions to the KP equation can be obtained without knowing its Lax pair. Researchers have paid much attention on the subject of bilinear identities because of its importance in Sato theory. By using the bilinear identities of soliton hierarchies [2–4, 9, 15], we can derive the Hirota bilinear forms for all the equations in the hierarchies. Recently, Lin and his collaborators give the bilinear identities for the wave functions of the KP hierarchy with a squared eigenfunction symmetry in [11]. Considering the squared eigenfunction symmetry as an auxiliary flow, they also give the bilinear identities for the extended KP hierarchy. They obtain the generating functions of the Hirota bilinear forms for the extended KP hierarchy by constructing the τ-function for the extended KP hierarchy.

This paper is organized as follows. In Section 2, we briefly recall the KP hierarchy and the (γ_n, σ_k)-KP hierarchy. In Section 3, the bilinear identities of the (γ_n, σ_k)-KP hierarchy are constructed. In Section 4, the τ-function of the (γ_n, σ_k)-KP hierarchy is introduced. The generation functions for the Hirota bilinear form of the (γ_n, σ_k)-KP hierarchy are obtained. In Section 5, we show the procedure of translating the Hirota bilinear forms into nonlinear partial differential equations. Conclusions are given in the last section.

2. The KP hierarchy and (γ_n, σ_k)-KP hierarchy

Let

$$L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \cdots$$

be a pseudo-differential operator whose coefficients are considered as generators of a differential algebra $\mathscr{A}$ [4].

The well-known KP hierarchy

$$L_{t_n} = [B_n, L], \quad n \in \mathbb{N} \tag{2.1}$$

can be constructed from the compatibility condition of the following linear systems [3,4]

$$L\psi = \lambda\psi, \tag{2.2a}$$

$$\frac{\partial \psi}{\partial t_n} = B_n\psi, \quad B_n = (L^n)_+, \quad n \in \mathbb{N}, \tag{2.2b}$$

where $\{t_n\}$ are the time variables with $t_1 = x$ and B_n stands for the differential part of L^n. The compatibility of t_n-flow and t_m-flow of the KP hierarchy (2.1) leads to the following zero-curvature equations

$$(B_m)_{t_n} - (B_n)_{t_m} = [B_n, B_m], \quad m,n \in \mathbb{N}. \tag{2.3}$$

Supposing that $W = 1 + \omega_1\partial^{-1} + \omega_2\partial^{-2} + \cdots$ is a dressing operator satisfying

$$\partial_{t_n} W = -(W\partial^n W^{-1})_- W, \quad n \in \mathbb{N}, \tag{2.4}$$

then the operator L defined by

$$L = W\partial W^{-1} \tag{2.5}$$

is a solution to the KP hierarchy (2.1).

Let the wave functions and the adjoint wave functions be

$$\psi(t,\lambda) = We^{\eta(t,\lambda)}, \tag{2.6a}$$

$$\psi^*(t,\lambda) = (W^*)^{-1}e^{-\eta(t,\lambda)}, \quad \eta(t,\lambda) = \sum_{i\geq 1} t_i\lambda^i, \tag{2.6b}$$

where W^* is the formal adjoint of W defined by $(\sum_i a_i\partial^i)^* := \sum_i (-\partial)^i a_i$, we find that the wave function (2.6a) satisfies the KP hierarchy (2.2) while the adjoint wave function satisfies

$$L^*\psi^* = \lambda\psi^*, \tag{2.7a}$$

$$\frac{\partial \psi^*}{\partial t_n} = -B_n^*\psi^*, \quad B_n^* = [(L^*)^n]_+, \quad n \in \mathbb{N}. \tag{2.7b}$$

Similarly, we can get the following hierarchy

$$L^*_{t_n} = [L^*, B_n^*], \quad n \in \mathbb{N} \tag{2.8}$$

from the linear systems (2.7a). If the operator W^* is a solution of

$$\partial_{t_n}[(W^*)^{-1}] = [(W^*)^{-1}\partial W^*]_{-1}(W^*)^{-1}, \quad n \in \mathbb{N}, \tag{2.9}$$

then the adjoint operator $L^* = -(W^*)^{-1}\partial W^*$ is a solution to the hierarchy (2.8).

For any fixed $k \in \mathbb{N}$, by defining a new variable τ_k whose vector field is given by

$$\partial_{\tau_k} = \partial_{t_k} - \sum_{i=1}^{N}\sum_{s\geq 0} \varsigma_i^{-s-1}\partial_{t_s},$$

Liu and his collaborators introduce a new extended KP hierarchy [8]

$$L_{t_n} = [B_n, L], (n \in \mathbb{N}, n \neq k), \tag{2.10a}$$

$$L_{\tau_k} = \left[B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i, L \right], \tag{2.10b}$$

$$q_{i,t_n} = B_n(q_i), \tag{2.10c}$$

$$r_{i,t_n} = -B_n^*(r_i), \tag{2.10d}$$

$$q_{i,\tau_k} = B_k(q_i), \tag{2.10e}$$

$$r_{i,\tau_k} = -B_k^*(r_i), i = 1, \ldots, N. \tag{2.10f}$$

The compatibility of t_n-flow and τ_k-flow of (2.10) gives rise to the following zero-curvature equations

$$B_{n,\tau_k} - \left(B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right)_{t_n} + \left[B_n, B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right] = 0.$$

For any fixed $n, k \in \mathbb{N}$, Yao and her collaborators propose the (γ_n, σ_k)-KP hierarchy with two generalized time series γ_n and σ_k in [18]

$$L_{t_s} = [B_s, L], \quad (n \in \mathbb{N}, s \neq n, \ s \neq k), \tag{2.11a}$$

$$L_{\gamma_n} = \left[B_n + \alpha_n \sum_{i=1}^{N} q_i \partial^{-1} r_i, L \right], \quad (n \neq k), \tag{2.11b}$$

$$L_{\sigma_k} = \left[B_k + \beta_k \sum_{i=1}^{N} q_i \partial^{-1} r_i, L \right], \tag{2.11c}$$

$$q_{i,t_s} = B_s(q_i), \tag{2.11d}$$

$$r_{i,t_s} = -B_s^*(r_i), \tag{2.11e}$$

$$\alpha_n (q_{i,\sigma_k} - B_k(q_i)) - \beta_k \left(q_{i,\gamma_n} - B_n(q_i) \right) = 0, \tag{2.11f}$$

$$\alpha_n (r_{i,\sigma_k} + B_k^*(r_i)) - \beta_k \left(r_{i,\gamma_n} + B_n^*(r_i) \right) = 0, \quad i = 1, \ldots, N, \tag{2.11g}$$

where α_n and β_k are constants, q_i and r_i $(i = 1, 2, \cdots, N)$ are *generalized* eigenfunctions and adjoint eigenfunctions. It's easy to see that the KP hierarchy can be derived from (2.11) by setting $\alpha_n = 0$ and $\beta_k = 0$. The commutativity of (2.11b) and (2.11c) under (2.11f) and (2.11g) gives rise to the following zero-curvature equations

$$B_{n,\sigma_k} - B_{k,\gamma_n} + [B_n, B_k] + \beta_k \left[B_n, \sum_{i=1}^{N} q_i \partial^{-1} r_i \right]_+ + \alpha_n \left[\sum_{i=1}^{N} q_i \partial^{-1} r_i, B_k \right]_+ = 0. \tag{2.12}$$

Supposing that the operator W in (2.5) satisfies the following evolution equations

$$\partial_{t_s} W = -(W \partial^s W^{-1})_- W, \quad (s \neq n, s \neq k) \tag{2.13a}$$

$$W_{\gamma_n} = -(W \partial^n W^{-1})_- W + \alpha_n \sum_{i=1}^{N} q_i \partial^{-1} r_i W, \quad (n \neq k) \tag{2.13b}$$

$$W_{\sigma_k} = -(W \partial^k W^{-1})_- W + \beta_k \sum_{i=1}^{N} q_i \partial^{-1} r_i W, \tag{2.13c}$$

we can prove that the operator L defined by (2.5) satisfies (2.11a) (2.11b) and (2.11c) (see [18] for the proof).

When we take $n=2$, $k=3$ and set $\gamma_2=y$, $\sigma_3=t$, $u_1=u$, the mixed KPESCS

$$\begin{aligned}&4u_t-3\partial^{-1}u_{yy}-12uu_x-u_{xxx}-3\alpha_2\sum_{i=1}^{N}(q_ir_i)_y+4\beta_3\sum_{i=1}^{N}(q_ir_i)_x\\&+3\alpha_2\sum_{i=1}^{N}(q_ir_{i,xx}-q_{i,xx}r_i)=0\end{aligned}, \tag{2.14a}$$

$$\begin{aligned}&\alpha_2\left(q_{i,t}-q_{i,xxx}-3uq_{i,x}-\tfrac{3}{2}q_i\partial^{-1}u_y-\tfrac{3}{2}q_iu_x-\tfrac{3}{2}\alpha_2q_i\sum_{j=1}^{N}q_jr_j\right)\\&-\beta_3(q_{i,y}-q_{i,xx}-2uq_i)=0\end{aligned}, \tag{2.14b}$$

$$\begin{aligned}&\alpha_2\left(r_{i,t}-r_{i,xxx}-3ur_{i,x}+\tfrac{3}{2}r_i\partial^{-1}u_y-\tfrac{3}{2}r_iu_x+\tfrac{3}{2}\alpha_2r_i\sum_{j=1}^{N}q_jr_j\right)\\&-\beta_3(r_{i,y}+r_{i,xx}+2ur_i)=0,\quad i=1,2,\ldots,N\end{aligned}, \tag{2.14c}$$

can be obtained from (2.12), (2.11f) and (2.11g).

In particular, if $\alpha_2=\beta_3=0$ (*resp.*, $\alpha_2=0,\beta_3=1$ or $\alpha_2=1,\beta_3=0$ or $\alpha_2=1,\beta_3=1$), the nonlinear equations (2.14) will be reduced to the KP equation [3,4](*resp.*, the first type [12,13,17], or the second type [5,8,12], or the mixed type of KP equation with self-consistent sources [18]). The KP equation with self-consistent sources has important applications in physics [10,13].

3. Bilinear Identities for the (γ_n,σ_k)-KP hierarchy

We introduce ∂_{z_i}-flows $(i=1,2,\cdots,N)$ as

$$\partial_{z_i}L=\left[q_i\partial^{-1}r_i,L\right],\quad i=1,2,\ldots,N, \tag{3.1}$$

where q_i and r_i are the eigenfunctions and their adjoint ones, respectively to construct the bilinear identities for (2.11). According to the results given in [1], the relation between the operator W and the auxiliary parameters z_i $(i=1,2,\cdots,N)$ satisfies

$$W_{z_i}=q_i\partial^{-1}r_iW,\quad i=1,2,\cdots,N. \tag{3.2}$$

Let $\xi(t,\lambda)=\sum\limits_{i\neq n,k}t_i\lambda^i+\gamma_n\lambda^n+\sigma_k\lambda^k$, the action of pseudo-differential operator on $\xi(t,\lambda)$ is defined by

$$\begin{aligned}&\partial^m\xi(t,\lambda)=\lambda^m,\\&\partial^m e^{\xi(t,\lambda)}=\lambda^m e^{\xi(t,\lambda)}\end{aligned}$$

for any integer m.

Denoting $z=(z_1,z_2,\ldots,z_N)$, $t=(t_1,\ldots,t_{n-1},\gamma_n,t_{n+1},\ldots,t_{k-1},\sigma_k,t_{k+1},\cdots)$ or $t=(t_1,\ldots,t_{n-1},\gamma_n,t_{n+1},\ldots,t_{k-1},\sigma_k,t_{k+1},\cdots)$, the wave function and the adjoint wave function with auxiliary parameters z_i $(i=1,2,\cdots,N)$ can be defined as

$$\omega(z,t,\lambda)=We^{\xi(t,\lambda)}, \tag{3.3a}$$

$$\omega^*(z,t,\lambda)=(W^*)^{-1}e^{-\xi(t,\lambda)}. \tag{3.3b}$$

Before giving the bilinear identities for (2.11), let's recall a useful lemma [3]:

Lemma 1. *Let P and Q be two pseudo-differential operators, Q^* is the formal adjoint of Q, then*

$$\mathrm{Res}_\partial P\cdot Q^* = \mathrm{Res}_\lambda P\left(e^{\xi(t,\lambda)}\right)\cdot Q\left(e^{-\xi(t,\lambda)}\right), \tag{3.4}$$

where $\mathrm{Res}_\partial\left(\sum\limits_i a_i\partial^i\right) = a_{-1}$ *and* $\mathrm{Res}_\lambda\left(\sum\limits_i a_i\lambda^i\right) = a_{-1}$.

Now we have the following theorems:

Theorem 1. *The (γ_n,σ_k)-KP hierarchy (2.11) is equivalent to the following bilinear identities with N auxiliary variables z_i, $i=1,2,3,...,N$,*

$$\mathrm{Res}_\lambda\,\omega(T,t,\lambda)\,\omega^*\left(T',t',\lambda\right)=0, \tag{3.5a}$$
$$\mathrm{Res}_\lambda\,\omega_{z_i}(T,t,\lambda)\,\omega^*\left(T',t',\lambda\right)=q_i(T,t)\,r_i\left(T',t'\right), \tag{3.5b}$$
$$\mathrm{Res}_\lambda\,\omega(T,t,\lambda)\left[\partial^{-1}q_i\left(T',t'\right)\omega^*\left(T',t',\lambda\right)\right]=-q_i(T,t), \tag{3.5c}$$
$$\mathrm{Res}_\lambda\left[\partial^{-1}r_i(T,t)\,\omega(T,t,\lambda)\right]\omega^*\left(T',t',\lambda\right)=r_i\left(T',t'\right), i=1,2,...,N, \tag{3.5d}$$

where

$$\begin{aligned}
&t=(t_1,\ldots,t_{n-1},\gamma_n,t_{n+1},\ldots,t_{k-1},\sigma_k,t_{k+1},\cdots),\\
&t=(t'_1,\ldots,t'_{n-1},\gamma'_n,t'_{n+1},\ldots,t'_{k-1},\sigma'_k,t'_{k+1},\cdots),\\
&T=(z_1-\alpha_n\gamma_n-\beta_k\sigma_k,\ z_2-\alpha_n\gamma_n-\beta_k\sigma_k,\ \cdots,\ z_N-\alpha_n\gamma_n-\beta_k\sigma_k),\\
&T'=(z_1-\alpha_n\gamma'_n-\beta_k\sigma'_k,z_2-\alpha_n\gamma'_n-\beta_k\sigma'_k,\cdots,z_N-\alpha_n\gamma'_n-\beta_k\sigma'_k),
\end{aligned}$$

and

$$f\left(T',t'\right)=\sum g_1\cdot g_2\cdot f(T,t),$$

$$\begin{aligned}
&g_1=(t'_1-t_1)^{i_1}\cdots(t'_{n-1}-t_{n-1})^{i_{n-1}}(t'_{n+1}-t_{n+1})^{i_{n+1}}\cdots(t'_{k-1}-t_{k-1})^{i_{k-1}}(t'_{k+1}-t_{k+1})^{i_{k+1}}\cdots,\\
&g_2=\tfrac{\partial_1^{i_1}\cdots\partial_{n-1}^{i_{n-1}}\partial_{n+1}^{i_{n+1}}\cdots\partial_{k-1}^{i_{k-1}}\partial_{k+1}^{i_{k+1}}\cdots}{i_1!\cdots i_{n-1}!i_{n+1}!\cdots i_{k-1}!i_{k+1}!\cdots}(-1)^{i_0-i}(\beta_k\gamma'_n-\beta_k\gamma_n)^i(\alpha_n\sigma'_k-\alpha_n\sigma_k)^{i_0-i}\tfrac{\partial_n^i\partial_k^{i_0-i}}{i!(i_0-i)!}.
\end{aligned}$$

The action of ∂^{-1} on the (adjoint) wave function is taken as pseudo-differential operator acting on the exponential part of the function, e.g., $\partial^{-1}(r\omega)=\left(\partial^{-1}rW\right)\left(e^{\xi(t,\lambda)}\right)$.

Proof. Let's prove the following observations first

$$\begin{aligned}
&\left(\beta_k\tfrac{d}{d\gamma_n}-\alpha_n\tfrac{d}{d\sigma_k}\right)^{m_0}\partial_{t_1}^{m_1}\cdots\partial_{t_{n-1}}^{m_{n-1}}\partial_{t_{n+1}}^{m_{n+1}}\cdots\partial_{t_{k-1}}^{m_{k-1}}\partial_{t_{k+1}}^{m_{k+1}}\cdots\partial_{t_l}^{m_l}[\omega^*(T,t,\lambda)]\\
&=P_{m_0m_1\cdots m_{n-1}m_{n+1}\cdots m_{k-1}m_{k+1}\cdots m_l}\,\omega^*(T,t,\lambda)
\end{aligned}, \tag{3.6a}$$

$$\begin{aligned}
&\left(\beta_k\tfrac{d}{d\gamma_n}-\alpha_n\tfrac{d}{d\sigma_k}\right)^{m_0}\partial_{t_1}^{m_1}\cdots\partial_{t_{n-1}}^{m_{n-1}}\partial_{t_{n+1}}^{m_{n+1}}\cdots\partial_{t_{k-1}}^{m_{k-1}}\partial_{t_{k+1}}^{m_{k+1}}\cdots\partial_{t_l}^{m_l}[r_i(T,t)]\\
&=P_{m_0m_1\cdots m_{n-1}m_{n+1}\cdots m_{k-1}m_{k+1}\cdots m_l}\,r_i(T,t)
\end{aligned}, \tag{3.6b}$$

where $P_{m_0m_1\cdots m_{n-1}m_{n+1}\cdots m_{k-1}m_{k+1}\cdots m_l}$ is a differential operator in ∂ since the actions of the partial derivatives ∂_{t_i} (for $i\neq n,k$) and $\frac{d}{d\gamma_n}$, $\frac{d}{d\sigma_k}$ can all be written as the actions of differential operators.

Indeed, applying $\partial_{t_s}, \frac{d}{d\gamma_n}, \frac{d}{d\sigma_k}$ to $\omega^*(T,t,\lambda)$, the following expression can be constructed

$$\partial_{t_s}[\omega^*(T,t,\lambda)] = -B_s^*[\omega^*(T,t,\lambda)], \tag{3.7a}$$

$$\frac{d}{d\gamma_n}[\omega^*(T,t,\lambda)] = \omega^*_{\gamma_n}(z,t,\lambda)|_{z=T} - \alpha_n \sum_{j=1}^{N} \omega^*_{z_j}(T,t,\lambda), \tag{3.7b}$$

$$\frac{d}{d\sigma_k}[\omega^*(T,t,\lambda)] = \omega^*_{\sigma_k}(z,t,\lambda)|_{z=T} - \beta_k \sum_{j=1}^{N} \omega^*_{z_j}(T,t,\lambda), \tag{3.7c}$$

which can be reduced to

$$\left(\beta_k \frac{d}{d\gamma_n} - \alpha_n \frac{d}{d\sigma_k}\right)[\omega^*(T,t,\lambda)] = [\alpha_n B_k^* - \beta_k B_n^*][\omega^*(T,t,\lambda)]. \tag{3.8}$$

Similarly, applying $\partial_{t_s}, \frac{d}{d\gamma_n}, \frac{d}{d\sigma_k}$ to $r_i(T,t)$ and taking (2.11e) (2.11g) into consideration, we have

$$\partial_{t_s}[r_i(T,t)] = -B_s^*[r_i(T,t)], i = 1, \cdots, N, \tag{3.9a}$$

$$\alpha_n\left[r_{i,\sigma_k}(z,t)|_{z=T} + B_k^* r_i(T,t)\right] - \beta_k\left[r_{i,\gamma_n}(z,t)|_{z=T} + B_n^* r(T,t)_i\right] = 0, \tag{3.9b}$$

$$\frac{d}{d\gamma_n}[r_i(T,t)] = r_{i,\gamma_n}(z,t)|_{z=T} - \alpha_n \sum_{j=1}^{N} r_{i,z_j}(T,t), \tag{3.9c}$$

$$\frac{d}{d\sigma_k}[r_i(T,t)] = r_{i,\sigma_k}(z,t)|_{z=T} - \beta_k \sum_{j=1}^{N} r_{i,z_j}(T,t). \tag{3.9d}$$

Substituting (3.9c) and (3.9d) into (3.9b), we obtain

$$\left(\beta_k \frac{d}{d\gamma_n} - \alpha_n \frac{d}{d\sigma_k}\right) r_i(T,t) = (\alpha_n B_k^* - \beta_k B_n^*)\, r_i(T,t). \tag{3.10}$$

So the observations (3.6) can be obtained with the help of (3.7a), (3.8) and (3.9a), (3.10).

Now we prove the bilinear identities (3.5) from (2.11)(2.13)(3.2):

To prove the bilinear identity (3.5a), it is sufficient to consider the following case

$$\mathrm{Res}_\lambda\, \omega(T,t,\lambda)\left(\beta_k \frac{d}{d\gamma_n} - \alpha_n \frac{d}{d\sigma_k}\right)^{m_0} \partial_{t_1}^{m_1} \cdots \partial_{t_{n-1}}^{m_{n-1}} \partial_{t_{n+1}}^{m_{n+1}} \cdots \partial_{t_{k-1}}^{m_{k-1}} \partial_{t_{k+1}}^{m_{k+1}} \cdots \partial_{t_l}^{m_l} \omega^*(T,t,\lambda) = 0$$

for every $m_j \geq 0$.

By using Lemma 1 and observation (3.6a), we have

$$\begin{aligned}
&\mathrm{Res}_\lambda\, \omega(T,t,\lambda)\left(\beta_k \tfrac{d}{d\gamma_n} - \alpha_n \tfrac{d}{d\sigma_k}\right)^{m_0} \partial_{t_1}^{m_1} \cdots \partial_{t_{n-1}}^{m_{n-1}} \partial_{t_{n+1}}^{m_{n+1}} \cdots \partial_{t_{k-1}}^{m_{k-1}} \partial_{t_{k+1}}^{m_{k+1}} \cdots \partial_{t_l}^{m_l} \omega^*(T,t,\lambda) \\
&= \mathrm{Res}_\lambda W e^{\xi(t,\lambda)} P_{m_0 m_1 \cdots m_{n-1} m_{n+1} \cdots m_{k-1} m_{k+1} \cdots m_l} (W^*)^{-1} e^{-\xi(t,\lambda)} \\
&= \mathrm{Res}_\partial W (W)^{-1} P^*_{m_0 m_1 \cdots m_{n-1} m_{n+1} \cdots m_{k-1} m_{k+1} \cdots m_l} \\
&= 0
\end{aligned},$$

so the bilinear identity (3.5a) holds.

Notice that $W_{z_i} = q_i\partial^{-1}r_iW, i = 1,2,\ldots,N$, we get

$$\begin{aligned}
&\mathrm{Res}_\lambda\,\omega_{z_i}(T,t,\lambda)\left(\beta_k\tfrac{d}{d\gamma_n}-\alpha_n\tfrac{d}{d\sigma_k}\right)^{m_0}\partial_{t_1}^{m_1}\cdots\partial_{t_{n-1}}^{m_{n-1}}\partial_{t_{n+1}}^{m_{n+1}}\cdots\partial_{t_{k-1}}^{m_{k-1}}\partial_{t_{k+1}}^{m_{k+1}}\cdots\partial_{t_l}^{m_l}\omega^*(T,t,\lambda)\\
&=\mathrm{Res}_\lambda\,\omega_{z_i}(T,t,\lambda)\,P_{m_0m_1\cdots m_{n-1}m_{n+1}\cdots m_{k-1}m_{k+1}\cdots m_l}\,\omega^*(T,t,\lambda)\\
&=\mathrm{Res}_\lambda\,q_i(T,t)\,\partial^{-1}r_i(T,t)\,We^{\xi(t,\lambda)}P_{m_0m_1\cdots m_{n-1}m_{n+1}\cdots m_{k-1}m_{k+1}\cdots m_l}(W^*)^{-1}e^{-\xi(t,\lambda)}\\
&=\mathrm{Res}_\partial\,q_i(T,t)\,\partial^{-1}r_i(T,t)\,P^*_{m_1m_2\cdots m_l}\\
&=q_i(T,t)\,P_{m_0m_1\cdots m_{n-1}m_{n+1}\cdots m_{k-1}m_{k+1}\cdots m_l}\,r_i(T,t),
\end{aligned}$$

so the bilinear identity (3.5b) is proved.

Similarly, we have the following bilinear identity

$$\mathrm{Res}_\lambda\,\omega(T,t,\lambda)\,\omega^*_{z_i}(T',t',\lambda) = q_i(T,t)\,r_i(T',t'). \tag{3.11}$$

By substituting $\omega^*_{z_i} = -r_i\partial^{-1}q_i\omega^*$ into (3.11) and $\omega_{z_i} = q_i\partial^{-1}r_i\omega$ into (3.5b) respectively, the bilinear identities (3.5c) and (3.5d) can be proved.

Theorem 2. *If $q_i(T,t)$, $r_i(T,t)$ $(i = 1,2,\cdots,N)$,*

$$\omega(T,t,\lambda) = \left(1+\sum_{i\geq 1}\omega_i\lambda^{-i}\right)e^{\xi(t,\lambda)},$$

and

$$\omega^*(T,t,\lambda) = \left(1+\sum_{i\geq 1}\omega_i^*\lambda^{-i}\right)e^{-\xi(t,\lambda)}$$

satisfy the bilinear identities (3.5), then the pseudo-differential operators $L = W\partial W^{-1}$ $(W = 1+\sum_i w_i\partial^{-i})$ and functions q_i and r_i are solutions to the (γ_n,σ_k)-KP hierarchy (2.11).

Proof. For any $m \geq 1$, denoting $\tilde{W} = 1+\sum_{i\geq 1}\omega_i^*\partial^{-i}$ and taking (3.5a) and Lemma 1 into account , we have

$$\begin{aligned}
&\mathrm{Res}_\partial W\tilde{W}^*\partial^m = \mathrm{Res}_\lambda We^{\xi(t,\lambda)}(-\partial)^m\tilde{W}e^{-\xi(t,\lambda)}\\
&=\mathrm{Res}_\lambda\,\omega(T,t,\lambda)\,(-\partial)^m\omega^*(T,t,\lambda)\\
&=0,
\end{aligned}$$

which implies that the negative part of $(W\tilde{W}^*)_-$ is 0. Noticing that the non-negative part of $(W\tilde{W}^*)_+$ is 1, we have $\tilde{W} = (W^*)^{-1}$.

For any $m > 0$, the following computation

$$\begin{aligned}
&\mathrm{Res}_\partial W_{z_i}W^{-1}(-\partial)^m = \mathrm{Res}_\lambda W_{z_i}e^{\xi(t,\lambda)}\partial^m\left(W^{-1}\right)^*e^{\xi(t,\lambda)}\\
&=\mathrm{Res}_\lambda\,\omega_{z_i}(T,t,\lambda)\,\partial^m\omega^*(T,t,\lambda) \qquad ,\\
&=q_i(T,t)\,\partial^m r_i(T,t)
\end{aligned}$$

leads to

$$\left(W_{z_i}W^{-1}\right)_- = q_i(T,\gamma)\,\partial^{-1}r_i(T,\gamma).$$

Since the non-negative part of $\left(W_{z_i}W^{-1}\right)_+$ is 0, then (3.2) holds.

From the definition of W, we know that $(W_{t_i}+L^i_-W)_+ = 0$.

For $m>0$ and $s\neq n, s\neq k$, we have the following computation

$$\begin{aligned}
&\mathrm{Res}_{\partial}(W_{t_s}W^{-1}+L^s_-)_-\partial^m=\mathrm{Res}_{\partial}(W_{t_s}W^{-1}+(W\partial^sW^{-1})_-)_-\partial^m\\
&=\mathrm{Res}_{\lambda}W_{t_s}\mathrm{e}^{\xi(t,\lambda)}\cdot(-\partial)^m(W^*)^{-1}\mathrm{e}^{-\xi(t,\lambda)}+\mathrm{Res}_{\partial}(W\partial^sW^{-1}-(W\partial^sW^{-1})_+)\partial^m\\
&=\mathrm{Res}_{\lambda}W_{t_s}\mathrm{e}^{\xi(t,\lambda)}\cdot(-\partial)^m(W^*)^{-1}\mathrm{e}^{-\xi(t,\lambda)}+\mathrm{Res}_{\lambda}W\partial^s\mathrm{e}^{\xi(t,\lambda)}\cdot(-\partial)^mW^{*-1}\mathrm{e}^{-\xi(t,\lambda)}\\
&=\mathrm{Res}_{\lambda}\omega_{t_s}(T,t,\lambda)(-\partial)^m\omega^*(T,t,\lambda)=0,
\end{aligned}$$

which means that (2.13a) holds.

Similarly, we can get

$$\frac{d}{d\gamma_n}[W(T,t)]=-L^n_-W, \tag{3.12a}$$

$$\frac{d}{d\sigma_k}[W(T,t)]=-L^k_-W, \tag{3.12b}$$

i.e.,

$$W_{\gamma_n}(z,t,)|_{z=T}-\alpha_n\sum_{i=1}^N q_i\partial r_iW=-L^n_-W,$$

$$W_{\sigma_k}(z,t)|_{z=T}-\beta_k\sum_{i=1}^N q_i\partial r_iW=-L^k_-W.$$

So (2.13b) and (2.13c) are proved.

Applying both sides of (3.12) to $e^{\xi(t,\lambda)}$ respectively, we can get the following equalities

$$\frac{d}{d\gamma_n}\omega(T,t,\lambda)=B_n\omega(T,t,\lambda),$$

$$\frac{d}{d\sigma_k}\omega(T,t,\lambda)=B_k\omega(T,t,\lambda).$$

Then according to (3.5c), we find

$$\begin{aligned}
\frac{d}{d\gamma_n}[q_i(T,t)]&=-\mathrm{Res}_{\lambda}\ \frac{d}{d\gamma_n}[\omega(T,t,\lambda)]\ \cdot\partial^{-1}\left(q_i(T',t')\omega^*(T',t',\lambda)\right)\\
&=L^n_+\left(-\mathrm{Res}_{\lambda}\,\omega(T,t,\lambda)\ \cdot\partial^{-1}\left(q_i(T',t')w^*(T',t',\lambda)\right)\right)\\
&=L^n_+(q_i(T,t)),
\end{aligned}$$

which leads to

$$\alpha_n\sum_{j=1}^N q_{i,z_j}(T,t)+q_{i,\gamma_n}(z,t)|_{z=T}-L^n_+q_i(T,t)=0. \tag{3.13}$$

Similarly, we have

$$\beta_k\sum_{j=1}^N q_{i,z_j}(T,t)+q_{i,\sigma_k}(z,t)|_{z=T}-L^k_+q_i(T,t)=0. \tag{3.14}$$

So equation (2.11f) can be proved by combining (3.13) and (3.14).

Equation (2.11g) can also be proved in the similar way from (3.5d). □

4. Tau-Function for (γ_n, σ_k)-KP hierarchy

Since the (adjoint) wave functions of (γ_n, σ_k)-KP hierarchy satisfy the same bilinear equation (3.5a) just as the (adjoint)wave functions of the original KP hierarchy do (if one considers z_i, $i = 1,2,\cdots,N$ as auxiliary parameters), it is reasonable to assume the existence of tau-function and make the following assumptions

$$\omega(T,t,\lambda) = \frac{\widetilde{\tau(z,t-[\lambda])}}{\widetilde{\tau(z,t)}} e^{\xi(t,\lambda)}, \tag{4.1a}$$

$$\omega^*(T,t,\lambda) = \frac{\widetilde{\tau(z,t+[\lambda])}}{\widetilde{\tau(z,t)}} e^{-\xi(t,\lambda)}, \tag{4.1b}$$

where the $\widetilde{\ }$ over a function $f(z,t)$ is defined as

$$\widetilde{f(z,t)} \equiv f(z_1 - \alpha_n\gamma_n - \beta_k\sigma_k, \ldots, z_N - \alpha_n\gamma_n - \beta_k\sigma_k, t) \tag{4.2}$$

with $[\lambda] = \left(\frac{1}{\lambda}, \frac{1}{2\lambda^2}, \cdots\right)$, $z = (z_1, z_2, \ldots, z_N)$, $t = (t_1, \ldots, t_{n-1}, \gamma_n, t_{n+1}, \ldots, t_{k-1}, \sigma_k, t_{k+1}, \cdots)$. For example,according to the definition (4.2), we have

$$\widetilde{\tau(z,t)} = \tau(T,t),$$

$$\tau\widetilde{(z,t-[\lambda])} = \tau(R,t-[\lambda]),$$

where

$$R = (R_1, R_2, \ldots, R_N), \quad R_i = z_i - \alpha_n\left(\gamma_n - \frac{1}{n\lambda^n}\right) - \beta_k\left(\sigma_k - \frac{1}{k\lambda^k}\right), \quad i = 1,2,\ldots,N.$$

According to [2, 11], assume

$$q_i(z,t) = \frac{\kappa_i(z,t)}{\tau(z,t)}, \quad r_i(z,t) = \frac{\rho_i(z,t)}{\tau(z,t)}, \quad i = 1,2,\ldots,N. \tag{4.3}$$

we can get the following results

$$\partial^{-1}(r_i(T,t)\,\omega(T,t,\lambda)) = \frac{\rho_i\widetilde{(z,t-[\lambda])}}{\lambda\tau(T,t)} e^{\xi(t,\lambda)}, \quad i = 1,2,\ldots,N, \tag{4.4a}$$

$$\partial^{-1}(q_i(T,t)\,\omega^*(T,t,\lambda)) = \frac{-\kappa_i\widetilde{(z,t+[\lambda])}}{\lambda\tau(T,t)} e^{-\xi(t,\lambda)}, \quad i = 1,2,\ldots,N. \tag{4.4b}$$

Substituting (4.1) (4.3) and (4.4) into (3.5), we have

$$\mathrm{Res}_\lambda\, \tau\widetilde{(z,t-[\lambda])}\tau\widetilde{(z,t'+[\lambda])} e^{\xi(t-t',\lambda)} = 0, \tag{4.5a}$$

$$\mathrm{Res}_\lambda\, \tau_{z_i}\widetilde{(z,t-[\lambda])}\tau\widetilde{(z,t'+[\lambda])} e^{\xi(t-t',\lambda)} \tag{4.5b}$$

$$-\mathrm{Res}_\lambda\, \tau\widetilde{(z,t-[\lambda])}(\partial_{z_i}\log\widetilde{\tau(z,t)})\tau\widetilde{(z,t'+[\lambda])} e^{\xi(t-t',\lambda)} = \kappa_i\widetilde{(z,t)}\rho_i\widetilde{(z,t')}, \tag{4.5c}$$

$$\mathrm{Res}_\lambda\, \lambda^{-1}\tau\widetilde{(z,t-[\lambda])}\kappa_i\widetilde{(z,t'+[\lambda])} e^{\xi(t-t',\lambda)} = \kappa_i\widetilde{(z,t)}\tau\widetilde{(z,t')}, \tag{4.5d}$$

$$\mathrm{Res}_\lambda\, \lambda^{-1}\rho_i\widetilde{(z,t-[\lambda])}\tau\widetilde{(z,t'+[\lambda])} e^{\xi(\gamma-t',\lambda)} = \rho_i\widetilde{(z,t')}\tau\widetilde{(z,t)}. \tag{4.5e}$$

Denoting $y=(y_1,y_2,\cdots)$ and setting t and t' as $t+y$ and $t-y$ respectively, we can write the equalities (4.5) as the following systems with Hirota bilinear derivatives $\tilde{D}$ and D_i's:

$$\sum_{i=0}^{\infty} p_i(2y)\, p_{i+1}\left(-\tilde{D}\right)\exp\left(\sum_{i=1}^{\infty} y_i D_i\right)\tau(T,t)\cdot\tau(T,t)=0, \tag{4.6a}$$

$$\begin{aligned}&-\sum_{i=0}^{\infty} p_i(2y)\left[\partial_{z_i}\log\tau(T,t+y)\right] p_{i+1}\left(-\tilde{D}\right)\tau(T,t+y)\cdot\tau(T,t-y)\\&+\sum_{i=0}^{\infty} p_i(2y)\, p_{i+1}\left(-\tilde{D}\right)\exp\left(\sum_{i=1}^{\infty} y_i D_i\right)\tau_{z_i}(T,t)\cdot\tau(T,t)\\&=\exp\left(\sum_{i=1}^{\infty} y_i D_i\right)\kappa_i(T,t)\cdot\rho_i(T,t)\end{aligned}, \tag{4.6b}$$

$$\begin{aligned}&\sum_{i=0}^{\infty} p_i(2y)\, p_i\left(-\tilde{D}\right)\exp\left(\sum_{i=1}^{\infty} y_i D_i\right)\tau(T,t)\cdot\kappa_i(T,t)\\&=\exp\left(\sum_{i=1}^{\infty} y_i D_i\right)\kappa_i(T,t)\cdot\tau(T,t)\end{aligned}, \tag{4.6c}$$

$$\begin{aligned}&\sum_{i=0}^{\infty} p_i(2y)\, p_i\left(-\tilde{D}\right)\exp\left(\sum_{i=1}^{\infty} y_i D_i\right)\rho_i(T,t)\cdot\tau(T,t)\\&=\exp\left(\sum_{i=1}^{\infty} y_i D_i\right)\tau(T,t)\cdot\rho_i(T,t)\end{aligned}. \tag{4.6d}$$

where $\tilde{D}=\left(D_1,\frac{1}{2}D_2,\frac{1}{3}D_3,\cdots\right)$, D_i is the Hirota bilinear derivative defined by $D_i f\cdot g=f_{t_i}g-fg_{t_i}$ and $p_i(y)$ is the $i-$th Schur polynomial given by $\exp\sum_{i=1}^{\infty} y_i\lambda^i=\sum_{i=0}^{\infty} p_i(y)\lambda^i$.

Let $y=0$, the equation (4.6b) can be converted into the following forms by using the Hirota bilinear operator

$$\begin{aligned}&\kappa_i(T,t)\rho_i(T,t)+D_x\tau_{z_i}(T,t)\cdot\tau(T,t)\\&=\kappa_i(T,t)\rho_i(T,t)+D_{z_i}\tau_x(T,t)\cdot\tau(T,t)\\&=\kappa_i(T,t)\rho_i(T,t)+\tfrac{1}{2}D_xD_{z_i}\tau(T,t)\cdot\tau(T,t)\\&=0\end{aligned}. \tag{4.7}$$

By setting $n=2$, $k=3$ and comparing the coefficient of y_3 in equation (4.6a) and the coefficient of y_2, y_3 in equation (4.6c) (4.6d), we can obtain

$$\kappa_i(z,t)\rho_i(z,t)+\frac{1}{2}D_1D_{z_i}\tau(z,t)\cdot\tau(z,t)=0, \tag{4.8a}$$

$$\left[D_1^4+3\left(D_2-\alpha_2\sum_{i=1}^{N}D_{z_i}\right)^2-4D_1\left(D_3-\beta_3\sum_{i=1}^{N}D_{z_i}\right)\right]\tau(z,t)\cdot\tau(z,t)=0, \tag{4.8b}$$

$$\left[4\left(D_3-\beta_3\sum_{i=1}^{N}D_{z_i}\right)+3D_1\left(D_2-\alpha_2\sum_{i=1}^{N}D_{z_i}\right)-D_1^3\right]\tau(z,t)\cdot\kappa_i(z,t)=0, \tag{4.8c}$$

$$\left[4\left(D_3-\beta_3\sum_{i=1}^{N}D_{z_i}\right)+3D_1\left(D_2-\alpha_2\sum_{i=1}^{N}D_{z_i}\right)-D_1^3\right]\rho_i(z,t)\cdot\tau(z,t)=0, \tag{4.8d}$$

$$\left[D_1^2+\left(D_2-\alpha_2\sum_{i=1}^{N}D_{z_i}\right)\right]\tau(z,t)\cdot\kappa_i(z,t)=0, \tag{4.8e}$$

$$\left[D_1^2+\left(D_2-\alpha_2\sum_{i=1}^{N}D_{z_i}\right)\right]\rho_i(z,t)\cdot\tau(z,t)=0,\quad i=1,2,\cdots,N. \tag{4.8f}$$

The bilinear equations (4.8) correspond to the mixed type of KP equation with self-consistent sources [5], which can be reduced to the first type or the second type of KP equation with self-consistent sources by setting $\alpha_2=0$ or $\beta_3=0$ respectively. The KP equation with self-consistent sources describes the interaction of a long wave with a short-wave packet propagating on the x,y plane at an angle to each other [13].

5. The Procedure of Getting Nonlinear Equation from Hirota's Bilinear Equation

At the beginning of this section, we recall two identities for arbitrary functions $\tau(t)$ and $\kappa(t)$, which is proved in [11]

$$\exp\left(\sum_i\delta_iD_i\right)\kappa\cdot\tau=e^{2\cosh\left(\sum_i\delta_i\partial_i\right)\log\tau}\cdot e^{\sum_i\delta_i\partial_i}(\kappa/\tau), \tag{5.1a}$$

$$\cosh\left(\sum_i\delta_iD_i\right)\tau\cdot\tau=e^{2\cosh\left(\sum_i\delta_i\partial_i\right)\log\tau}. \tag{5.1b}$$

By defining $u=\partial_x^2\log\tau(x\equiv t_1)$, $q=\frac{\kappa}{\tau}$, $r=\frac{\rho}{\tau}$, we can get [11]

$$\frac{1}{\tau^2}\sum_{n=0}^{\infty}\frac{\left(\sum_i\delta_iD_i\right)^n}{n!}\kappa\cdot\tau=\exp\left[2\sum_{n=1}^{\infty}\frac{\left(\sum_i\delta_i\partial_i\right)^{2n}}{(2n)!}\partial^{-2}u\right]\cdot e^{\sum_i\delta_i\partial_i}q, \tag{5.2a}$$

$$\frac{1}{\tau^2}\sum_{n=1}^{\infty}\frac{\left(\sum_i\delta_iD_i\right)^{2n}}{(2n)!}\tau\cdot\tau=\exp\left[2\sum_{n=1}^{\infty}\frac{\left(\sum_i\delta_i\partial_i\right)^{2n}}{(2n)!}\partial^{-2}u\right], \tag{5.2b}$$

and similarly, we have

$$\frac{1}{\tau^2}\sum_{n=0}^{\infty}\frac{\left(\sum_i \delta_i D_i\right)^n}{n!}\rho\cdot\tau = \exp\left[2\sum_{n=0}^{\infty}\frac{\left(\sum_i \delta_i \partial_i\right)^{2n}}{2n!}\partial^{-2}u\right]\cdot e^{\sum_i \delta_i\partial_i} r. \tag{5.2c}$$

Comparing the coefficient of $(\delta_i)^j, j \geq 0$, we can get the relations between the Hirota bilinear derivatives and the usual partial derivatives. Here are some of them

$$\begin{cases} \frac{D_1^4\tau\cdot\tau}{\tau^2} = 2u_{1,1} + 12u^2, \frac{D_1^2\tau\cdot\tau}{\tau^2} = 2u, \frac{D_1 D_3\tau\cdot\tau}{\tau^2} = 2\partial^{-1}u_3 \\ \frac{D_1^3\rho\cdot\tau}{\tau^2} = r_{1,1,1} + 6ur_1, \frac{D_1 D_2\rho\cdot\tau}{\tau^2} = r_{1,2} + 2r\partial^{-1}u_2 \\ \frac{D_2\rho\cdot\tau}{\tau^2} = r_2, \frac{D_3\rho\cdot\tau}{\tau^2} = r_3, \frac{D_1^2\rho\cdot\tau}{\tau^2} = r_{1,1} + 2ur \\ \frac{D_1^3\kappa\cdot\tau}{\tau^2} = q_{1,1,1} + 6uq_1, \frac{D_1 D_2\kappa\cdot\tau}{\tau^2} = q_{1,2} + 2q\partial^{-1}u_2 \\ \frac{D_2\kappa\cdot\tau}{\tau^2} = q_2, \frac{D_3\kappa\cdot\tau}{\tau^2} = q_3, \frac{D_1^2\kappa\cdot\tau}{\tau^2} = q_{1,1} + 2uq \end{cases} \tag{5.3}$$

where the subscripts $i, j, \cdots$ of $u_{i,j}, ..., r_{i,j}, ..., q_{i,j}, ...$ denote the derivatives with respect to the variables $t_i, t_j, \cdots$.

We also have the following expressions

$$\begin{cases} \frac{D_1 D_{z_i}\tau\cdot\tau}{\tau^2} = 2\partial^{-1}u_{z_i}, \frac{D_2 D_{z_i}\tau\cdot\tau}{\tau^2} = 2\partial^{-1}u_{2,z_i}, \frac{D_{z_i}^2\tau\cdot\tau}{\tau^2} = 2\partial^{-1}u_{z_i,z_i} \\ \frac{D_{z_i}\rho\cdot\tau}{\tau^2} = r_{z_i}, \frac{D_1 D_{z_i}\rho\cdot\tau}{\tau^2} = r_{1,z_i} + 2r\partial^{-1}u_{z_i} \\ \frac{D_{z_i}\kappa\cdot\tau}{\tau^2} = q_{z_i}, \frac{D_1 D_{z_i}\kappa\cdot\tau}{\tau^2} = q_{1,z_i} + 2q\partial^{-1}u_{z_i} \end{cases} \tag{5.4}$$

where the subscripts z_i $(i = 1, 2, 3, ...N)$ of u denote the derivatives with respect to the variables z_i $(i = 1, 2, 3, ..., N)$.

By using (5.3) and (5.4), we can write (4.8) as the following nonlinear partial differential equations

$$\begin{aligned} &\partial^{-1}\partial_{z_i}u + r_i q_i = 0, \\ &\left(u_{xxx} + 12uu_x + 4\beta_3\sum_{k=1}^{N}u_{z_k} - 4u_t\right)_x + 3\left(\partial_y - \alpha_2\sum_{k=1}^{N}\partial_{z_k}\right)^2 u = 0, \\ &-4q_{i,t} + 4\beta_3 q_{i,z_k} + 3q_{i,xy} + 6q_i\left(\partial_y - \alpha_2\sum_{k=1}^{N}\partial_{z_k}\right)\partial^{-1}u - 3\alpha_2\sum_{k=1}^{N}q_{i,xz_k} + q_{i,xxx} + 6uq_{i,x} = 0, \\ &4r_{i,t} - 4\beta_3 r_{i,z_k} + 3r_{i,xy} + 6r_i\left(\partial_y - \alpha_2\sum_{k=1}^{N}\partial_{z_k}\right)\partial^{-1}u - 3\alpha_2\sum_{k=1}^{N}r_{i,xz_k} - r_{i,xxx} - 6ur_{i,x} = 0, \\ &q_{i,xx} + 2uq_i - q_{i,y} + \alpha_2\sum_{k=1}^{N}q_{i,z_k} = 0, \\ &r_{i,xx} + 2ur_i + r_{i,y} - \alpha_2\sum_{k=1}^{N}r_{i,z_k} = 0, \quad i = 1, 2, 3, ..., N. \end{aligned} \tag{5.5}$$

Equations (2.14) can be obtained by eliminating the auxiliary variables z_i in (5.5). So we can see that the Hirota bilinear equations (4.8) correspond to the mixed type of KP equation with self-consistent sources (2.14).

6. Conclusions

The bilinear identities for the (γ_n, σ_k)-KP hierarchy [18] are constructed in this paper, which could be seen as the generating functions of all the Hirota's bilinear equations for the zero-curvature forms in the (γ_n, σ_k)-KP Hierarchy. Many integrable 2+1 dimensional equations with self-consistent sources are included as special cases of this hierarchy. We have shown that the Hirota's bilinear forms (4.8) correspond to the mixed type of KPESCS, which can be reduced to the first and the second type of KPESCS.

With the help of N auxiliary flows ($\partial_{z_i}-$flow), we obtain the bilinear identities of the whole (γ_n, σ_k)-KP hierarchy, which have many important applications. For example, taking the intimate relation between quasi-periodic solutions and bilinear identity into account, we investigate the quasi-periodic solutions for the (γ_n, σ_k)-KP Hierarchy. Under proper constraints, the (γ_n, σ_k)-KP hierarchy can be reduced to Gelfand-Dickey hierarchy (GDH), KdV equation, Bonssinesq equation and many other equations with self-consistent sources. So the bilinear identities of the (γ_n, σ_k)-KP hierarchy can help us learn the relation among these equations' bilinear identities. We will investigate these problems in future.

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