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In this paper, we apply Fokas unified method to study initial-boundary value problems for the two-component Gerdjikov-Ivanov equation formulated on the finite interval with 3×3 Lax pairs. The solution can be expressed in terms of the solution of a 3×3 Riemann-Hilbert problem. The relevant jump matrices are explicitly given in terms of three matrix-value spectral functions $s(\lambda)$, $S(\lambda)$ and $S_L(\lambda)$, which arising from the initial values at $t = 0$, boundary values at $x = 0$ and boundary values at $x = L$, respectively. Moreover, The associated Dirichlet to Neumann map is analyzed via the global relation. The relevant formulae for boundary value problems on the finite interval can reduce to ones on the half-line as the length of the interval tends to infinity.

Keywords: Two-component Gerdjikov-Ivanov equation, initial-boundary value problem, Fokas unified method, Riemann-Hilbert problem.

1. Introduction

The Gerdjikov-Ivanov (GI) equation takes in the form [10]

$$q_t = iq_{xx} + q^2 \bar{q}_x + \frac{i}{2} q^3 \bar{q}^2. \tag{1.1}$$

In these years, there has been much work on the GI equation, including Hamiltonian structures [2], Darboux transformation [3], rouge wave and breather soliton [19], algebro-geometric solutions [11],

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envelope bright and dark soliton solution [17]. Recently, Zhang,Cheng and He obtained the N-soliton solutions with Riemann-Hilbert method about the two-component (2-GI) equation [22]

$$
\begin{cases}
q_{1t} = iq_{1xx} + q_1(q_1\bar{q}_{1x} + q_2\bar{q}_{2x}) + \frac{i}{2}q_1(|q_1|^4 + |q_2|^4) + i|q_1q_2|^2q_1, \\
q_{2t} = iq_{2xx} + q_2(q_1\bar{q}_{1x} + q_2\bar{q}_{2x}) + \frac{i}{2}q_2(|q_1|^4 + |q_2|^4) + i|q_1q_2|^2q_2.
\end{cases}
\tag{1.2}
$$

In 1997, Fokas announced the unified transform for the analysis of initial boundary value (IBV) problems for linear and nonlinear integrable PDEs [4]. The Fokas method was usually used to analyze the IBV problem for integrable PDEs with 2×2 Lax pair on the half-line and the finite interval, such as nonlinear Schröding equation [5,6], sine-Gordon equation [7,15], KdV equation [8], mKdV equation [1,9], derivative nonlinear Schröding equation [12]. In 2012, Lenells extended this method to the IBV problem of integrable systems with 3×3 Lax pair on the half-line [13]. After that, several important integrable equations with 3×3 Lax pair have been investigated, including Degasperis-Procesi [14], Sasa-satuma [18]. However, there has been still less work on the IBV problems on the finite interval of integrable equations with 3×3 Lax pair except to the two-component NLS [20], general coupled NLS [16] and the integrable spin-1 Gross-Pitaevskii [21] equations.

In this paper, we apply Fokas method to consider 2-GI equation with the following initial boundary value data:

$$
\begin{array}{llll}
\text{Initial value:} & q_1(x,t=0) = q_{10}(x), & q_2(x,t=0) = q_{20}(x), \\
\text{Dirichlet boundary value:} & q_1(x=0,t) = g_{01}(t), & q_2(x=0,t) = g_{02}(t), \\
& q_1(x=L,t) = f_{01}(t), & q_2(x=L,t) = f_{02}(t), \\
\text{Neumann boundary value:} & q_{1x}(x=0,t) = g_{11}(t), & q_{2x}(x=0,t) = g_{12}(t), \\
& q_{1x}(x=L,t) = f11(t), & q_{2x}(x=L,t) = f_{12}(t),
\end{array}
\tag{1.3}
$$

where $q_1(x,t)$ and $q_2(x,t)$ are complex-valued functions of $(x,t) \in \Omega$, and Ω denotes the finite interval domain

$$
\Omega = \{(x,t)|0 \le x \le L, 0 \le t \le T\},
$$

here $L > 0$ is a positive fixed constant and $T > 0$ being a fixed final time.

Comparing with two-component NLS equation [20], the IBV problem of the 2-GI equation (1.2) also presents some distinctive features in the use of Fokas method: (i) The order of spectral variable k in the Lax pair (2.1) is higher than that of 2-NLS equation. In order to make the results on the interval reduce to the ones on the half-line, we should first introduce transformation $\psi(x,t,k) = k^{\frac{1}{2}\Lambda}\phi(x,t,k)k^{-\frac{1}{2}\Lambda}$ so that the Lax pairs are even functions of k. (ii) The 2-GI equation admits a generalized Wadati-Konno-Ichikawa (WKI) type Lax pair, which admits a gauge transformation to AKNS-type Lax pair, but this gauge transformation can not be used to analyze the IBV problem by mapping it into 2-NLS equation. We need to introduce a matrix-value function $G(x,t)$ to transform the WKI-type Lax pair into AKNS-type Lax pair.

Organization of this paper is as follows. In the following section 2, we perform the spectral analysis of the associated Lax pair for the 2-GI equation (1.2). In the section 3, we give the corresponding matrix RH problem associated with the IBV problem of 2-GI equation. In section 4, we get the map between the Dirichlet and the Neumann boundary problem through analysising the global relation. Especially, the relevant formulae for boundary value problems on the finite interval can reduce to ones on the half-line as the length of the interval approaches to infinity.

2. Spectral analysis

2.1. *Lax pair*

The 2-GI equation admits a 3×3 Lax pair [22]

$$\psi_x + ik^2 \Lambda \psi = U_1 \psi, \tag{2.1a}$$

$$\psi_t + 2ik^4 \Lambda \psi = U_2 \psi, \tag{2.1b}$$

where $\psi(x,t,k)$ is a $3 \times 3-$matrix valued eigenfunction, $k \in \mathbb{C}$ is the spectral parameter, and $U_1(x,t), U_2(x,t)$ are $3 \times 3-$matrix valued functions given by

$$U_1 = -kQ\Lambda + \frac{i}{2}Q^2\Lambda, \qquad U_2 = -2k^3 Q\Lambda + ik^2 \Lambda Q^2 + ikQ_x - \frac{1}{2}[Q_x, Q] + \frac{i}{4}Q^4\Lambda, \tag{2.2}$$

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & q_1 & q_2 \\ \bar{q}_1 & 0 & 0 \\ \bar{q}_2 & 0 & 0 \end{pmatrix}. \tag{2.3}$$

There are both odd power and even power of k in the Lax pair (2.1), to make (2.1) are even functions of k for analyzing the large L limit, we introduce a transformation

$$\psi(x,t,k) = k^{\frac{1}{2}\Lambda} \phi(x,t,k) k^{-\frac{1}{2}\Lambda}, \tag{2.4}$$

and get an equivalent Lax pair

$$\phi_x + ik^2 \Lambda \phi = \tilde{U}_1 \phi, \tag{2.5a}$$

$$\phi_t + 2ik^4 \Lambda \phi = \tilde{U}_2 \phi, \tag{2.5b}$$

where

$$\begin{aligned} \tilde{U}_1 &= -Q_1\Lambda - k^2 Q_2\Lambda + \frac{i}{2}Q^2\Lambda, \\ \tilde{U}_2 &= -2k^4 Q_2\Lambda + k^2(i\Lambda Q^2 + iQ_{2x} - 2Q_1\Lambda) + (iQ_{1x} - \frac{1}{2}[Q_x, Q] + \frac{i}{4}Q^4\Lambda), \end{aligned} \tag{2.6}$$

$$Q_1 = \begin{pmatrix} 0 & q_1 & q_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ \bar{q}_1 & 0 & 0 \\ \bar{q}_2 & 0 & 0 \end{pmatrix}, \qquad Q = Q_1 + Q_2. \tag{2.7}$$

Let $\lambda = k^2$, Lax pair (2.5) becomes

$$\phi_x + i\lambda \Lambda \phi = \tilde{U}_1 \phi, \tag{2.8a}$$

$$\phi_t + 2i\lambda^2 \Lambda \phi = \tilde{U}_2 \phi, \tag{2.8b}$$

where $\tilde{U}_1, \tilde{U}_2$ are given by (2.6) with k^2 replaced with λ.

2.2. *The closed one-form*

Defining a 3×3 matrix-value function

$$G(x,t) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2i}\bar{q}_1 & 1 & 0 \\ \frac{1}{2i}\bar{q}_2 & 0 & 1 \end{pmatrix}, \qquad (2.9)$$

and making a transformation

$$\phi(x,t,k) = G(x,t)\mu(x,t,k)e^{-i\lambda\Lambda x - 2i\lambda^2\Lambda t}, \qquad (2.10)$$

then we get a new Lax pair for $\mu(x,t,\lambda)$

$$\mu_x + i\lambda[\Lambda,\mu] = V_1\mu, \qquad (2.11a)$$

$$\mu_t + 2i\lambda^2[\Lambda,\mu] = V_2\mu, \qquad (2.11b)$$

where

$$V_1 = G^{-1}(-Q_1\Lambda + \frac{i}{2}Q^2\Lambda)G - G^{-1}G_x, \qquad (2.12a)$$

$$V_2 = \lambda G^{-1}(i\Lambda Q^2 + iQ_{2x} - 2Q_1\Lambda)G + G^{-1}(iQ_{1x} - \frac{1}{2}[Q_x,Q] + \frac{i}{4}Q^4\Lambda)G - G^{-1}G_t. \qquad (2.12b)$$

Letting $\hat{A}$ denotes the operators which acts on a 3×3 matrix X by $\hat{A}X = [A,X]$, then the equations (2.11) can be rewritten in a differential form

$$d(e^{(i\lambda x + 2i\lambda^2 t)\hat{\Lambda}}\mu) = W, \qquad (2.13)$$

where the closed one-form $W(x,t,k)$ is defined by

$$W = e^{(i\lambda x + 2i\lambda^2 t)\hat{\Lambda}}(V_1 dx + V_2 dt)\mu. \qquad (2.14)$$

2.3. *The eigenfunctions μ_j's*

We define four eigenfunctions $\{\mu_j\}_1^4$ of (2.11) by the Volterra integral equations

$$\mu_j(x,t,k) = \mathbb{I} + \int_{\gamma_j} e^{-(i\lambda x + 2i\lambda^2 t)\hat{\Lambda}}W_j(x',t',k). \qquad j = 1,2,3,4. \qquad (2.15)$$

where W_j is given by (2.14) with μ replaced by μ_j, and the contours $\{\gamma_j\}_1^4$ can be given by the following inequalities (see Figure 1):

$$\begin{aligned} \gamma_1 &: x - x' \geq 0, t - t' \leq 0, \\ \gamma_2 &: x - x' \geq 0, t - t' \geq 0, \\ \gamma_3 &: x - x' \leq 0, t - t' \geq 0, \\ \gamma_4 &: x - x' \leq 0, t - t' \leq 0. \end{aligned} \qquad (2.16)$$
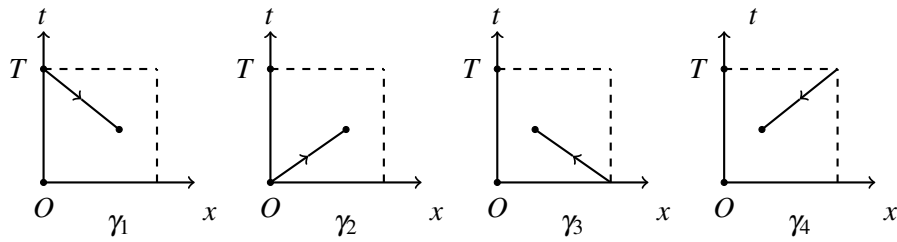
Fig. 1. The four contours $\gamma_1, \gamma_2, \gamma_3$ and γ_4 in the $(x,t)-$domain.
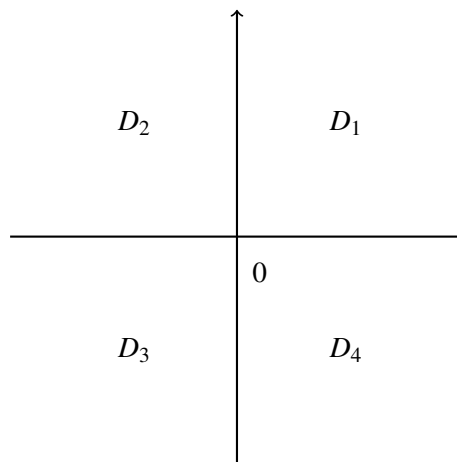

Fig. 2. The sets $D_n, n = 1, \ldots, 4$, which decompose the complex λ-plane.

and the matrix equation (2.15) involves the exponentials

$$
\begin{aligned}
&[\mu_j]_1:\ e^{2i\lambda(x-x')+4i\lambda^2(t-t')}, e^{2i\lambda(x-x')+4i\lambda^2(t-t')}\\
&[\mu_j]_2:\ e^{-2i\lambda(x-x')-4i\lambda^2(t-t')},\\
&[\mu_j]_3:\ e^{-2i\lambda(x-x')-4i\lambda^2(t-t')}.
\end{aligned}
\tag{2.17}
$$

from which, we find that the functions $\{\mu_j\}_1^4$ are bounded and analytic for $\lambda \in \mathbb{C}$ such that λ belongs to

$$
\begin{aligned}
&\mu_1 : (D_2, D_3, D_3),\\
&\mu_2 : (D_1, D_4, D_4),\\
&\mu_3 : (D_3, D_2, D_2),\\
&\mu_4 : (D_4, D_1, D_1),
\end{aligned}
\tag{2.18}
$$

where $\{D_n\}_1^4$ denote four open, pairwisely disjoint subsets of the complex $\lambda-$plane showed in Figure 2.

And the sets $\{D_n\}_1^4$ admit the following properties:

$$
\begin{aligned}
D_1 &= \{k \in \mathbb{C} | \mathrm{Re}l_1 > \mathrm{Re}l_2 = \mathrm{Re}l_3, \mathrm{Re}z_1 > \mathrm{Re}z_2 = \mathrm{Re}z_3\}, \\
D_2 &= \{k \in \mathbb{C} | \mathrm{Re}l_1 > \mathrm{Re}l_2 = \mathrm{Re}l_3, \mathrm{Re}z_1 < \mathrm{Re}z_2 = \mathrm{Re}z_3\}, \\
D_3 &= \{k \in \mathbb{C} | \mathrm{Re}l_1 < \mathrm{Re}l_2 = \mathrm{Re}l_3, \mathrm{Re}z_1 > \mathrm{Re}z_2 = \mathrm{Re}z_3\}, \\
D_4 &= \{k \in \mathbb{C} | \mathrm{Re}l_1 < \mathrm{Re}l_2 = \mathrm{Re}l_3, \mathrm{Re}z_1 < \mathrm{Re}z_2 = \mathrm{Re}z_3\},
\end{aligned}
$$

where $l_i(\lambda)$ and $z_i(\lambda)$ are the diagonal entries of matrices $i\lambda\Lambda$ and $2i\lambda^2\Lambda$, respectively.

2.4. *The spectral functions M_n's*

For each $n = 1, \ldots, 4$, a solution $M_n(x, t, \lambda)$ of (2.11) can be defined by the following system of integral equations:

$$
(M_n)_{ij}(x, t, \lambda) = \delta_{ij} + \int_{\gamma_{ij}^n} \left(e^{-(i\lambda x + 2i\lambda^2 t)\hat{\Lambda}} W_n(x', t', \lambda) \right)_{ij}, \quad \lambda \in D_n, \quad i, j = 1, 2, 3. \tag{2.19}
$$

where W_n is given by (2.14) with μ replaced with M_n, and the contours γ_{ij}^n, $n = 1, \ldots, 4$, $i, j = 1, 2, 3$ are defined by

$$
\gamma_{ij}^n = \begin{cases} \gamma_1 \text{ if } \mathrm{Re}l_i(\lambda) < \mathrm{Re}l_j(\lambda) \text{ and } \mathrm{Re}z_i(\lambda) \geq \mathrm{Re}z_j(\lambda), \\ \gamma_2 \text{ if } \mathrm{Re}l_i(\lambda) < \mathrm{Re}l_j(\lambda) \text{ and } \mathrm{Re}z_i(\lambda) < \mathrm{Re}z_j(\lambda), \\ \gamma_3 \text{ if } \mathrm{Re}l_i(\lambda) \geq \mathrm{Re}l_j(\lambda) \text{ and } \mathrm{Re}z_i(\lambda) \leq \mathrm{Re}z_j(\lambda), \\ \gamma_4 \text{ if } \mathrm{Re}l_i(\lambda) \geq \mathrm{Re}l_j(\lambda) \text{ and } \mathrm{Re}z_i(\lambda) \geq \mathrm{Re}z_j(\lambda). \end{cases} \quad \text{for} \quad \lambda \in D_n. \tag{2.20}
$$

Here, we make a distinction between the contours γ_3 and γ_4 as follows,

$$
\gamma_{ij}^n = \begin{cases} \gamma_3, \text{ if } \prod_{1 \leq i < j \leq 3} (\mathrm{Re}l_i(\lambda) - \mathrm{Re}l_j(\lambda))(\mathrm{Re}z_i(\lambda) - \mathrm{Re}z_j(\lambda)) < 0, \\ \gamma_4, \text{ if } \prod_{1 \leq i < j \leq 3} (\mathrm{Re}l_i(\lambda) - \mathrm{Re}l_j(\lambda))(\mathrm{Re}z_i(\lambda) - \mathrm{Re}z_j(\lambda)) > 0. \end{cases} \tag{2.21}
$$

The rule chosen in the produce is if $l_m = l_n$, m may not equals n, we just choose the subscript is smaller one.

According to the definition of the γ^n, one find that

$$
\gamma^1 = \begin{pmatrix} \gamma_4 & \gamma_4 & \gamma_4 \\ \gamma_2 & \gamma_4 & \gamma_4 \\ \gamma_2 & \gamma_4 & \gamma_4 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_1 & \gamma_3 & \gamma_3 \\ \gamma_1 & \gamma_3 & \gamma_3 \end{pmatrix}
$$

$$
\gamma^3 = \begin{pmatrix} \gamma_3 & \gamma_1 & \gamma_1 \\ \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_3 & \gamma_3 & \gamma_3 \end{pmatrix} \quad \gamma^4 = \begin{pmatrix} \gamma_4 & \gamma_2 & \gamma_2 \\ \gamma_4 & \gamma_4 & \gamma_4 \\ \gamma_4 & \gamma_4 & \gamma_4 \end{pmatrix}. \tag{2.22}
$$

The following proposition ascertains that the M_n's defined in this way have the properties required for the formulation of a Riemann-Hilbert problem.

Proposition 2.1. *For each $n = 1, \ldots, 4$, the function $M_n(x, t, \lambda)$ is well-defined by equation (2.19) for $\lambda \in \bar{D}_n$ and $(x, t) \in \Omega$. Moreover, M_n admits a bounded and contious extension to $\bar{D}_n$ and*

$$
M_n(x, t, \lambda) = \mathbb{I} + O(\frac{1}{\lambda}), \qquad \lambda \to \infty, \quad \lambda \in D_n. \tag{2.23}
$$

Proof. Analogous to the proof provided in [13] □

Remark 2.1. Of course, for any fixed point (x,t), M_n is bounded and analytic as a function of $k \in D_n$ away from a possible discrete set of singularities $\{k_j\}$ at which the Fredholm determinant vanishes. The bounedness and analyticity properties are established in appendix B in [13].

2.5. *The jump matrices*

The spectral functions $\{S_n(\lambda)\}_1^4$ can be defined by

$$S_n(\lambda) = M_n(0,0,\lambda), \qquad \lambda \in D_n, \quad n = 1,\dots,4. \tag{2.24}$$

Let M denote the sectionally analytic function on the Riemann λ−plane which equals M_n for $\lambda \in D_n$. Then M satisfies the jump conditions

$$M_n = M_m J_{m,n}, \qquad k \in \bar{D}_n \cap \bar{D}_m, \qquad n,m = 1,\dots,4, \quad n \neq m, \tag{2.25}$$

where the jump matrices $J_{m,n}(x,t,\lambda)$ are given by

$$J_{m,n} = e^{-(i\lambda x + 2i\lambda^2 t)\hat{\Lambda}}(S_m^{-1} S_n). \tag{2.26}$$

2.6. *The adjugated eigenfunctions*

As the expressions of $S_n(\lambda)$ will involve the adjugate matrix of $\{s(\lambda), S(\lambda), S_L(\lambda)\}$ defined in the next subsection. We will also need the analyticity and boundedness of the the matrices $\{\mu_j(x,t,\lambda)\}_1^4$. We recall that the adjugate matrix X^A of a 3×3 matrix X is defined by

$$X^A = \begin{pmatrix} m_{11}(X) & -m_{12}(X) & m_{13}(X) \\ -m_{21}(X) & m_{22}(X) & -m_{23}(X) \\ m_{31}(X) & -m_{32}(X) & m_{33}(X) \end{pmatrix},$$

where $m_{ij}(X)$ denote the (ij)th minor of X.

It follows from (2.11) that the adjugated eigenfunction μ^A satisfies the Lax pair

$$\begin{cases} \mu_x^A - i\lambda [\Lambda, \mu^A] = -V_1^T \mu^A, \\ \mu_t^A - 2i\lambda^2 [\Lambda, \mu^A] = -V_2^T \mu^A. \end{cases} \tag{2.27}$$

where V^T denotes the transform of a matrix V. Thus, the eigenfunctions $\{\mu_j^A\}_1^4$ are solutions of the integral equations

$$\mu_j^A(x,t,\lambda) = \mathbb{I} - \int_{\gamma_j} e^{i\lambda(x-x')+2i\lambda^2(t-t')\hat{\Lambda}}(V_1^T dx + V_2^T dt)\mu_j^A, \quad j = 1,2,3. \tag{2.28}$$

Then we can get the following analyticity and boundedness properties:

$$\begin{aligned} &\mu_1^A : (D_3, D_2, D_2), \\ &\mu_2^A : (D_4, D_1, D_1), \\ &\mu_3^A : (D_2, D_3, D_3), \\ &\mu_4^A : (D_1, D_4, D_4). \end{aligned} \tag{2.29}$$

2.7. *Symmetries*

We will show that the eigenfunctions $\mu_j(x,t,k)$ satisfy an important symmetry.

Proposition 2.2. *The eigenfunction $\psi(x,t,k)$ of the Lax pair (2.1) satisfies the following symmetry:*

$$\psi^{-1}(x,t,k) = \overline{\psi(x,t,\bar{k})}^T = -\Lambda\psi(x,t,-k)\Lambda, \tag{2.30}$$

here the superscript T denotes a matrix transpose.

Proof. The matrices $U(x,t,k)$ and $V(x,t,k)$ in the Lax pair (2.1) written in the form

$$\psi_x = U\psi, \qquad \psi_t = V\psi,$$

satisfy the following symmetry relations

$$U(x,t,k)^T = -\overline{U(x,t,\bar{k})}, \qquad V(x,t,k)^T = -\overline{V(x,t,\bar{k})}, \tag{2.31}$$

and

$$U(x,t,k) = \Lambda U(x,t,-k)\Lambda, \qquad V(x,t,k) = \Lambda V(x,t,-k)\Lambda. \tag{2.32}$$

In turn, relations (2.31) and (2.32) imply

$$\psi_x^A(x,t,k) = \overline{U(x,t,\bar{k})}\,\psi^A(x,t,k), \qquad \psi_t^A(x,t,k) = \overline{V(x,t,\bar{k})}\,\psi^A(x,t,k), \tag{2.33}$$

and

$$\psi_x^A(x,t,k) = -\Lambda U^T(x,t,-k)\Lambda\psi^A(x,t,k), \qquad \psi_t^A(x,t,k) = -\Lambda V(x,t,-k)\Lambda\psi^A(x,t,k). \tag{2.34}$$

$$\square$$

Remark 2.2. From proposition 2.3, one can show that the eigenfunctions $\mu_j(x,t,\lambda)$ of Lax pair equations (2.11) satisfy the same symmetry.

2.8. *The $J_{m,n}$'s computation*

Let us define the $3 \times 3-$matrix value spectral functions $s(\lambda)$, $S(\lambda)$ and $S_L(\lambda)$ by

$$\mu_3(x,t,\lambda) = \mu_2(x,t,\lambda)e^{-(i\lambda x + 2i\lambda^2 t)\hat{\Lambda}}s(\lambda), \tag{2.35a}$$

$$\mu_1(x,t,\lambda) = \mu_2(x,t,\lambda)e^{-(i\lambda x + 2i\lambda^2 t)\hat{\Lambda}}S(\lambda), \tag{2.35b}$$

$$\mu_4(x,t,\lambda) = \mu_3(x,t,\lambda)e^{-(i\lambda(x-L) + 2i\lambda^2 t)\hat{\Lambda}}S_L(\lambda), \tag{2.35c}$$

Thus,

$$s(\lambda) = \mu_3(0,0,\lambda), \tag{2.36a}$$

$$S(\lambda) = \mu_1(0,0,\lambda) = e^{2i\lambda^2 T\hat{\Lambda}}\mu_2^{-1}(0,T,\lambda), \tag{2.36b}$$

$$S_L(\lambda) = \mu_4(L,0,\lambda) = e^{2i\lambda^2 T\hat{\Lambda}}\mu_3^{-1}(L,T,\lambda), \tag{2.36c}$$

And we can deduce from the properties of μ_j and μ_j^A that $\{s(\lambda), S(\lambda), S_L(\lambda)\}$ and $\{s^A(\lambda), S^A(\lambda), S_L^A(\lambda)\}$ have the following boundedness properties:

$$
\begin{aligned}
s(\lambda) &: \quad (D_3 \cup D_4, D_1 \cup D_2, D_1 \cup D_2), \\
S(\lambda) &: \quad (D_2 \cup D_4, D_1 \cup D_3, D_1 \cup D_3), \\
S_L(\lambda) &: (D_2 \cup D_4, D_1 \cup D_3, D_1 \cup D_3), \\
s^A(\lambda) &: (D_1 \cup D_2, D_3 \cup D_4, D_3 \cup D_4), \\
S^A(\lambda) &: (D_1 \cup D_3, D_2 \cup D_4, D_2 \cup D_4), \\
S_L^A(\lambda) &: (D_1 \cup D_3, D_2 \cup D_4, D_2 \cup D_4).
\end{aligned}
$$

Moreover,

$$
M_n(x,t,\lambda) = \mu_2(x,t,\lambda) e^{-(i\lambda x + 2i\lambda^2 t)\hat{\Lambda}} S_n(\lambda), \quad \lambda \in D_n. \tag{2.37}
$$

Proposition 2.3. *The S_n can be expressed in terms of the entries of $s(\lambda), S(\lambda)$ and $S_L(\lambda)$ as follows:*

$$
S_1 = \begin{pmatrix} \frac{1}{m_{11}(\mathscr{A})} & \mathscr{A}_{12} & \mathscr{A}_{13} \\ 0 & \mathscr{A}_{22} & \mathscr{A}_{23} \\ 0 & \mathscr{A}_{32} & \mathscr{A}_{33} \end{pmatrix}, \; S_2 = \begin{pmatrix} \frac{S_{11}}{(S^T s^A)_{11}} & s_{12} & s_{13} \\ \frac{S_{21}}{(S^T s^A)_{11}} & s_{22} & s_{23} \\ \frac{S_{31}}{(S^T s^A)_{11}} & s_{32} & s_{33} \end{pmatrix}, \tag{2.38a}
$$

$$
S_3 = \begin{pmatrix} s_{11} & \frac{m_{33}(s)m_{21}(S) - m_{23}(s)m_{31}(S)}{(s^T S^A)_{11}} & \frac{m_{32}(s)m_{21}(S) - m_{22}(s)m_{31}(S)}{(s^T S^A)_{11}} \\ s_{21} & \frac{m_{33}(s)m_{11}(S) - m_{13}(s)m_{31}(S)}{(s^T S^A)_{11}} & \frac{m_{32}(s)m_{11}(S) - m_{12}(s)m_{31}(S)}{(s^T S^A)_{11}} \\ s_{31} & \frac{m_{23}(s)m_{11}(S) - m_{13}(s)m_{21}(S)}{(s^T S^A)_{11}} & \frac{m_{22}(s)m_{11}(S) - m_{12}(s)m_{21}(S)}{(s^T S^A)_{11}} \end{pmatrix},
$$
$$
S_4 = \begin{pmatrix} \mathscr{A}_{11} & 0 & 0 \\ \mathscr{A}_{21} & \frac{m_{33}(\mathscr{A})}{\mathscr{A}_{11}} & \frac{m_{32}(\mathscr{A})}{\mathscr{A}_{11}} \\ \mathscr{A}_{31} & \frac{m_{23}(\mathscr{A})}{\mathscr{A}_{11}} & \frac{m_{22}(\mathscr{A})}{\mathscr{A}_{11}} \end{pmatrix}. \tag{2.38b}
$$

where $\mathscr{A} = (\mathscr{A}_{ij})_{i,j=1}^3$ is a 3×3 matrix, which is defined as $\mathscr{A} = s(\lambda) e^{-i\lambda L \hat{\Lambda}} S_L(\lambda)$. And the functions

$$
(S^T s^A)_{11} = S_{11} m_{11}(s) - S_{21} m_{21}(s) + S_{31} m_{31}(s),
$$

$$
(s^T S^A)_{11} = s_{11} m_{11}(S) - s_{21} m_{21}(S) + s_{31} m_{31}(S).
$$

Proof. Firstly, we define $R_n(\lambda), T_n(\lambda)$ and $Q_n(\lambda)$ as follows:

$$
R_n(\lambda) = e^{2i\lambda^2 T \hat{\Lambda}} M_n(0, T, \lambda), \tag{2.39a}
$$

$$
T_n(\lambda) = e^{i\lambda L \hat{\Lambda}} M_n(L, 0, \lambda), \tag{2.39b}
$$

$$
Q_n(\lambda) = e^{(i\lambda L + 2i\lambda^2 T)\hat{\Lambda}} M_n(L, T, \lambda). \tag{2.39c}
$$

Then, we have the following relations:

$$
\begin{cases} M_n(x,t,\lambda) = \mu_1(x,t,\lambda) e^{(i\lambda x + 2i\lambda^2 t)\hat{\Lambda}} R_n(\lambda), \\ M_n(x,t,\lambda) = \mu_2(x,t,\lambda) e^{(i\lambda x + 2i\lambda^2 t)\hat{\Lambda}} S_n(\lambda), \\ M_n(x,t,\lambda) = \mu_3(x,t,\lambda) e^{(i\lambda x + 2i\lambda^2 t)\hat{\Lambda}} T_n(\lambda), \\ M_n(x,t,\lambda) = \mu_4(x,t,\lambda) e^{(i\lambda x + 2i\lambda^2 t)\hat{\Lambda}} Q_n(\lambda). \end{cases} \tag{2.40}
$$

The relations (2.40) imply that

$$
\begin{aligned}
s(\lambda) &= S_n(\lambda) T_n^{-1}(\lambda), \\
S(\lambda) &= S_n(\lambda) R_n^{-1}(\lambda), \\
\mathscr{A}(\lambda) &= S_n(\lambda) Q_n^{-1}(\lambda).
\end{aligned}
\tag{2.41}
$$

These equations constitute a matrix factorization problem which, given $\{s(\lambda), S(\lambda), S_L(\lambda)\}$ can be solved for the $\{R_n, S_n, T_n, Q_n\}$. Indeed, the integral equations (2.19) together with the definitions of $\{R_n, S_n, T_n, Q_n\}$ imply that

$$
\begin{cases}
(R_n(\lambda))_{ij} = 0 & if \ \gamma_{ij}^n = \gamma_1, \\
(S_n(\lambda))_{ij} = 0 & if \ \gamma_{ij}^n = \gamma_2, \\
(T_n(\lambda))_{ij} = \delta_{ij} & if \ \gamma_{ij}^n = \gamma_3, \\
(Q_n(\lambda))_{ij} = \delta_{ij} & if \ \gamma_{ij}^n = \gamma_4.
\end{cases}
\tag{2.42}
$$

It follows that (2.41) are 27 scalar equations for 27 unknowns. By computing the explicit solution of this algebraic system, we arrive at (2.38). $\qquad\square$

Remark 2.3. Due to our symmetry, see Lemma 2.30, the representation of the functions $S_n(\lambda)$ can be become simple. It leads to much more simple to compute the jump matrices $J_{m,n}(x,t,\lambda)$.

2.9. *The residue conditions*

Since μ_2 is an entire function, it follows from (2.37) that M can only have sigularities at the points where the $S_n's$ have singularities. We denote the possible zeros by $\{\lambda_j\}_1^N$ and assume they satisfy the following assumption. We assume that

- $m_{11}(\mathscr{A})(\lambda)$ has n_0 possible simple zeros in D_1 denoted by $\{\lambda_j\}_1^{n_0}$;
- $(S^T s^A)_{11}(k)$ has $n_1 - n_0$ possible simple zeros in D_2 denoted by $\{\lambda_j\}_{n_0+1}^{n_1}$;
- $(s^T S^A)_{11}(k)$ has $n_2 - n_1$ possible simple zeros in D_3 denoted by $\{\lambda_j\}_{n_1+1}^{n_2}$;
- $\mathscr{A}_{11}(k)$ has $N - n_2$ possible simple zeros in D_4 denoted by $\{\lambda_j\}_{n_2+1}^{N}$;

and that none of these zeros coincide. Moreover, we assume that none of these functions have zeros on the boundaries of the D_n's. We determine the residue conditions at these zeros in the following:

Proposition 2.4. *Let $\{M_n\}_1^4$ be the eigenfunctions defined by (2.19) and assume that the set $\{\lambda_j\}_1^N$ of singularities are as the above assumption. Then the following residue conditions hold:*

$$
Res_{\lambda=\lambda_j}[M]_1 = \frac{\mathscr{A}_{33}(\lambda_j)[M(\lambda_j)]_2 - \mathscr{A}_{23}(\lambda_j)[M(\lambda_j)]_3}{\dot{m}_{11}(\mathscr{A})(\lambda_j) m_{21}(\mathscr{A})(\lambda_j)} e^{2\theta(\lambda_j)}, \quad 1 \leq j \leq n_0, \lambda_j \in D_1
\tag{2.43a}
$$

$$
\begin{aligned}
Res_{\lambda=\lambda_j}[M]_1 &= \frac{S_{21}(\lambda_j) s_{33}(\lambda_j) - S_{31}(\lambda_j) s_{23}(\lambda_j)}{(S^T s^A)_{11}(\lambda_j) m_{11}(\lambda_j)} e^{2\theta(\lambda_j)} [M(\lambda_j)]_2 \\
&+ \frac{S_{31}(\lambda_j) s_{22}(\lambda_j) - S_{21}(\lambda_j) s_{32}(\lambda_j)}{(S^T s^A)_{11}(\lambda_j) m_{11}(\lambda_j)} e^{2\theta(\lambda_j)} [M(\lambda_j)]_3 \\
&\qquad n_0 + 1 \leq j \leq n_1, \lambda_j \in D_2,
\end{aligned}
\tag{2.43b}
$$

$$
\begin{aligned}
Res_{\lambda=\lambda_j}[M]_2 &= \frac{m_{33}(s)(\lambda_j) m_{21}(S)(\lambda_j) - m_{23}(s)(\lambda_j) m_{31}(S)(\lambda_j)}{(s^T S^A)_{11}(\lambda_j) s_{11}(\lambda_j)} e^{-2\theta(\lambda_j)} [M(\lambda_j)]_1 \\
&\qquad n_1 + 1 \leq j \leq n_2, \lambda_j \in D_3,
\end{aligned}
\tag{2.43c}
$$

$$Res_{\lambda=\lambda_j}[M]_3 = \frac{m_{32}(s)(\lambda_j)m_{21}(S)(\lambda_j)-m_{22}(s)(\lambda_j)m_{31}(S)(\lambda_j)}{(s^TS^A)_{11}(\lambda_j)s_{11}(\lambda_j)}e^{-2\theta(\lambda_j)}[M(\lambda_j)]_1 \tag{2.43d}$$
$$n_1+1 \le j \le n_2, \lambda_j \in D_3.$$

$$Res_{\lambda=\lambda_j}[M]_2 = \frac{m_{33}(\mathscr{A})(\lambda_j)}{\dot{\mathscr{A}}_{11}(\lambda_j)\mathscr{A}_{21}(\lambda_j)}e^{-2\theta(\lambda_j)}[M(\lambda_j)]_1, \quad n_2+1 \le j \le N, \lambda_j \in D_4. \tag{2.43e}$$

$$Res_{\lambda=\lambda_j}[M]_3 = \frac{m_{22}(\mathscr{A})(\lambda_j)}{\dot{\mathscr{A}}_{11}(\lambda_j)\mathscr{A}_{21}(\lambda_j)}e^{-2\theta(\lambda_j)}[M(\lambda_j)]_1, \quad n_2+1 \le j \le N, \lambda_j \in D_4. \tag{2.43f}$$

where $\dot{f} = \frac{df}{d\lambda}$, and θ is defined by

$$\theta(x,t,\lambda) = i\lambda x + 2i\lambda^2 t. \tag{2.44}$$

Proof. We will prove (2.43a), (2.43c), the other conditions follow by similar arguments. Equation (2.37) implies the relation

$$M_1 = \mu_2 e^{(i\lambda x+2i\lambda^2 t)\hat{\Lambda}}S_1, \tag{2.45a}$$

$$M_3 = \mu_2 e^{(i\lambda x+2i\lambda^2 t)\hat{\Lambda}}S_3, \tag{2.45b}$$

In view of the expressions for S_1 and S_3 given in (2.38), the three columns of (2.45a) read:

$$[M_1]_1 = [\mu_2]_1 \frac{1}{m_{11}(\mathscr{A})}, \tag{2.46a}$$

$$[M_1]_2 = [\mu_2]_1 e^{-2\theta}\mathscr{A}_{12} + [\mu_2]_2\mathscr{A}_{22} + [\mu_2]_3\mathscr{A}_{32}, \tag{2.46b}$$

$$[M_1]_3 = [\mu_2]_1 e^{-2\theta}\mathscr{A}_{13} + [\mu_2]_2\mathscr{A}_{23} + [\mu_2]_3\mathscr{A}_{33}. \tag{2.46c}$$

while the three columns of (2.45b) read:

$$[M_3]_1 = [\mu_2]_1 s_{11} + [\mu_2]_2 s_{21} e^{2\theta} + [\mu_2]_3 s_{31} e^{2\theta} \tag{2.47a}$$

$$[M_3]_2 = [\mu_2]_1 \frac{m_{33}(s)m_{21}(S)-m_{23}(s)m_{31}(S)}{(s^TS^A)_{11}}e^{-2\theta} \\ +[\mu_2]_2\frac{m_{33}(s)m_{11}(S)-m_{13}(s)m_{31}(S)}{(s^TS^A)_{11}} \\ +[\mu_2]_3\frac{m_{23}(s)m_{11}(S)-m_{13}(s)m_{21}(S)}{(s^TS^A)_{11}} \tag{2.47b}$$

$$[M_3]_3 = [\mu_2]_1 \frac{m_{32}(s)m_{21}(S)-m_{22}(s)m_{31}(S)}{(s^TS^A)_{11}}e^{-2\theta} \\ +[\mu_2]_2\frac{m_{32}(s)m_{11}(S)-m_{12}(s)m_{31}(S)}{(s^TS^A)_{11}} \\ +[\mu_2]_3\frac{m_{22}(s)m_{11}(S)-m_{12}(s)m_{21}(S)}{(s^TS^A)_{11}}. \tag{2.47c}$$

We first suppose that $\lambda_j \in D_1$ is a simple zero of $m_{11}(\mathscr{A})(\lambda)$. Solving (2.46b) and (2.46c) for $[\mu_2]_1, [\mu_2]_3$ and substituting the result in to (2.46a), we find

$$[M_1]_1 = \frac{\mathscr{A}_{33}[M_1]_2 - \mathscr{A}_{32}[M_1]_3}{m_{11}(\mathscr{A})m_{21}(\mathscr{A})}e^{2\theta} - \frac{[\mu_2]_2}{m_{21}(\mathscr{A})}e^{2\theta}.$$

Taking the residue of this equation at λ_j, we find the condition (2.43a) in the case when $\lambda_j \in D_1$.

In order to prove (2.43c), we solve (2.47a) for $[\mu_2]_1$, then substituting the result into (2.47b) and (2.47c), we find

$$[M_3]_2 = \frac{m_{33}(s)}{s_{11}}[\mu_2]_2 + \frac{m_{23}(s)}{s_{11}}[\mu_2]_3 + \frac{m_{33}(s)m_{21}(S) - m_{23}(s)m_{31}(S)}{(s^T\dot{S}^A)_{11}s_{11}}e^{-2\theta}[M_3]_1, \qquad (2.48a)$$

$$[M_3]_3 = \frac{m_{32}(s)}{s_{11}}[\mu_2]_2 + \frac{m_{22}(s)}{s_{11}}[\mu_2]_3 + \frac{m_{32}(s)m_{21}(S) - m_{22}(s)m_{31}(S)}{(s^T\dot{S}^A)_{11}s_{11}}e^{-2\theta}[M_3]_1. \qquad (2.48b)$$

Taking the residue of this equation at λ_j, we find the condition (2.43c) in the case when $\lambda_j \in D_3$. $\quad\square$

2.10. *The global relation*

The spectral functions $S(\lambda), S_L(\lambda)$ and $s(\lambda)$ are not independent but satisfy an important relation. Indeed, it follows from (2.35) that

$$\mu_1(x,t,\lambda)e^{-(i\lambda x + 2i\lambda^2 t)\hat{\Lambda}}\{S^{-1}(\lambda)s(\lambda)e^{i\lambda L\hat{\Lambda}}S_L(\lambda)\} = \mu_4(x,t,\lambda). \qquad (2.49)$$

Since $\mu_1(0,T,\lambda) = \mathbb{I}$, evaluation at $(0,T)$ yields the following global relation:

$$S^{-1}(\lambda)s(\lambda)e^{i\lambda L\hat{\Lambda}}S_L(\lambda) = e^{2i\lambda^2 T\hat{\Lambda}}c(T,\lambda), \qquad (2.50)$$

where $c(T,\lambda) = \mu_4(0,T,\lambda)$.

3. The Riemann-Hilbert problem

The sectionally analytic function $M(x,t,\lambda)$ defined in section 2 satisfies a Riemann-Hilbert problem which can be formulated in terms of the initial and boundary values of $q_1(x,t)$ and $q_2(x,t)$. By solving this Riemann-Hilbert problem, the solution of (1.2) can be recovered for all values of x,t.

Theorem 3.1. *Suppose that $q_1(x,t)$ and $q_2(x,t)$ are a pair of solutions of (1.2) in the interval domain Ω. Then $q_1(x,t)$ and $q_2(x,t)$ can be reconstructed from the initial value $\{q_{10}(x),q_{20}(x)\}$ and boundary values $\{g_{01}(t),g_{02}(t),g_{11}(t),g_{12}(t)\}$, $\{f_{01}(t),f_{02}(t),f_{11}(t),f_{12}(t)\}$ defined as follows,*

$$\begin{aligned}
&q_{10}(x) = q_1(x,t=0), \;\; q_{20}(x) = q_2(x,t=0),\\
&g_{01}(t) = q_1(x=0,t), \;\; g_{02}(t) = q_2(x=0,t),\\
&f_{01}(t) = q_1(x=L,t), \;\; f_{02}(t) = q_2(x=L,t),\\
&g_{11}(t) = q_{1x}(x=0,t), \; g_{12}(t) = q_{2x}(x=0,t),\\
&f_{11}(t) = q_{1x}(x=L,t), \; f_{12}(t) = q_{2x}(x=L,t).
\end{aligned} \qquad (3.1)$$

Use the initial and boundary data to define the jump matrices $J_{m,n}(x,t,\lambda)$ in terms of the spectral functions $s(\lambda)$ and $S(\lambda), S_L(\lambda)$ by equation (2.35).

Assume that the possible zeros $\{\lambda_j\}_1^N$ of the functions $m_{11}(\mathscr{A})(\lambda)$, $(S^T s^A)_{11}(\lambda)$, $(s^T S^A)_{11}(\lambda)$ and $\mathscr{A}_{11}(\lambda)$ are as the assumption in subsection 2.8.

Then the solution $\{q_1(x,t), q_2(x,t)\}$ is given by

$$q_1(x,t) = 2i \lim_{\lambda \to \infty} (\lambda M(x,t,\lambda))_{12}, \quad q_2(x,t) = 2i \lim_{\lambda \to \infty} (\lambda M(x,t,\lambda))_{13}. \tag{3.2}$$

where $M(x,t,\lambda)$ satisfies the following 3×3 matrix Riemann-Hilbert problem:

- M is sectionally meromorphic on the Riemann $\lambda-$sphere with jumps across the contours $\bar{D}_n \cap \bar{D}_m, n,m = 1, \cdots, 4$, see Figure 2.
- Across the contours $\bar{D}_n \cap \bar{D}_m$, M satisfies the jump condition

$$M_n(x,t,\lambda) = M_m(x,t,\lambda)J_{m,n}(x,t,\lambda), \quad \lambda \in \bar{D}_n \cap \bar{D}_m, n,m = 1,2,3,4. \tag{3.3}$$

- $M(x,t,\lambda) = \mathbb{I} + O(\frac{1}{\lambda})$, $\lambda \to \infty$.
- The residue condition of M is showed in Proposition 2.4.

Proof. It only remains to prove (3.2) and this equation follows from the large λ asymptotics of the eigenfunctions. $\square$

4. Non-linearizable Boundary Conditions

A key difficulty of initial-boundary value problems is that some of the boundary values are unkown for a well-posed problem. While we need all boundary values to define the spectral functions $S(\lambda)$ and $S_L(\lambda)$, and hence for the formulation of the Riemann-Hilbert problem. Our main result, Theorem 4.3, expresses the unknown boundary data in terms of the prescribed boundary data and the initial data in terms of the solution of a system of nonlinear integral equations.

4.1. *Asymptotics*

An analysis of (2.11) shows that the eigenfunctions $\{\mu_j\}_1^4$ have the following asymptotics as $\lambda \to \infty$:

$$\mu_j(x,t,\lambda) = \mathbb{I} + \frac{1}{\lambda}\begin{pmatrix} \mu_{11}^{(1)} & \mu_{12}^{(1)} & \mu_{13}^{(1)} \\ \mu_{21}^{(1)} & \mu_{22}^{(1)} & \mu_{23}^{(1)} \\ \mu_{31}^{(1)} & \mu_{32}^{(1)} & \mu_{33}^{(1)} \end{pmatrix} + \frac{1}{\lambda^2}\begin{pmatrix} \mu_{11}^{(2)} & \mu_{12}^{(2)} & \mu_{13}^{(2)} \\ \mu_{21}^{(2)} & \mu_{22}^{(2)} & \mu_{23}^{(2)} \\ \mu_{31}^{(2)} & \mu_{32}^{(2)} & \mu_{33}^{(2)} \end{pmatrix} + O(\frac{1}{\lambda^3})$$

$$= \mathbb{I} + \frac{1}{\lambda}\begin{pmatrix} \int_{(x_j,t_j)}^{(x,t)} \Delta_{11}dx + \eta_{11}dt & \frac{1}{2i}q_1 & \frac{1}{2i}q_2 \\ -\frac{1}{4}\bar{q}_{1x} + \frac{q_1}{8i}|q|^2 & \int_{(x_j,t_j)}^{(x,t)} \Delta_{22}dx + \eta_{22}dt & \int_{(x_j,t_j)}^{(x,t)} \Delta_{23}dx + \eta_{23}dt \\ -\frac{1}{4}\bar{q}_{2x} + \frac{q_2}{8i}|q|^2 & \int_{(x_j,t_j)}^{(x,t)} \Delta_{32}dx + \eta_{32}dt & \int_{(x_j,t_j)}^{(x,t)} \Delta_{33}dx + \eta_{33}dt \end{pmatrix} \tag{4.1}$$

$$+ \frac{1}{\lambda^2}\begin{pmatrix} \mu_{11}^{(2)} & \frac{1}{4}\bar{q}_{1x} + \frac{1}{2i}(q_1\mu_{22}^{(1)} + q_2\mu_{32}^{(1)}) & \frac{1}{4}\bar{q}_{2x} + \frac{1}{2i}(q_1\mu_{23}^{(1)} + q_2\mu_{33}^{(1)}) \\ \mu_{21}^{(2)} & \mu_{22}^{(2)} & \mu_{23}^{(2)} \\ \mu_{31}^{(2)} & \mu_{32}^{(2)} & \mu_{33}^{(2)} \end{pmatrix} + O(\frac{1}{\lambda^3}).$$

where

$$|q|^2 = |q_1|^2 + |q_2|^2, \tag{4.2}$$

$$\Delta_{11} = \tfrac{1}{8i}|q|^4 - \tfrac{1}{4}(q_1\bar{q}_{1x} + q_2\bar{q}_{2x}),$$
$$\Delta_{22} = \tfrac{i}{8}|q_1|^2|q|^2 + \tfrac{1}{4}q_1\bar{q}_{1x},$$
$$\Delta_{23} = \tfrac{i}{8}\bar{q}_1 q_2|q|^2 + \tfrac{1}{4}q_2\bar{q}_{1x}, \qquad (4.3a)$$
$$\Delta_{32} = \tfrac{i}{8}q_1\bar{q}_2|q|^2 + \tfrac{1}{4}q_1\bar{q}_{2x},$$
$$\Delta_{33} = \tfrac{i}{8}|q_2|^2|q|^2 + \tfrac{1}{4}q_2\bar{q}_{2x}.$$

$$\eta_{11} = \tfrac{1}{8i}|q|^6 + \tfrac{1}{4}(\bar{q}_1 q_{1x} + \bar{q}_2 q_{2x} - q_1\bar{q}_{1x} - q_2\bar{q}_{2x})|q|^2 + \tfrac{1}{4i}(|q_{1x}|^2 + |q_{2x}|^2) - \tfrac{1}{4}(q_1\bar{q}_{1t} + q_2\bar{q}_{2t}),$$

$$\eta_{22} = -\tfrac{1}{8i}|q_1|^2|q|^4 + \tfrac{1}{8}|q|^2(q_1\bar{q}_{1x} - \bar{q}_1 q_{1x}) + \tfrac{1}{8}(q_1\bar{q}_{1x} - \bar{q}_2 q_{2x} + q_2\bar{q}_{2x} - \bar{q}_1 q_{1x})|q_1|^2 + \tfrac{i}{4}|q_{1x}|^2 + \tfrac{1}{4}q_1\bar{q}_{1t},$$

$$\eta_{23} = -\tfrac{1}{8i}\bar{q}_1 q_2|q|^4 + \tfrac{1}{8}|q|^2(q_2\bar{q}_{1x} - \bar{q}_1 q_{2x}) + \tfrac{1}{8}(q_1\bar{q}_{1x} - \bar{q}_2 q_{2x} + q_2\bar{q}_{2x} - \bar{q}_1 q_{1x})\bar{q}_1 q_2 + \tfrac{i}{4}\bar{q}_{1x} q_{2x} + \tfrac{1}{4}q_2\bar{q}_{1t},$$

$$\eta_{32} = -\tfrac{1}{8i}q_1\bar{q}_2|q|^4 + \tfrac{1}{8}|q|^2(q_1\bar{q}_{2x} - \bar{q}_2 q_{1x}) + \tfrac{1}{8}(q_1\bar{q}_{1x} - \bar{q}_2 q_{2x} + q_2\bar{q}_{2x} - \bar{q}_1 q_{1x})q_1\bar{q}_2 + \tfrac{i}{4}q_{1x}\bar{q}_{2x} + \tfrac{1}{4}q_1\bar{q}_{2t},$$

$$\eta_{33} = -\tfrac{1}{8i}|q_2|^2|q|^4 + \tfrac{1}{8}|q|^2(q_2\bar{q}_{2x} - \bar{q}_2 q_{2x}) + \tfrac{1}{8}(q_1\bar{q}_{1x} - \bar{q}_2 q_{2x} + q_2\bar{q}_{2x} - \bar{q}_1 q_{1x})|q_2|^2 + \tfrac{i}{4}|q_{2x}|^2 + \tfrac{1}{4}q_2\bar{q}_{2t}.$$
$$(4.3b)$$

Remark 4.1. The explicit formulas of $\mu_{11}^{(2)}$ and $\mu_{ij}^{(2)}, i,j = 2,3$ are not presented in the following analysis, we do not write down the asymptotic expressions of these functions.

Next, we define functions $\{\Phi_{ij}(t,\lambda)\}_{i,j=1}^3$ and $\phi_{ij}(t,\lambda)_{i,j=1}^3$ by:

$$\mu_2(0,t,\lambda) = \begin{pmatrix} \Phi_{11}(t,\lambda) & \Phi_{12}(t,\lambda) & \Phi_{13}(t,\lambda) \\ \Phi_{21}(t,\lambda) & \Phi_{22}(t,\lambda) & \Phi_{23}(t,\lambda) \\ \Phi_{31}(t,\lambda) & \Phi_{32}(t,\lambda) & \Phi_{33}(t,\lambda) \end{pmatrix}, \qquad (4.4)$$

$$\mu_3(L,t,\lambda) = \begin{pmatrix} \phi_{11}(t,\lambda) & \phi_{12}(t,\lambda) & \phi_{13}(t,\lambda) \\ \phi_{21}(t,\lambda) & \phi_{22}(t,\lambda) & \phi_{23}(t,\lambda) \\ \phi_{31}(t,\lambda) & \phi_{32}(t,\lambda) & \phi_{33}(t,\lambda) \end{pmatrix}. \qquad (4.5)$$

From the asymptotic of $\mu_j(x,t,\lambda)$ in (4.1) we have

$$\mu_2(0,t,\lambda) = \mathbb{I} + \tfrac{1}{\lambda}\begin{pmatrix} \Phi_{11}^{(1)}(t) & \Phi_{12}^{(1)}(t) & \Phi_{13}^{(1)}(t) \\ \Phi_{21}^{(1)}(t) & \Phi_{22}^{(1)}(t) & \Phi_{23}^{(1)}(t) \\ \Phi_{31}^{(1)}(t) & \Phi_{32}^{(1)}(t) & \Phi_{33}(t)^{(1)} \end{pmatrix}$$
$$+ \tfrac{1}{\lambda^2}\begin{pmatrix} \Phi_{11}^{(2)}(t) & \Phi_{12}^{(2)}(t) & \Phi_{13}^{(2)}(t) \\ \Phi_{21}^{(2)}(t) & \Phi_{22}^{(2)}(t) & \Phi_{23}^{(2)}(t) \\ \Phi_{31}^{(2)}(t) & \Phi_{32}^{(2)}(t) & \Phi_{33}^{(2)}(t) \end{pmatrix} + O(\tfrac{1}{\lambda^3}). \qquad (4.6)$$

Recalling the definition of the boundary data at $x = 0$, we have

$$\Phi_{12}^{(1)}(t) = \tfrac{1}{2i}g_{01}(t), \quad \Phi_{12}^{(2)}(t) = \tfrac{1}{4}g_{11} + \tfrac{1}{2i}(g_{01}\Phi_{22}^{(1)} + g_{02}\Phi_{32}^{(1)}),$$
$$\Phi_{13}^{(1)}(t) = \tfrac{1}{2i}g_{02}(t), \quad \Phi_{13}^{(2)}(t) = \tfrac{1}{4}g_{12} + \tfrac{1}{2i}(g_{01}\Phi_{23}^{(1)} + g_{02}\Phi_{33}^{(1)}). \qquad (4.7)$$

In particular, we find the following expressions for the boundary values at $x = 0$:

$$g_{01}(t) = 2i\Phi_{12}^{(1)}(t), \qquad g_{02}(t) = 2i\Phi_{13}^{(1)}(t) \qquad (4.8a)$$

$$g_{11}(t) = 4\Phi_{12}^{(2)}(t) + 2i(g_{01}(t)\Phi_{22}^{(1)}(t) + g_{02}\Phi_{32}^{(1)}(t)),$$
$$g_{12}(t) = 4\Phi_{13}^{(2)}(t) + 2i(g_{01}\Phi_{23}^{(1)}(t) + g_{02}(t)\Phi_{33}^{(1)}(t)). \tag{4.8b}$$

Similarly, we have the asymptotic formulas for $\mu_3(L, t, \lambda) = \phi_{ij}(t, \lambda)_{i,j=1}^3$,

$$\mu_3(L, t, \lambda) = \mathbb{I} + \frac{1}{\lambda} \begin{pmatrix} \phi_{11}^{(1)}(t) & \phi_{12}^{(1)}(t) & \phi_{13}^{(1)}(t) \\ \phi_{21}^{(1)}(t) & \phi_{22}^{(1)}(t) & \phi_{23}^{(1)}(t) \\ \phi_{31}^{(1)}(t) & \phi_{32}^{(1)}(t) & \phi_{33}(t)^{(1)} \end{pmatrix}$$
$$+ \frac{1}{\lambda^2} \begin{pmatrix} \phi_{11}^{(2)}(t) & \phi_{12}^{(2)}(t) & \phi_{13}^{(2)}(t) \\ \phi_{21}^{(2)}(t) & \phi_{22}^{(2)}(t) & \phi_{23}^{(2)}(t) \\ \phi_{31}^{(2)}(t) & \phi_{32}^{(2)}(t) & \phi_{33}^{(2)}(t) \end{pmatrix} + O(\frac{1}{\lambda^3}). \tag{4.9}$$

Recalling that the definition of the boundary data at $x = L$, we have

$$\phi_{12}^{(1)}(t) = \tfrac{1}{2i}f_{01}(t), \quad \phi_{12}^{(2)}(t) = \tfrac{1}{4}f_{11} + \tfrac{1}{2i}(f_{01}\phi_{22}^{(1)} + f_{02}\phi_{32}^{(1)}),$$
$$\phi_{13}^{(1)}(t) = \tfrac{1}{2i}f_{02}(t), \quad \phi_{13}^{(2)}(t) = \tfrac{1}{4}f_{12} + \tfrac{1}{2i}(f_{01}\phi_{23}^{(1)} + f_{02}\phi_{33}^{(1)}). \tag{4.10}$$

In particular, we find the following expressions for the boundary values at $x = L$:

$$f_{01}(t) = 2i\phi_{12}^{(1)}(t), \qquad f_{02}(t) = 2i\phi_{13}^{(1)}(t) \tag{4.11a}$$

$$f_{11}(t) = 4\phi_{12}^{(2)}(t) + 2i(f_{01}(t)\phi_{22}^{(1)}(t) + f_{02}\phi_{32}^{(1)}(t)),$$
$$f_{12}(t) = 4\phi_{13}^{(2)}(t) + 2i(f_{01}\phi_{23}^{(1)} + f_{02}(t)\phi_{33}^{(1)}(t)). \tag{4.11b}$$

From the global relation (2.50)and replacing T by t, we find

$$\mu_2(0, t, \lambda)e^{-2i\lambda^2 t\hat{\Lambda}}\{s(\lambda)e^{i\lambda L\hat{\Lambda}}S_L(\lambda)\} = c(t, \lambda), \quad \lambda \in (D_3 \cup D_4, D_1 \cup D_2, D_1 \cup D_2). \tag{4.12}$$

Lemma 4.1. *We assuming that the initial value and boundary value are compatible at $x = 0$ and $x = L$, then in the vanishing initial value case, the global relation (A.3) implies that the large λ behavior of $c_{j1}(t, \lambda), j = 2, 3$ satisfy*

$$c_{21}(t, \lambda) = \frac{\Phi_{21}^{(1)}(t)}{\lambda} + \frac{\Phi_{21}^{(2)}(t) + \Phi_{21}^{(1)}(t)\bar{\phi}_{11}^{(1)}(t)}{\lambda^2} + O(\frac{1}{\lambda^3})$$
$$+ \left[\frac{\bar{\phi}_{12}^{(1)}(t)}{\lambda} + \frac{\bar{\phi}_{12}^{(2)}(t) + \Phi_{22}^{(1)}(t)\bar{\phi}_{12}^{(1)}(t) + \Phi_{23}^{(1)}(t)\bar{\phi}_{13}^{(1)}(t)}{\lambda^2} + O(\frac{1}{\lambda^3}) \right] e^{-2i\lambda L}, \quad \lambda \to \infty, \tag{4.13a}$$

$$c_{31}(t, \lambda) = \frac{\Phi_{31}^{(1)}(t)}{\lambda} + \frac{\Phi_{31}^{(2)}(t) + \Phi_{31}^{(1)}(t)\bar{\phi}_{11}^{(1)}(t)}{\lambda^2} + O(\frac{1}{\lambda^3})$$
$$+ \left[\frac{\bar{\phi}_{13}^{(1)}(t)}{\lambda} + \frac{\bar{\phi}_{13}^{(2)}(t) + \Phi_{32}^{(1)}(t)\bar{\phi}_{12}^{(1)}(t) + \Phi_{33}^{(1)}(t)\bar{\phi}_{13}^{(1)}(t)}{\lambda^2} + O(\frac{1}{\lambda^3}) \right] e^{-2i\lambda L}, \quad \lambda \to \infty, \tag{4.13b}$$

Proof. The global relation shows that under the assumption of vanishing initial value

$$c_{21}(t,\lambda) = \Phi_{21}(t,\lambda)\bar{\phi}_{11}(t,\bar{\lambda}) + \Phi_{22}(t,\lambda)\bar{\phi}_{12}(t,\bar{\lambda})e^{-2i\lambda L} + \Phi_{23}(t,\lambda)\bar{\phi}_{13}(t,\bar{\lambda})e^{-2i\lambda L}, \quad (4.14a)$$

$$c_{31}(t,\lambda) = \Phi_{31}(t,\lambda)\bar{\phi}_{11}(t,\bar{\lambda}) + \Phi_{32}(t,\lambda)\bar{\phi}_{12}(t,\bar{\lambda})e^{-2i\lambda L} + \Phi_{33}(t,\lambda)\bar{\phi}_{13}(t,\bar{\lambda})e^{-2i\lambda L}, \quad (4.14b)$$

Recalling the equation

$$\mu_t + 2i\lambda^2[\Lambda,\mu] = V_2\mu. \quad (4.15)$$

From the first column of the equation (4.15) we get

$$
\begin{cases}
\Phi_{11t} = \left[\frac{1}{2}(g_{01}\bar{g}_{11} + g_{02}\bar{g}_{12}) + \frac{i}{4}(|g_{01}|^2 + |g_{02}|^2)^2\right]\Phi_{11} + (2\lambda g_{01} + ig_{11})\Phi_{21} + (2\lambda g_{02} + ig_{12})\Phi_{31}, \\[2mm]
\Phi_{21t} = 4i\lambda^2\Phi_{21} + \left[-\frac{1}{2}\lambda(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{01} + i\lambda\bar{g}_{11} + \frac{i}{4}\left((2|g_{01}|^2 + |g_{02}|^2)\bar{g}_{11} + \bar{g}_{01}g_{02}\bar{g}_{12} - \bar{g}_{01}^2 g_{11}\right.\right. \\
\qquad \left.\left. -\bar{g}_{01}\bar{g}_{02}g_{12}\right) - \frac{1}{4}\bar{g}_{01}(|g_{01}|^2 + |g_{02}|^2)^2 - \frac{1}{2i}\bar{g}_{01t}\right]\Phi_{11} - \frac{i}{4}(|g_{01}|^2 + |g_{02}|^2)|g_{01}|^2\Phi_{21} \\
\qquad -\frac{1}{2}g_{01}\bar{g}_{11}\Phi_{21} - \left(\frac{1}{2}g_{02}\bar{g}_{11} + \frac{i}{4}(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{01}g_{02}\right)\Phi_{31}, \\[2mm]
\Phi_{31t} = 4i\lambda^2\Phi_{31} + \left[-\frac{1}{2}\lambda(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{02} + i\lambda\bar{g}_{12} + \frac{i}{4}\left((2|g_{02}|^2 + |g_{01}|^2)\bar{g}_{12} + \bar{g}_{02}g_{01}\bar{g}_{11} - \bar{g}_{02}^2 g_{12}\right.\right. \\
\qquad \left.\left. -\bar{g}_{01}g_{02}g_{11}\right) - \frac{1}{4}\bar{g}_{02}(|g_{01}|^2 + |g_{02}|^2)^2 - \frac{1}{2i}\bar{g}_{02t}\right]\Phi_{11} - \frac{i}{4}(|g_{01}|^2 + |g_{02}|^2)g_{01}\bar{g}_{02}\Phi_{21} \\
\qquad -\frac{1}{2}g_{01}\bar{g}_{12}\Phi_{21} - \left(\frac{1}{2}g_{02}\bar{g}_{12} + \frac{i}{4}(|g_{01}|^2 + |g_{02}|^2)|g_{02}|^2\right)\Phi_{31}.
\end{cases}
\tag{4.16a}
$$

From the second column of the equation (4.15) we get

$$
\begin{cases}
\Phi_{12t} = -4i\lambda^2\Phi_{12} + \left[\frac{1}{2}(g_{01}\bar{g}_{11} + g_{02}\bar{g}_{12}) + \frac{i}{4}(|g_{01}|^2 + |g_{02}|^2)^2\right]\Phi_{12} + (2\lambda g_{01} + ig_{11})\Phi_{22} \\
\qquad +(2\lambda g_{02} + ig_{12})\Phi_{32}, \\[2mm]
\Phi_{22t} = \left[-\frac{1}{2}\lambda(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{01} + i\lambda\bar{g}_{11} + \frac{i}{4}\left((2|g_{01}|^2 + |g_{02}|^2)\bar{g}_{11} + \bar{g}_{01}g_{02}\bar{g}_{12} - \bar{g}_{01}^2 g_{11}\right.\right. \\
\qquad \left.\left. -\bar{g}_{01}\bar{g}_{02}g_{12}\right) - \frac{1}{4}\bar{g}_{01}(|g_{01}|^2 + |g_{02}|^2)^2 - \frac{1}{2i}\bar{g}_{01t}\right]\Phi_{12} - \frac{i}{4}(|g_{01}|^2 + |g_{02}|^2)|g_{01}|^2\Phi_{22} \\
\qquad -\frac{1}{2}g_{01}\bar{g}_{11}\Phi_{22} - \left(\frac{1}{2}g_{02}\bar{g}_{11} + \frac{i}{4}(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{01}g_{02}\right)\Phi_{32}, \\[2mm]
\Phi_{32t} = \left[-\frac{1}{2}\lambda(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{02} + i\lambda\bar{g}_{12} + \frac{i}{4}\left((2|g_{02}|^2 + |g_{01}|^2)\bar{g}_{12} + \bar{g}_{02}g_{01}\bar{g}_{11} - \bar{g}_{02}^2 g_{12}\right.\right. \\
\qquad \left.\left. -\bar{g}_{01}g_{02}g_{11}\right) - \frac{1}{4}\bar{g}_{02}(|g_{01}|^2 + |g_{02}|^2)^2 - \frac{1}{2i}\bar{g}_{02t}\right]\Phi_{12} - \frac{i}{4}(|g_{01}|^2 + |g_{02}|^2)g_{01}\bar{g}_{02}\Phi_{22} \\
\qquad -\frac{1}{2}g_{01}\bar{g}_{12}\Phi_{22} - \left(\frac{1}{2}g_{02}\bar{g}_{12} + \frac{i}{4}(|g_{01}|^2 + |g_{02}|^2)|g_{02}|^2\right)\Phi_{32}.
\end{cases}
\tag{4.16b}
$$

From the third column of the equation (4.15) we get

$$
\begin{cases}
\Phi_{13t} = -4i\lambda^2\Phi_{13} + \left[\frac{1}{2}(g_{01}\bar{g}_{11} + g_{02}\bar{g}_{12}) + \frac{i}{4}(|g_{01}|^2 + |g_{02}|^2)^2\right]\Phi_{13} + (2\lambda g_{01} + ig_{11})\Phi_{23} \\
\qquad + (2\lambda g_{02} + ig_{12})\Phi_{33}, \\[2mm]
\Phi_{23t} = \left[-\frac{1}{2}\lambda(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{01} + i\lambda\bar{g}_{11} + \frac{i}{4}\left((2|g_{01}|^2 + |g_{02}|^2)\bar{g}_{11} + \bar{g}_{01}g_{02}\bar{g}_{12} - \bar{g}_{01}^2 g_{11}\right.\right. \\
\qquad \left.\left. -\bar{g}_{01}\bar{g}_{02}g_{12}\right) - \frac{1}{4}\bar{g}_{01}(|g_{01}|^2 + |g_{02}|^2)^2 - \frac{1}{2i}\bar{g}_{01t}\right]\Phi_{13} - \frac{i}{4}(|g_{01}|^2 + |g_{02}|^2)|g_{01}|^2\Phi_{23} \\
\qquad - \frac{1}{2}g_{01}\bar{g}_{11}\Phi_{23} - \left(\frac{1}{2}g_{02}\bar{g}_{11} + \frac{i}{4}(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{01}g_{02}\right)\Phi_{33}, \\[2mm]
\Phi_{33t} = \left[-\frac{1}{2}\lambda(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{02} + i\lambda\bar{g}_{12} + \frac{i}{4}\left((2|g_{02}|^2 + |g_{01}|^2)\bar{g}_{12} + \bar{g}_{02}g_{01}\bar{g}_{11} - \bar{g}_{02}^2 g_{12}\right.\right. \\
\qquad \left.\left. -\bar{g}_{01}g_{02}g_{11}\right) - \frac{1}{4}\bar{g}_{02}(|g_{01}|^2 + |g_{02}|^2)^2 - \frac{1}{2i}\bar{g}_{02t}\right]\Phi_{13} - \frac{i}{4}(|g_{01}|^2 + |g_{02}|^2)g_{01}\bar{g}_{02}\Phi_{23} \\
\qquad - \frac{1}{2}g_{01}\bar{g}_{12}\Phi_{23} - \left(\frac{1}{2}g_{02}\bar{g}_{12} + \frac{i}{4}(|g_{01}|^2 + |g_{02}|^2)|g_{02}|^2\right)\Phi_{33}.
\end{cases}
\tag{4.16c}
$$

Suppose

$$
\begin{pmatrix}\Phi_{11}\\\Phi_{21}\\\Phi_{31}\end{pmatrix} = \left(\alpha_0(t) + \frac{\alpha_1(t)}{\lambda} + \frac{\alpha_2(t)}{\lambda^2} + \cdots\right) + \left(\beta_0(t) + \frac{\beta_1(t)}{\lambda} + \frac{\beta_2(t)}{\lambda^2} + \cdots\right)e^{4i\lambda^2 t}, \tag{4.17}
$$

where the coefficients $\alpha_j(t)$ and $\beta_j(t)$, $j = 0, 1, 2, \cdots$, are independent of k and are 3×1 matrix functions.

To determine these coefficients, we substitute the above equation into equation (4.16a) and use the initial conditions

$$
\alpha_0(0) + \beta_0(0) = (1\ 0\ 0)^T, \quad \alpha_1(0) + \beta_1(0) = (0\ 0\ 0)^T.
$$

Then we get

$$
\begin{aligned}
\begin{pmatrix}\Phi_{11}\\\Phi_{21}\\\Phi_{31}\end{pmatrix} &= \begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{1}{\lambda}\begin{pmatrix}\Phi_{11}^{(1)}\\\Phi_{21}^{(1)}\\\Phi_{31}^{(1)}\end{pmatrix} + \frac{1}{\lambda^2}\begin{pmatrix}\Phi_{11}^{(2)}\\\Phi_{21}^{(2)}\\\Phi_{31}^{(2)}\end{pmatrix} + O(\tfrac{1}{\lambda^3}) \\
&\quad + \left[\frac{1}{\lambda}\begin{pmatrix}0\\-\Phi_{21}^{(1)}(0)\\-\Phi_{31}^{(1)}(0)\end{pmatrix} + O(\tfrac{1}{\lambda^2})\right]e^{4i\lambda^2 t}
\end{aligned}
\tag{4.18}
$$

Similarly, suppose

$$
\begin{pmatrix}\Phi_{12}\\\Phi_{22}\\\Phi_{32}\end{pmatrix} = \left(\alpha_0(t) + \frac{\alpha_1(t)}{\lambda} + \frac{\alpha_2(t)}{\lambda^2} + \cdots\right) + \left(\beta_0(t) + \frac{\beta_1(t)}{\lambda} + \frac{\beta_2(t)}{\lambda^2} + \cdots\right)e^{-4i\lambda^2 t}, \tag{4.19}
$$

where the coefficients $\alpha_j(t)$ and $\beta_j(t)$, $j = 0, 1, 2, \cdots$, are independent of k and are 3×1 matrix functions.

To determine these coefficients,we substitute the above equation into equation (4.16b) and use the initial conditions

$$\alpha_0(0) + \beta_0(0) = (0\ 1\ 0)^T, \quad \alpha_1(0) + \beta_1(0) = (0\ 0\ 0)^T.$$

Then we get

$$
\begin{pmatrix} \Phi_{12} \\ \Phi_{22} \\ \Phi_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} \Phi_{12}^{(1)} \\ \Phi_{22}^{(1)} \\ \Phi_{32}^{(1)} \end{pmatrix} + \frac{1}{\lambda^2} \begin{pmatrix} \Phi_{12}^{(2)} \\ \Phi_{22}^{(2)} \\ \Phi_{32}^{(2)} \end{pmatrix} + O(\tfrac{1}{\lambda^3})
$$
$$
+ \left[\frac{1}{\lambda} \begin{pmatrix} -\Phi_{12}^{(1)}(0) \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\lambda^2} \begin{pmatrix} -\Phi_{12}^{(2)}(0) + \Phi_{12}^{(1)}(0)\Phi_{22}^{(1)} + \Phi_{12}^{(1)}(0)\Phi_{32}^{(1)} \\ \frac{1}{4i}\left(-\frac{1}{2}(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{01} + i\bar{g}_{11}\right)\Phi_{12}^{(1)}(0) \\ \frac{1}{4i}\left(-\frac{1}{2}(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{02} + i\bar{g}_{12}\right)\Phi_{12}^{(1)}(0) \end{pmatrix} + O(\tfrac{1}{\lambda^2}) \right] e^{-4i\lambda^2 t}
$$

(4.20)

Similar to the derivation of $\Phi_{i2}, i = 1,2,3$, from (4.16c) we can get the asymptotic formulas of $\Phi_{i3}, i = 1,2,3$

$$
\begin{pmatrix} \Phi_{13} \\ \Phi_{23} \\ \Phi_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} \Phi_{13}^{(1)} \\ \Phi_{23}^{(1)} \\ \Phi_{33}^{(1)} \end{pmatrix} + \frac{1}{\lambda^2} \begin{pmatrix} \Phi_{13}^{(2)} \\ \Phi_{23}^{(2)} \\ \Phi_{33}^{(2)} \end{pmatrix} + O(\tfrac{1}{\lambda^3})
$$
$$
+ \left[\frac{1}{\lambda} \begin{pmatrix} -\Phi_{13}^{(1)}(0) \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\lambda^2} \begin{pmatrix} -\Phi_{13}^{(2)}(0) + \Phi_{13}^{(1)}(0)\Phi_{23}^{(1)} + \Phi_{13}^{(1)}(0)\Phi_{33}^{(1)} \\ \frac{1}{4i}\left(-\frac{1}{2}(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{01} + i\bar{g}_{11}\right)\Phi_{13}^{(1)}(0) \\ \frac{1}{4i}\left(-\frac{1}{2}(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{02} + i\bar{g}_{12}\right)\Phi_{13}^{(1)}(0) \end{pmatrix} + O(\tfrac{1}{\lambda^2}) \right] e^{-4i\lambda^2 t}
$$

(4.21)

Similar to (4.16), we also know that $\{\phi_{ij}\}_{i,j=1}^3$ satisfy the similar partial derivative equations. Substituting these formulas into the equation (4.14a) and noticing that we assume that the initial value and boundary value are compatible at $x = 0$ and $x = L$, we get the asymptotic behavior (4.13a) of $c_{j1}(t,\lambda)$ as $\lambda \to \infty$. Similar to prove the formula (4.13b).

$\square$

4.2. *The Dirichlet and Neumann problems*

In what follows, we can derive the effective characterizations of spectral function $S(\lambda), S_L(\lambda)$ for the Dirichlet ($\{g_{01}(t), g_{02}(t)\}$ and $\{f_{01}(t), f_{02}(t)\}$ prescribed), the Neumann ($\{g_{11}(t), g_{12}(t)\}$ and $\{f_{11}(t), f_{12}(t)\}$ prescribed) problems.

Define the following new functions as

$$f_-(t,\lambda) = f(t,\lambda) - f(t,-\lambda), \quad f_+(t,\lambda) = f(t,\lambda) + f(t,-\lambda),$$

(4.22)

Introducing

$$\Delta(k) = e^{2i\lambda L} - e^{-2i\lambda L}, \quad \Sigma(k) = e^{2i\lambda L} + e^{-2i\lambda L}$$

(4.23)

Denoting ∂D_3^0 as the boundary contour which is not included the zeros of $\Delta(\lambda)$.

Theorem 4.1. *Let $T < \infty$. Let $q_0(x) = (q_{10}(x), q_{20}(x)), 0 \leq x \geq L$, be two initial functions.*

For the Dirichlet problem it is assumed that the function $\{g_{01}(t), g_{02}(t)\}, 0 \leq t < T$, *has sufficient smoothness and is compatible with* $\{q_{10}(x), q_{20}(x)\}$ *at* $x = t = 0$, *that is*

$$q_{10}(0) = g_{01}(0), \quad q_{20}(0) = g_{02}(0).$$

the function $\{f_{01}(t), f)02(t)\}, 0 \leq t < T$, *has sufficient smoothness and is compatible with* $q_{10}(x), q_{20}(x)$ *at* $x = L$, *that is*

$$q_{10}(L) = f_{01}(0), \quad q_{20}(L) = f_{02}(0).$$

For the Neumann problem it is assumed that the functions $\{g_{11}(t), g12(t)\}, 0 \leq t < T$, *has sufficient smoothness and is compatible with* $q_0(x)$ *at* $x = t = 0$. *The functions* $\{f_{11}(t), f_{12}(t)\}, 0 \leq t < T$, *has sufficient smoothness and is compatible with* $q_0(x)$ *at* $x = L$.

Then the spectral function $S(\lambda), S_L(\lambda)$ *is given by*

$$S(\lambda) = \begin{pmatrix} \overline{\Phi_{11}(\bar\lambda)} & e^{4i\lambda^2 T}\overline{\Phi_{21}(\bar\lambda)} & e^{4i\lambda^2 T}\overline{\Phi_{31}(\bar\lambda)} \\ e^{-4i\lambda^2 T}\overline{\Phi_{12}(\bar\lambda)} & \overline{\Phi_{22}(\bar\lambda)} & \overline{\Phi_{32}(\bar\lambda)} \\ e^{-4i\lambda^2 T}\overline{\Phi_{13}(\bar\lambda)} & \overline{\Phi_{23}(\bar\lambda)} & \overline{\Phi_{33}(\bar\lambda)} \end{pmatrix} \tag{4.24}$$

$$S_L(\lambda) = \begin{pmatrix} \overline{\phi_{11}(\bar\lambda)} & e^{4i\lambda^2 T}\overline{\phi_{21}(\bar\lambda)} & e^{4i\lambda^2 T}\overline{\phi_{31}(\bar\lambda)} \\ e^{-4i\lambda^2 T}\overline{\phi_{12}(\bar\lambda)} & \overline{\phi_{22}(\bar\lambda)} & \overline{\phi_{32}(\bar\lambda)} \\ e^{-4i\lambda^2 T}\overline{\phi_{13}(\bar\lambda)} & \overline{\phi_{23}(\bar\lambda)} & \overline{\phi_{33}(\bar\lambda)} \end{pmatrix} \tag{4.25}$$

and the complex-value functions $\{\Phi_{l3}(t,\lambda)\}_{l=1}^3$ *satisfy the following system of integral equations:*

$$\begin{cases} \Phi_{13}(t,\lambda) = \int_0^t e^{-4i\lambda^2(t-t')}\left\{\left[\frac{1}{2}(g_{01}\bar g_{11}+g_{02}\bar g_{12})+\frac{i}{4}(|g_{01}|^2+|g_{02}|^2)^2\right]\Phi_{13}+(2\lambda g_{01}+ig_{11})\Phi_{23}\right. \\ \qquad\qquad \left.+(2\lambda g_{02}+ig_{12})\Phi_{33}\right\}(t',\lambda)dt', \\[2mm] \Phi_{23}(t,\lambda) = \int_0^t\left\{\left[-\frac{1}{2}\lambda(|g_{01}|^2+|g_{02}|^2)\bar g_{01}+i\lambda\bar g_{11}+\frac{i}{4}((2|g_{01}|^2+|g_{02}|^2)\bar g_{11}+\bar g_{01}g_{02}\bar g_{12}\right.\right. \\ \qquad\qquad \left.-\bar g_{01}^2 g_{11}-\bar g_{01}\bar g_{02}g_{12})-\frac{1}{4}\bar g_{01}(|g_{01}|^2+|g_{02}|^2)^2-\frac{1}{2i}\bar g_{01t}\right]\Phi_{13}-\frac{1}{2}g_{01}\bar g_{11}\Phi_{23} \\ \qquad\qquad \left.-\frac{i}{4}(|g_{01}|^2+|g_{02}|^2)|g_{01}|^2\Phi_{23}-\left(\frac{1}{2}g_{02}\bar g_{11}+\frac{i}{4}(|g_{01}|^2+|g_{02}|^2)\bar g_{01}g_{02}\right)\Phi_{33}\right\}(t',\lambda)dt', \\[2mm] \Phi_{33}(t,\lambda) = 1+\int_0^t\left\{\left[-\frac{1}{2}\lambda(|g_{01}|^2+|g_{02}|^2)\bar g_{02}+i\lambda\bar g_{12}+\frac{i}{4}((2|g_{02}|^2+|g_{01}|^2)\bar g_{12}+\bar g_{02}g_{01}\bar g_{11}\right.\right. \\ \qquad\qquad \left.-\bar g_{02}^2 g_{12}-\bar g_{01}g_{02}g_{11})-\frac{1}{4}\bar g_{02}(|g_{01}|^2+|g_{02}|^2)^2-\frac{1}{2i}\bar g_{02t}\right]\Phi_{13}-\frac{1}{2}g_{01}\bar g_{12}\Phi_{23} \\ \qquad\qquad \left.-\frac{i}{4}(|g_{01}|^2+|g_{02}|^2)g_{01}\bar g_{02}\Phi_{23}-\left(\frac{1}{2}g_{02}\bar g_{12}+\frac{i}{4}(|g_{01}|^2+|g_{02}|^2)|g_{02}|^2\right)\Phi_{33}\right\}(t',\lambda)dt'. \end{cases} \tag{4.26}$$

and $\{\Phi_{l1}(t,\lambda)\}_{l=1}^3, \{\Phi_{l2}(t,\lambda)\}_{l=1}^3$ *satisfy the following system of integral equations:*

$$
\begin{cases}
\begin{aligned}
\Phi_{11}(t,\lambda) =& 1 + \int_0^t \left\{ \left[\tfrac{1}{2}(g_{01}\bar{g}_{11} + g_{02}\bar{g}_{12}) + \tfrac{i}{4}(|g_{01}|^2 + |g_{02}|^2)^2 \right] \Phi_{11} + (2\lambda g_{01} + ig_{11})\Phi_{21} \right. \\
& \left. + (2\lambda g_{02} + ig_{12})\Phi_{31} \right\}(t',\lambda)dt', \\[2mm]
\Phi_{21}(t,\lambda) =& \int_0^t e^{4i\lambda^2(t-t')} \left\{ \left[-\tfrac{1}{2}\lambda(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{01} + i\lambda\bar{g}_{11} + \tfrac{i}{4}\left((2|g_{01}|^2 + |g_{02}|^2)\bar{g}_{11} + \bar{g}_{01}g_{02}\bar{g}_{12} \right. \right. \right. \\
& \left. -\bar{g}_{01}^2 g_{11} - \bar{g}_{01}\bar{g}_{02}g_{12} \right) - \tfrac{1}{4}\bar{g}_{01}(|g_{01}|^2 + |g_{02}|^2)^2 - \tfrac{1}{2i}\bar{g}_{01t} \right]\Phi_{11} - \tfrac{1}{2}g_{01}\bar{g}_{11}\Phi_{21} \\
& \left. -\tfrac{i}{4}(|g_{01}|^2 + |g_{02}|^2)|g_{01}|^2\Phi_{21} - \left(\tfrac{1}{2}g_{02}\bar{g}_{11} + \tfrac{i}{4}(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{01}g_{02} \right)\Phi_{31} \right\}(t',\lambda)dt', \\[2mm]
\Phi_{31}(t,\lambda) =& \int_0^t e^{4i\lambda^2(t-t')} \left\{ \left[-\tfrac{1}{2}\lambda(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{02} + i\lambda\bar{g}_{12} + \tfrac{i}{4}\left((2|g_{02}|^2 + |g_{01}|^2)\bar{g}_{12} + \bar{g}_{02}g_{01}\bar{g}_{11} \right. \right. \right. \\
& \left. -\bar{g}_{02}^2 g_{12} - \bar{g}_{01}g_{02}g_{11} \right) - \tfrac{1}{4}\bar{g}_{02}(|g_{01}|^2 + |g_{02}|^2)^2 - \tfrac{1}{2i}\bar{g}_{02t} \right]\Phi_{11} - \tfrac{1}{2}g_{01}\bar{g}_{12}\Phi_{21} \\
& \left. -\tfrac{i}{4}(|g_{01}|^2 + |g_{02}|^2)g_{01}\bar{g}_{02}\Phi_{21} - \left(\tfrac{1}{2}g_{02}\bar{g}_{12} + \tfrac{i}{4}(|g_{01}|^2 + |g_{02}|^2)|g_{02}|^2 \right)\Phi_{31} \right\}(t',\lambda)dt'.
\end{aligned}
\end{cases}
\tag{4.27}
$$

$$
\begin{cases}
\begin{aligned}
\Phi_{12}(t,\lambda) =& \int_0^t e^{-4i\lambda^2(t-t')} \left\{ \left[\tfrac{1}{2}(g_{01}\bar{g}_{11} + g_{02}\bar{g}_{12}) + \tfrac{i}{4}(|g_{01}|^2 + |g_{02}|^2)^2 \right] \Phi_{12} + (2\lambda g_{01} + ig_{11})\Phi_{22} \right. \\
& \left. + (2\lambda g_{02} + ig_{12})\Phi_{32} \right\}(t',\lambda)dt', \\[2mm]
\Phi_{22}(t,\lambda) =& 1 + \int_0^t \left\{ \left[-\tfrac{1}{2}\lambda(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{01} + i\lambda\bar{g}_{11} + \tfrac{i}{4}\left((2|g_{01}|^2 + |g_{02}|^2)\bar{g}_{11} + \bar{g}_{01}g_{02}\bar{g}_{12} \right. \right. \right. \\
& \left. -\bar{g}_{01}^2 g_{11} - \bar{g}_{01}\bar{g}_{02}g_{12} \right) - \tfrac{1}{4}\bar{g}_{01}(|g_{01}|^2 + |g_{02}|^2)^2 - \tfrac{1}{2i}\bar{g}_{01t} \right]\Phi_{12} - \tfrac{1}{2}g_{01}\bar{g}_{11}\Phi_{22} \\
& \left. -\tfrac{i}{4}(|g_{01}|^2 + |g_{02}|^2)|g_{01}|^2\Phi_{22} - \left(\tfrac{1}{2}g_{02}\bar{g}_{11} + \tfrac{i}{4}(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{01}g_{02} \right)\Phi_{32} \right\}(t',\lambda)dt', \\[2mm]
\Phi_{32}(t,\lambda) =& 1 + \int_0^t \left\{ \left[-\tfrac{1}{2}\lambda(|g_{01}|^2 + |g_{02}|^2)\bar{g}_{02} + i\lambda\bar{g}_{12} + \tfrac{i}{4}\left((2|g_{02}|^2 + |g_{01}|^2)\bar{g}_{12} + \bar{g}_{02}g_{01}\bar{g}_{11} \right. \right. \right. \\
& \left. -\bar{g}_{02}^2 g_{12} - \bar{g}_{01}g_{02}g_{11} \right) - \tfrac{1}{4}\bar{g}_{02}(|g_{01}|^2 + |g_{02}|^2)^2 - \tfrac{1}{2i}\bar{g}_{02t} \right]\Phi_{12} - \tfrac{1}{2}g_{01}\bar{g}_{12}\Phi_{22} \\
& \left. -\tfrac{i}{4}(|g_{01}|^2 + |g_{02}|^2)g_{01}\bar{g}_{02}\Phi_{22} - \left(\tfrac{1}{2}g_{02}\bar{g}_{12} + \tfrac{i}{4}(|g_{01}|^2 + |g_{02}|^2)|g_{02}|^2 \right)\Phi_{32} \right\}(t',\lambda)dt'.
\end{aligned}
\end{cases}
\tag{4.28}
$$

(i) For the Dirichlet problem, the unknown Neumann boundary value $\{g_{11}(t), g_{12}(t)\}$ and $\{f_{11}(t), f_{12}(t)\}$ are given by

$$
\begin{aligned}
g_{11}(t) =& \frac{2}{i\pi} \int_{\partial D_3} \frac{\Sigma}{\Delta}(\lambda\Phi_{12-} + ig_{01})d\lambda - \frac{1}{\pi}\int_{\partial D_3}(g_{01}\bar{\phi}_{22-} + g_{02}\bar{\phi}_{23-})d\lambda \\
& + \frac{4}{i\pi}\int_{\partial D_3}\frac{1}{\Delta}(\lambda\bar{\phi}_{21-} - 2\bar{\phi}_{21}^{(1)})d\lambda + \frac{2}{\pi}\int_{\partial D_3}(g_{01}\Phi_{22-} + g_{02}\Phi_{32-})d\lambda \\
& + \frac{4}{i\pi}\int_{\partial D_3}\frac{\lambda}{\Delta}[(\Phi_{11}-1)\bar{\phi}_{21} + \Phi_{12}(\bar{\phi}_{22}-1)e^{-2i\lambda L} + \Phi_{13}\bar{\phi}_{23}e^{-2i\lambda L}]_- d\lambda,
\end{aligned}
\tag{4.29a}
$$

$$
\begin{aligned}
g_{12}(t) =& \frac{2}{i\pi} \int_{\partial D_3} \frac{\Sigma}{\Delta}(\lambda\Phi_{13-} + ig_{02})d\lambda - \frac{1}{\pi}\int_{\partial D_3}(g_{01}\bar{\phi}_{32-} + g_{02}\bar{\phi}_{33-})d\lambda \\
& + \frac{4}{i\pi}\int_{\partial D_3}\frac{1}{\Delta}(\lambda\bar{\phi}_{31-} - 2\bar{\phi}_{31}^{(1)})d\lambda + \frac{2}{\pi}\int_{\partial D_3}(g_{01}\Phi_{23-} + g_{02}\Phi_{33-})d\lambda \\
& + \frac{4}{i\pi}\int_{\partial D_3}\frac{\lambda}{\Delta}[(\Phi_{11}-1)\bar{\phi}_{31} + \Phi_{12}\bar{\phi}_{32}e^{-2i\lambda L} + \Phi_{13}(\bar{\phi}_{33}-1)e^{-2i\lambda L}]_- d\lambda,
\end{aligned}
\tag{4.29b}
$$

and

$$f_{11}(t) = -\frac{2}{i\pi} \int_{\partial D_3} \frac{\Sigma}{\Delta} (\lambda \phi_{12-} + i f_{01}) d\lambda - \frac{1}{\pi} \int_{\partial D_3} (f_{01} \bar{\Phi}_{22-} + f_{02} \bar{\Phi}_{23-}) d\lambda$$
$$- \frac{4}{i\pi} \int_{\partial D_3} \frac{1}{\Delta} (\lambda \bar{\Phi}_{21-} - 2\bar{\Phi}_{21}^{(1)}) d\lambda + \frac{2}{\pi} \int_{\partial D_3} (f_{01} \phi_{22-} + f_{02} \phi_{32-}) d\lambda \qquad (4.30a)$$
$$- \frac{4}{i\pi} \int_{\partial D_3} \frac{\lambda}{\Delta} [(\phi_{11} - 1)\bar{\Phi}_{21} + \phi_{12}(\bar{\Phi}_{22} - 1)e^{2i\lambda L} + \phi_{13}\bar{\phi}_{23}e^{2i\lambda L}]_- d\lambda,$$

$$f_{12}(t) = -\frac{2}{i\pi} \int_{\partial D_3} \frac{\Sigma}{\Delta} (\lambda \phi_{13-} + i f_{02}) d\lambda - \frac{1}{\pi} \int_{\partial D_3} (f_{01} \bar{\Phi}_{32-} + f_{02} \bar{\Phi}_{33-}) d\lambda$$
$$- \frac{4}{i\pi} \int_{\partial D_3} \frac{1}{\Delta} (\lambda \bar{\Phi}_{31-} - 2\bar{\Phi}_{31}^{(1)}) d\lambda + \frac{2}{\pi} \int_{\partial D_3} (f_{01} \phi_{23-} + f_{02} \phi_{33-}) d\lambda \qquad (4.30b)$$
$$- \frac{4}{i\pi} \int_{\partial D_3} \frac{\lambda}{\Delta} [(\phi_{11} - 1)\bar{\Phi}_{31} + \phi_{12}\bar{\Phi}_{32}e^{2i\lambda L} + \phi_{13}(\bar{\Phi}_{33} - 1)e^{2i\lambda L}]_- d\lambda.$$

where the conjugate of a function h denotes $\bar{h} = \overline{h(\bar{\lambda})}$.

(ii) For the Neumann problem, the unknown boundary values $\{g_{01}(t), g_{02}(t)\}$ and $\{f_{01}(t), f_{02}(t)\}$ are given by

$$g_{01}(t) = \frac{1}{\pi} \int_{\partial D_3} \frac{\Sigma}{\Delta} \Phi_{12+} d\lambda + \frac{2}{\pi} \int_{\partial D_3} \frac{1}{\Delta} \bar{\phi}_{21+} d\lambda$$
$$+ \frac{2}{\pi} \int_{\partial D_3} \frac{1}{\Delta} \left[\bar{\phi}_{21}(\Phi_{11} - 1) + (\bar{\phi}_{22} - 1)\Phi_{12}e^{-2i\lambda L} + \bar{\phi}_{23}\Phi_{13}e^{-2i\lambda L} \right]_+ d\lambda, \qquad (4.31a)$$

$$g_{02}(t) = \frac{1}{\pi} \int_{\partial D_3} \frac{\Sigma}{\Delta} \Phi_{13+} d\lambda + \frac{2}{\pi} \int_{\partial D_3} \frac{1}{\Delta} \bar{\phi}_{31+} d\lambda$$
$$+ \frac{2}{\pi} \int_{\partial D_3} \frac{1}{\Delta} \left[\bar{\phi}_{31}(\Phi_{11} - 1) + \bar{\phi}_{32}\Phi_{12}e^{-2i\lambda L} + (\bar{\phi}_{33} - 1)\Phi_{13}e^{-2i\lambda L} \right]_+ d\lambda, \qquad (4.31b)$$

and

$$f_{01}(t) = -\frac{1}{\pi} \int_{\partial D_3} \frac{\Sigma}{\Delta} \phi_{12+} d\lambda - \frac{2}{\pi} \int_{\partial D_3} \frac{1}{\Delta} \bar{\Phi}_{21+} d\lambda$$
$$- \frac{2}{\pi} \int_{\partial D_3} \frac{1}{\Delta} \left[\bar{\Phi}_{21}(\phi_{11} - 1) + (\bar{\Phi}_{22} - 1)\phi_{12}e^{2i\lambda L} + \bar{\Phi}_{23}\phi_{13}e^{2i\lambda L} \right]_+ d\lambda, \qquad (4.32a)$$

$$f_{02}(t) = -\frac{1}{\pi} \int_{\partial D_3} \frac{\Sigma}{\Delta} \phi_{13+} d\lambda - \frac{2}{\pi} \int_{\partial D_3} \frac{1}{\Delta} \bar{\Phi}_{31+} d\lambda$$
$$- \frac{2}{\pi} \int_{\partial D_3} \frac{1}{\Delta} \left[\bar{\Phi}_{31}(\phi_{11} - 1) + \bar{\Phi}_{32}\phi_{12}e^{2i\lambda L} + (\bar{\Phi}_{33} - 1)\phi_{13}e^{2i\lambda L} \right]_+ d\lambda. \qquad (4.32b)$$

Proof. The representations (4.24) follow from the relation $S(k) = e^{2i\lambda^2 T\hat{\Lambda}}\mu_2^{-1}(0, T, k)$. And the system (4.26) is the direct result of the Volteral integral equations of $\mu_2(0, t, k)$.

(i) In order to derive (4.29a) we note that equation (4.8b) expresses g_{11} in terms of $\Phi_{12}^{(2)}$ and $\Phi_{22}^{(1)}, \Phi_{32}^{(1)}$. Furthermore, equation (A.4) and Cauchy theorem imply

$$-\frac{i\pi}{2}\Phi_{22}^{(1)}(t) = \int_{\partial D_2} \left(\Phi_{22}(t,\lambda)-1\right)d\lambda = \int_{\partial D_4} \left(\Phi_{22}(t,\lambda)-1\right)d\lambda.$$

$$-\frac{i\pi}{2}\Phi_{32}^{(1)}(t) = \int_{\partial D_2} \Phi_{32}(t,\lambda)d\lambda = \int_{\partial D_4} \Phi_{32}(t,\lambda)d\lambda.$$

and

$$-\frac{i\pi}{2}\Phi_{12}^{(2)}(t) = \int_{\partial D_2} \left(\lambda\Phi_{12}(t,\lambda)-\Phi_{12}^{(1)}(t)\right)d\lambda = \int_{\partial D_4} \left(\lambda\Phi_{12}(t,\lambda)-\Phi_{12}^{(1)}(t)\right)d\lambda,$$

Thus,

$$i\pi\Phi_{22}^{(1)} = \int_{\partial D_3} \Phi_{22-}(t,\lambda)d\lambda, \qquad i\pi\Phi_{32}^{(1)} = \int_{\partial D_3} \Phi_{32-}(t,\lambda)d\lambda, \qquad (4.33)$$

$$\begin{aligned}
i\pi\Phi_{12}^{(2)}(t) &= -\left(\int_{\partial D_2}+\int_{\partial D_4}\right)\left[\lambda\Phi_{12}(t,\lambda)-\Phi_{12}^{(1)}(t)\right]d\lambda \\
&= \left(\int_{\partial D_3}+\int_{\partial D_1}\right)\left[\lambda\Phi_{12}(t,\lambda)-\Phi_{12}^{(1)}(t)\right]d\lambda \\
&= \int_{\partial D_3}\left[\lambda\Phi_{12}(t,\lambda)-\Phi_{12}^{(1)}(t)\right]_- d\lambda \\
&= \int_{\partial D_3^0}\left\{\lambda\Phi_{12}(t,\lambda)-\frac{g_{01}}{2i}+\frac{2e^{-2i\lambda L}}{\Delta}[\lambda\Phi_{12}(t,\lambda)-\frac{g_{01}}{2i}]\right\}_- d\lambda + I(t).
\end{aligned} \qquad (4.34)$$

where $I(t)$ is defined by

$$I(t) = -\int_{\partial D_3^0}\left\{\frac{2e^{-2i\lambda L}}{\Delta}[\lambda\Phi_{12}(t,\lambda)-\frac{g_{01}}{2i}]\right\}_- d\lambda.$$

The last step involves using the global relation (4.14a) to compute $I(t)$, that is

$$\begin{aligned}
I(t) &= \int_{\partial D_3^0}\left\{-\frac{2e^{-2i\lambda L}}{\Delta}\left[\lambda c_{12}-\Phi_{12}^{(1)}-\frac{\Phi_{12}^{(1)}\bar{\phi}_{22}^{(1)}+\Phi_{13}^{(1)}\bar{\phi}_{23}^{(1)}}{\lambda}-\bar{\phi}_{21}^{(1)}e^{2i\lambda L}\right]\right\}_- d\lambda \\
&\quad +\int_{\partial D_3^0}\left\{-\frac{2e^{-2i\lambda L}}{\Delta}\left[\frac{\Phi_{12}^{(1)}\bar{\phi}_{22}^{(1)}+\Phi_{13}^{(1)}\bar{\phi}_{23}^{(1)}}{\lambda}-(\lambda\bar{\phi}_{21}-\bar{\phi}_{21}^{(1)})e^{2i\lambda L}\right]\right\}_- d\lambda \qquad (4.35) \\
&\quad +\int_{\partial D_3^0}\left\{\frac{2e^{-2i\lambda L}}{\Delta}\left[\bar{\phi}_{21}(\Phi_{11}-1)e^{2i\lambda L}+(\bar{\phi}_{22}-1)\Phi_{12}+\bar{\phi}_{23}\Phi_{13}\right]\right\}_- d\lambda.
\end{aligned}$$

Using the asymptotic (4.13a) and Cauchy theorem to compute the first term on the right-hand side of equation (A.13), we find

$$\begin{aligned}
I(t) &= -i\pi\Phi_{12}^{(2)}(t)+\int_{\partial D_3^0}(\frac{g_{01}}{2i}\bar{\phi}_{22-}+\frac{g_{02}}{2i}\bar{\phi}_{23-})d\lambda+\int_{\partial D_3^0}\frac{2}{\Delta}(\lambda\bar{\phi}_{21-}-2\bar{\phi}_{21}^{(1)})]d\lambda \\
&\quad +\int_{\partial D_3^0}\frac{2\lambda}{\Delta}\left[\bar{\phi}_{21}(\Phi_{11}-1)e^{2i\lambda L}+(\bar{\phi}_{22}-1)\Phi_{12}e^{-2i\lambda L}+\bar{\phi}_{23}\Phi_{13}e^{-2i\lambda L}\right]_- d\lambda.
\end{aligned} \qquad (4.36)$$

Equations (4.34) and (A.14) imply

$$\Phi_{12}^{(2)}(t) = \frac{1}{2i\pi} \int_{\partial D_3^0} \frac{\Sigma}{\Delta}(\lambda \Phi_{12-} + ig_{01})d\lambda - \frac{1}{4\pi} \int_{\partial D_3^0} (g_{01}\bar{\phi}_{22-} + g_{02}\bar{\phi}_{23-})d\lambda$$
$$+ \frac{1}{i\pi} \int_{\partial D_3^0} \frac{1}{\Delta}(\lambda \bar{\phi}_{21-} - 2\bar{\phi}_{21}^{(1)})]d\lambda$$
$$+ \frac{1}{i\pi} \int_{\partial D_3^0} \frac{2\lambda}{\Delta}\left[\bar{\phi}_{21}(\Phi_{11}-1) + (\bar{\phi}_{22}-1)\Phi_{12}e^{-2i\lambda L} + \bar{\phi}_{23}\Phi_{13}e^{-2i\lambda L}\right]_- d\lambda.$$
$$(4.37)$$

Equations (4.33) and (4.37) together with (4.8b) yield (4.29a). Similarly, we can prove (4.29b).

The expressions (4.30a) for $f_{11}(t)$ can be derived in a similar way. Indeed, we note that equation (4.11b) expresses f_{11} in terms of $\phi_{12}^{(2)}$ and $\phi_{22}^{(1)}, \phi_{32}^{(1)}$. These three equations satisfy the analog of equations (4.33) and (4.34). In particular, $\phi_{21}^{(2)}$ satisfies

$$i\pi\phi_{12}^{(2)} = -\int_{\partial D_3^0}\left(\frac{\Sigma}{\Delta}(\lambda \phi_{12-} - 2\phi_{12}^{(1)})\right)d\lambda + J(t),\qquad (4.38)$$

where

$$J(t) = \int_{\partial D_3^0}\left\{\frac{2e^{2i\lambda L}}{\Delta}[\lambda \phi_{12}(t,\lambda) - \frac{f_{01}(t)}{2i}]\right\}_- d\lambda.$$

Then using the global relation to compute $J(t)$, that is

$$J(t) = -i\pi\phi_{12}^{(2)}(t) + \int_{\partial D_3^0}(\frac{f_{01}}{2i}\bar{\Phi}_{22-} + \frac{f_{02}}{2i}\bar{\Phi}_{23-})d\lambda - \int_{\partial D_3^0}\frac{2}{\Delta}(\lambda\bar{\Phi}_{21-} - 2\bar{\Phi}_{21}^{(1)})]d\lambda$$
$$- \int_{\partial D_3^0}\frac{2\lambda}{\Delta}\left[\bar{\Phi}_{21}(\phi_{11}-1) + (\bar{\Phi}_{22}-1)\phi_{12}e^{2i\lambda L} + \bar{\Phi}_{23}\phi_{13}e^{2i\lambda L}\right]_- d\lambda.$$
$$(4.39)$$

The equation (4.38) and (4.39) together with the asymptotics of $c_{12}(t,\lambda)$ yield (4.30a). The proof of (4.30b) is similar.

(ii) In order to derive the representations (4.31a) relevant for the Neumann problem, we note that equation (4.8a) expresses g_{01} and g_{02} in terms of $\Phi_{12}^{(1)}$ and $\Phi_{13}^{(1)}$, respectively. Furthermore, equation (A.4) and Cauchy's theorem imply

$$-\frac{i\pi}{2}\Phi_{12}^{(1)}(t) = \int_{\partial D_2}\Phi_{12}(t,\lambda)d\lambda = \int_{\partial D_4}\Phi_{12}(t,\lambda)d\lambda,$$

Thus,

$$i\pi\Phi_{12}^{(1)}(t) = \left(\int_{\partial D_3} + \int_{\partial D_1}\right)\Phi_{12}(t,\lambda)d\lambda$$
$$= \int_{\partial D_3}\Phi_{12-}(t,\lambda)d\lambda \qquad (4.40)$$
$$= \int_{\partial D_3}\left(\frac{\Sigma}{\Delta}\Phi_{12+}(t,\lambda)\right)d\lambda + K(t),$$

where

$$K(t) = -\int_{\partial D_3^0} \frac{2}{\Delta} \left(e^{-2i\lambda L} \Phi_{12}(t,\lambda) \right)_+ d\lambda,$$

using the global relation and the asymptotic formulas of $c_{21}(t,\lambda)$, we have

$$K(t) = -i\pi \Phi_{12}^{(1)}(t) + 2\int_{\partial D_3^0} \left\{ \frac{1}{\Delta} \bar{\phi}_{21+} + \frac{1}{\Delta} \left[\bar{\phi}_{21}(\Phi_{11}-1)e^{2i\lambda L} + (\bar{\phi}_{22}-1)\Phi_{12} + \bar{\phi}_{23}\Phi_{13} \right]_+ \right\} d\lambda.$$

(4.41)

Equations (4.8a), (4.40) and (4.41) yields (4.31a). The proof of the other formulas is similar.

□

4.3. *Effective characterizations*

Substituting into the system (4.26), (4.27) and (4.28) the expressions

$$\Phi_{ij} = \Phi_{ij,0} + \varepsilon \Phi_{ij,1} + \varepsilon^2 \Phi_{ij,2} + \cdots, \quad i,j = 1,2,3.$$

(4.42a)

$$\phi_{ij} = \phi_{ij,0} + \varepsilon \phi_{ij,1} + \varepsilon^2 \phi_{ij,2} + \cdots, \quad i,j = 1,2,3.$$

(4.42b)

$$g_{01} = \varepsilon g_{01}^{(1)} + \varepsilon^2 g_{01}^{(2)} + \cdots, g_{02} = \varepsilon g_{02}^{(1)} + \varepsilon^2 g_{02}^{(2)} + \cdots,$$

(4.42c)

$$f_{01} = \varepsilon f_{01}^{(1)} + \varepsilon^2 f_{01}^{(2)} + \cdots, f_{02} = \varepsilon f_{02}^{(1)} + \varepsilon^2 f_{02}^{(2)} + \cdots,$$

(4.42d)

$$g_{11} = \varepsilon g_{11}^{(1)} + \varepsilon^2 g_{11}^{(2)} + \cdots, g_{12} = \varepsilon g_{12}^{(1)} + \varepsilon^2 g_{12}^{(2)} + \cdots,$$

(4.42e)

$$f_{11} = \varepsilon f_{11}^{(1)} + \varepsilon^2 f_{11}^{(2)} + \cdots, f_{12} = \varepsilon f_{12}^{(1)} + \varepsilon^2 f_{12}^{(2)} + \cdots,$$

(4.42f)

where $\varepsilon > 0$ is a small parameter, we find that the terms of $O(1)$ give

$$O(1) : \begin{cases} \Phi_{13,0} = 0 \ \Phi_{23,0} = 0 \ \Phi_{33,0} = 1, \\ \Phi_{11,0} = 1 \ \Phi_{21,0} = 0 \ \Phi_{31,0} = 0, \\ \Phi_{12,0} = 0 \ \Phi_{22,0} = 1 \ \Phi_{32,0} = 0. \end{cases}$$

(4.43)

Moreover, the terms of $O(\varepsilon)$ give

$$O(\varepsilon) : \begin{cases} \Phi_{33,1} = 0 \quad \Phi_{23,1} = 0, \\ \Phi_{13,1}(t,k) = \int_0^t e^{-4i\lambda^2(t-t')}(2\lambda g_{02}^{(1)} + ig_{12}^{(1)})(t')dt', \\ \Phi_{11,1} = 0, \\ \Phi_{21,1} = \int_0^t e^{4i\lambda^2(t-t')}(i\lambda \bar{g}_{11}^{(1)})(t')dt', \\ \Phi_{31,1} = \int_0^t e^{4i\lambda^2(t-t')}(i\lambda \bar{g}_{12}^{(1)})(t')dt', \\ \Phi_{12,1} = \int_0^t e^{-4i\lambda^2(t-t')}(2\lambda g_{01}^{(1)} + ig_{11}^{(1)})(t')dt', \\ \Phi_{22,1} = 0, \quad \Phi_{32,1} = 0. \end{cases}$$

(4.44)

the terms of $O(\varepsilon^2)$ give

$$
O(\varepsilon^2): \begin{cases}
\Phi_{13,2} = \int_0^t e^{-4i\lambda^2(t-t')}(2\lambda g_{02}^{(2)} + i\lambda g_{12}^{(2)})(t')dt', \\[4pt]
\Phi_{23,2} = \int_0^t \left[i\lambda \bar{g}_{11}^{(1)}(t')\Phi_{13,1}(t',k) - \tfrac{1}{2}\bar{g}_{11}^{(1)}g_{02}^{(1)})(t') \right] dt', \\[4pt]
\Phi_{33,2} = \int_0^t \left[i\lambda \bar{g}_{12}^{(1)})(t')\Phi_{13,1}(t',k) - \tfrac{1}{2}\bar{g}_{12}^{(1)}g_{02}^{(1)}(t') \right] dt', \\[4pt]
\Phi_{11,2} = \int_0^t \left[\tfrac{1}{2}(\bar{g}_{11}^{(1)}g_{01}^{(1)} + \bar{g}_{12}^{(1)}g_{02}^{(1)})(t') + (2\lambda g_{01}^{(1)} + ig_{11}^{(1)})\Phi_{21,1}(t',\lambda) + (2\lambda g_{02}^{(1)} + ig_{12}^{(1)})\Phi_{31,1}(t',\lambda) \right] dt', \\[4pt]
\Phi_{21,2} = \int_0^t e^{4i\lambda^2(t-t')}(i\lambda \bar{g}_{11}^{(2)})(t')dt', \\[4pt]
\Phi_{31,2} = \int_0^t e^{4i\lambda^2(t-t')}(i\lambda \bar{g}_{12}^{(2)})(t')dt', \\[4pt]
\Phi_{12,2} = \int_0^t e^{-4i\lambda^2(t-t')}(2\lambda g_{01}^{(2)} + ig_{11}^{(2)})(t')dt', \\[4pt]
\Phi_{22,2} = \int_0^t \left[(i\lambda \bar{g}_{11}^{(1)}(t')\Phi_{12,1}(t',k) - \tfrac{1}{2}\bar{g}_{11}^{(1)}g_{01}^{(1)})(t') \right] dt', \\[4pt]
\Phi_{32,2} = \int_0^t \left[(i\lambda \bar{g}_{12}^{(1)}(t')\Phi_{12,1}(t',k) - \tfrac{1}{2}\bar{g}_{12}^{(1)}g_{01}^{(1)})(t') \right] dt'.
\end{cases}
\tag{4.45}
$$

Similarly, we will have the analogue formulas for $\{\phi_{ij,l}\}_{i,j=1}^3, l = 0,1,2$ expressed in terms of the boundary data at $x = L$, that is $\{f_{ij}^{(l)}\}_{i=0,1}^{j=1,2}, l = 1,2$.

On the other hand, expanding (4.29), (4.30) and assuming for simplicity that $m_{11}(\mathscr{A})(\lambda)$ has no zeros, we find

$$
g_{11}^{(1)}(t) = \frac{2}{i\pi}\int_{\partial D_3^0}(\lambda\Phi_{12,1-}(t,\lambda) + ig_{01}^{(1)})d\lambda + \frac{4}{i\pi}\int_{\partial D_3^0}\frac{1}{\Delta}(\lambda\bar{\phi}_{21,1-}(t,\lambda) - 2\bar{\phi}_{21}^{(1)})d\lambda, \tag{4.46a}
$$

$$
g_{12}^{(1)}(t) = \frac{2}{i\pi}\int_{\partial D_3^0}(\lambda\Phi_{13,1-}(t,\lambda) + ig_{02}^{(1)})d\lambda + \frac{4}{i\pi}\int_{\partial D_3^0}\frac{1}{\Delta}(\lambda\bar{\phi}_{31,1-}(t,\lambda) - 2\bar{\phi}_{31}^{(1)})d\lambda, \tag{4.46b}
$$

$$
f_{11}^{(1)}(t) = -\frac{2}{i\pi}\int_{\partial D_3^0}(\lambda\phi_{12,1-}(t,\lambda) + if_{01}^{(1)})d\lambda - \frac{4}{i\pi}\int_{\partial D_3^0}\frac{1}{\Delta}(\lambda\bar{\Phi}_{21,1-}(t,\lambda) - 2\bar{\Phi}_{21}^{(1)})d\lambda, \tag{4.46c}
$$

$$
f_{12}^{(1)}(t) = -\frac{2}{i\pi}\int_{\partial D_3^0}(\lambda\phi_{13,1-}(t,\lambda) + if_{02}^{(1)})d\lambda - \frac{4}{i\pi}\int_{\partial D_3^0}\frac{1}{\Delta}(\lambda\bar{\Phi}_{31,1-}(t,\lambda) - 2\bar{\Phi}_{31}^{(1)})d\lambda, \tag{4.46d}
$$

we also find that

$$
\begin{aligned}
\Phi_{12,1-} &= 4\lambda\int_0^t e^{-4i\lambda^2(t-t')}g_{01}^{(1)}(t')dt', \\
\Phi_{13,1-} &= 4\lambda\int_0^t e^{-4i\lambda^2(t-t')}g_{02}^{(1)}(t')dt', \\
\phi_{21,1-} &= 2i\lambda\int_0^t e^{4i\lambda^2(t-t')}\bar{f}_{11}^{(1)}(t')dt', \\
\phi_{31,1-} &= 2i\lambda\int_0^t e^{4i\lambda^2(t-t')}\bar{f}_{12}^{(1)}(t')dt'.
\end{aligned}
\tag{4.47}
$$

The Dirichlet problem can now be solved perturbatively as follows: assuming for simplicity that $m_{11}(\mathscr{A})(\lambda)$ has no zeros and given $g_{01}^{(1)}, g_{02}^{(1)}$ and $\bar{f}_{11}^{(1)}, \bar{f}_{12}^{(1)}$, we can use equation (4.47) to determine $\Phi_{1j,1-}, \phi_{j1,1-}, j = 2,3$. We can then compute $g_{11}^{(1)}, g_{12}^{(1)}$ from (4.46a), (4.46b) and then $\Phi_{1j,1}, j = 2,3$ from (4.44) and the analogue results for $\phi_{j1,1}, j = 2,3$. In the same way we can determine $\Phi_{1j,2}, j = 2,3$ from (4.45) and the analogue results for $\phi_{j1,2}, j = 2,3$, then compute $g_{11}^{(2)}, g_{12}^{(2)}$ and $f_{11}^{(2)}, f_{12}^{(2)}$.

These arguments can be extended to the higher order and also can be extended to the systems (4.26), (4.27) and (4.28) thus yields a constructive scheme for computing $S(k)$ to all orders. The construction of $S_L(\lambda)$ is similar.

Similarly, these arguments also can be used to the Neumann problem. That is to say, in all cases, the system can be solved perturbatively to all orders.

4.4. *The large L limit*

In the limit $L \to \infty$, the representations for $g_{11}(t), g_{12}(t)$ and $g_{01}(t), g_{02}(t)$ of theorem 4.3 reduce to the corresponding representations on the half-line. Indeed, as $L \to \infty$,

$$f_{01} \to 0, \quad f_{02} \to 0, \quad f_{11} \to 0, \quad f_{12} \to 0,$$
$$\phi_{ij} \to \delta_{ij}, \quad \tfrac{\Sigma}{\Delta} \to 1 \text{ as } \lambda \to \infty \text{ in } D_3$$

Thus, the $L \to \infty$ limits of the representations (4.29a), (4.29b) and (4.31a), (4.31b) are

$$g_{11}(t) = \tfrac{2}{i\pi} \int_{\partial D_3^0} (\lambda \Phi_{12-} + i g_{01}) d\lambda + \tfrac{2}{\pi} \int_{\partial D_3^0} (g_{01}\Phi_{22-} + g_{02}\Phi_{32-}) d\lambda,$$
$$g_{12}(t) = \tfrac{2}{i\pi} \int_{\partial D_3^0} (\lambda \Phi_{13-} + i g_{02}) d\lambda + \tfrac{2}{\pi} \int_{\partial D_3^0} (g_{01}\Phi_{23-} + g_{02}\Phi_{33-}) d\lambda. \tag{4.48}$$

and

$$g_{01}(t) = \tfrac{1}{\pi} \int_{\partial D_3^0} \Phi_{12+} d\lambda, \quad g_{02}(t) = \tfrac{1}{\pi} \int_{\partial D_3^0} \Phi_{13+} d\lambda, \tag{4.49}$$

respectively. And these formulas coincide with the corresponding half-line formulas, see (A.9), (A.10).

Appendix A. Some formulas on the half-line

For the convenience of reader, we show the half-line formulas of $g_{11}(t), g_{12}(t)$ and $g_{01}(t), g_{02}(t)$ on the $\lambda-$plane.

From the global relation (2.50)and replacing T by t, we find

$$\mu_2(0,t,\lambda) e^{2i\lambda^2 t \hat{\Lambda}} s(\lambda) = c(t,\lambda), \quad \lambda \in (D_3 \cup D_4, D_1 \cup D_2, D_1 \cup D_2). \tag{A.1}$$

We partition matrix as following,

$$\mu_2(0,t,\lambda) = \begin{pmatrix} \Phi_{11} & \Phi_{1j} \\ \Phi_{j1} & \Phi_{2\times 2} \end{pmatrix}, \quad j = 2, 3, \tag{A.2}$$

where $\Phi_{2\times 2}$ denotes a 2×2 matrix, Φ_{1j} denotes a 1×2 vector, Φ_{j1} denotes a 2×1 vector. Then, we can write the second column of the global relation, undering the matrix partitioned as (A.2), as

$$\Phi_{11}(t,\lambda) s_{1j}(\lambda) s_{2\times 2}^{-1}(\lambda) e^{-4i\lambda^2 t} + \Phi_{1j}(t,\lambda) = c_{1j}(t,\lambda), \quad \lambda \in D_1 \cup D_2, \tag{A.3a}$$

$$\Phi_{j1}(t,\lambda) s_{1j}(\lambda) s_{2\times 2}^{-1}(\lambda) e^{-4i\lambda^2 t} + \Phi_{2\times 2}(t,\lambda) = c_{2\times 2}(t,\lambda), \quad \lambda \in D_1 \cup D_2, \tag{A.3b}$$

The functions $c_{1j}(t,\lambda), c_{2\times 2}(t,\lambda)$ are analytic and bounded in $D_1 \cup D_2$ away from the possible zeros of $m_{11}(\lambda)$ and of order $O(\tfrac{1}{\lambda})$ as $k \to \infty$.

From the asymptotic of $\mu_j(x,t,\lambda)$ in (4.1) we have

$$\mu_2(0,t,\lambda) = \mathbb{I} + \frac{1}{\lambda} \begin{pmatrix} \int_{(0,0)}^{(0,t)} \Delta_{11} dx' + \eta_{11} dt' & \frac{1}{2i}Q \\ \frac{1}{8i}|q|^2 Q^T - \frac{1}{4}\bar{Q}_x^T & \int_{(0,0)}^{(0,t)} \Delta dx' + \eta dt' \end{pmatrix}$$
$$+ \frac{1}{\lambda^2} \begin{pmatrix} \mu_{11}^{(2)} & \frac{1}{4}Q_x + \frac{1}{2i}Q\mu_{2\times 2}^{(1)} \\ \mu_{j1}^{(2)} & \mu_{2\times 2}^{(2)} \end{pmatrix} + O(\frac{1}{\lambda^3}) \tag{A.4}$$

where $Q = (q_1, q_2)$, Δ_{11} is defined by first identities of (4.3a), η_{11} is defined by (4.3b), Δ and η are 2×2 matrices defined as following,

$$\Delta = \begin{pmatrix} \Delta_{22} & \Delta_{23} \\ \Delta_{32} & \Delta_{33} \end{pmatrix}, \qquad \eta = \begin{pmatrix} \eta_{22} & \eta_{23} \\ \eta_{32} & \eta_{33} \end{pmatrix}, \tag{A.5}$$

Also, we have

$$\Phi_{1j}(t,\lambda) = \frac{\Phi_{1j}^{(1)}(t)}{\lambda} + \frac{\Phi_{1j}^{(2)}(t)}{\lambda^2} + O(\frac{1}{\lambda^3}), \quad \lambda \to \infty, \lambda \in D_1 \cup D_2 \tag{A.6a}$$

$$\Phi_{2\times 2}(t,\lambda) = \mathbb{I}_{2\times 2} + \frac{\Phi_{2\times 2}^{(1)}(t)}{\lambda} + \frac{\Phi_{2\times 2}^{(2)}(t)}{\lambda^2} + O(\frac{1}{\lambda^3}), \quad \lambda \to \infty, \lambda \in D_1 \cup D_2. \tag{A.6b}$$

where

$$\Phi_{1j}^{(1)}(t) = \frac{1}{2i}g_0(t), \quad \Phi_{1j}^{(2)}(t) = \frac{1}{4}g_1(t) - \frac{i}{2}g_0 \Phi_{2\times 2}^{(1)}(t)$$
$$\Phi_{2\times 2}^{(1)}(t) = \int_0^t \eta \, dt'.$$

here $g_0(t)$ and $g_1(t)$ are vector boundary functions defined by the boundary data of (1.3) as $g_0(t) = (g_{01}(t), g_{02}(t))$ and $g_1(t) = (g_{11}(t), g_{12}(t))$.

In particular, we find the following expressions for the boudary values:

$$g_0 = 2i\Phi_{1j}^{(1)}(t), \tag{A.7a}$$

$$g_1 = 2ig_0 \Phi_{2\times 2}^{(1)}(t) + 4\Phi_{1j}^{(2)}(t), \tag{A.7b}$$

We will also need the asymptotic of $c_{1j}(t,\lambda)$,

Lemma A.1. *The global relation (A.3) implies that the large λ behavior of $c_{1j}(t,\lambda), c_{2\times 2}(t,\lambda)$ satisfies*

$$c_{1j}(t,\lambda) = \frac{\Phi_{1j}^{(1)}(t)}{\lambda} + \frac{\Phi_{1j}^{(2)}(t)}{\lambda^2} + O(\frac{1}{\lambda^3}), \quad \lambda \to \infty, \lambda \in D_1. \tag{A.8}$$

Proof. Analogous to the proof provided in Lemma 4.2. $\qquad \square$

We can now derive the maps between Dirichlet boundary condition and the Neumann boundary condition as follows:

(i) For the Dirichlet problem, the unknown Neumann boundary value $g_1(t)$ is given by

$$g_1(t) = \frac{2}{\pi i} \int_{\partial D_3} (\lambda \Phi_{1j-}(t,\lambda) + ig_0(t)) + \frac{2g_0}{\pi} \int_{\partial D_3} \Phi_{2\times 2-} d\lambda$$
$$- \frac{4}{i\pi} \int_{\partial D_3} \lambda e^{-4i\lambda^2 t} \Phi_{11}(-\lambda) s_{1j}(-\lambda) s_{2\times 2}^{-1}(-\lambda) d\lambda. \tag{A.9}$$

(ii) For the Neumann problem, the unknown boundary values $g_0(t)$ is given by

$$g_0(t) = \frac{1}{\pi} \int_{\partial D_3} \Phi_{1j+}(t,\lambda) d\lambda + \frac{2}{\pi} \int_{\partial D_3} e^{-4i\lambda^2 t} \Phi_{11}(-\lambda) s_{1j}(-\lambda) s_{2\times 2}^{-1}(-\lambda) d\lambda. \tag{A.10}$$

Proof.

(i) In order to derive (A.9) we note that equation (A.7b) expresses g_1 in terms of $\Phi_{2\times 2}^{(1)}$ and $\Phi_{1j}^{(2)}$. Furthermore, equation (A.6) and Cauchy theorem imply

$$-\frac{\pi i}{2} \Phi_{2\times 2}^{(1)}(t) = \int_{\partial D_2} [\Phi_{2\times 2}(t,\lambda) - \mathbb{I}_{2\times 2}] d\lambda = \int_{\partial D_4} [\Phi_{2\times 2}(t,\lambda) - \mathbb{I}_{2\times 2}] d\lambda$$

and

$$-\frac{\pi i}{2} \Phi_{1j}^{(2)}(t) = \int_{\partial D_2} \left[\lambda \Phi_{1j}(t,\lambda) - \frac{g_0(t)}{2i} \right] d\lambda = \int_{\partial D_4} \left[\lambda \Phi_{1j}(t,\lambda) - \frac{g_0(t)}{2i} \right] d\lambda.$$

Thus,

$$\begin{aligned} i\pi \Phi_{2\times 2}^{(1)}(t) &= -\left(\int_{\partial D_2} + \int_{\partial D_4} \right) [\Phi_{2\times 2}(t,\lambda) - \mathbb{I}_{2\times 2}] d\lambda \\ &= \left(\int_{\partial D_1} + \int_{\partial D_3} \right) [\Phi_{2\times 2}(t,\lambda) - \mathbb{I}_{2\times 2}] d\lambda \\ &= \int_{\partial D_3} [\Phi_{2\times 2}(t,\lambda) - \mathbb{I}_{2\times 2}] d\lambda - \int_{\partial D_3} [\Phi_{2\times 2}(t,-\lambda) - \mathbb{I}_{2\times 2}] d\lambda \\ &= \int_{\partial D_3} \Phi_{2\times 2-}(t,\lambda) d\lambda. \end{aligned} \tag{A.11}$$

Similarly,

$$\begin{aligned} i\pi \Phi_{1j}^{(2)}(t) &= \left(\int_{\partial D_3} + \int_{\partial D_1} \right) \left[\lambda \Phi_{1j}(t,\lambda) - \frac{g_0(t)}{2i} \right] d\lambda \\ &= \left(\int_{\partial D_3} - \int_{\partial D_1} \right) \left[\lambda \Phi_{1j}(t,-\lambda) - \frac{g_0(t)}{2i} \right] d\lambda + I(t) \\ &= \int_{\partial D_3} [\lambda \Phi_{1j-}(t,\lambda) + ig_0(t)] d\lambda + I(t). \end{aligned} \tag{A.12}$$

where $I(t)$ is defined by

$$I(t) = 2 \int_{\partial D_1} \left[\lambda \Phi_{1j}(t,\lambda) - \frac{g_0(t)}{2i} \right] d\lambda$$

The last step involves using the global relation to compute $I(t)$

$$I(t) = 2 \int_{\partial D_1} \left[\lambda (c_{1j} s_{2\times 2}^{-1} - \Phi_{11} s_{1j} s_{2\times 2}^{-1} e^{-4i\lambda^2 t}) - \frac{g_0(t)}{2i} \right] d\lambda \tag{A.13}$$

Using the asymptotic (A.8) and Cauchy theorem to compute the first term on the right-hand side of equation (A.13), we find

$$I(t) = -i\pi \Phi_{1j}^{(2)} - 2 \int_{\partial D_3} \lambda \Phi_{11}(-\lambda) s_{1j}(-\lambda) s_{2\times 2}^{-1}(-\lambda) e^{-4i\lambda^2 t} d\lambda. \tag{A.14}$$

Equations (A.12) and (A.14) imply

$$\Phi_{1j}^{(2)}(t) = \frac{1}{2\pi i} \int_{\partial D_3} [\lambda \Phi_{1j-}(t,\lambda) + ig_0(t)] \, d\lambda$$
$$- \frac{1}{\pi i} \int_{\partial D_3} \lambda \Phi_{11}(-\lambda) s_{1j}(-\lambda) s_{2\times 2}^{-1}(-\lambda) e^{-4i\lambda^2 t} d\lambda.$$

This equation together with (A.7b) and (A.11) yields (A.9).

(ii) In order to derive the representations (A.10) relevant for the Neumann problem, we note that equation (A.7a) expresses g_0 in terms of $\Phi_{1j}^{(1)}$. Furthermore, equation (A.6a) and Cauchy's theorem imply

$$-\frac{\pi i}{2} \Phi_{1j}^{(1)}(t) = \int_{\partial D_2} \Phi_{1j}(t,\lambda) d\lambda = \int_{\partial D_4} \Phi_{1j}(t,\lambda) d\lambda, \tag{A.15}$$

Thus,

$$i\pi \Phi_{1j}^{(1)}(t) = \left(\int_{\partial D_3} + \int_{\partial D_1} \right) \Phi_{1j}(t,\lambda) d\lambda$$
$$= \left(\int_{\partial D_3} - \int_{\partial D_1} \right) \Phi_{1j}(t,\lambda) d\lambda + 2 \int_{\partial D_1} \Phi_{1j}(t,\lambda) d\lambda \tag{A.16}$$
$$= \int_{\partial D_3} \Phi_{1j+}(t,\lambda) d\lambda + 2 \int_{\partial D_1} \Phi_{1j}(t,\lambda) d\lambda,$$

and using the global relation, we have

$$2 \int_{\partial D_1} \Phi_{1j}(t,\lambda) d\lambda = 2 \int_{\partial D_1} (c_{1j} s_{2\times 2}^{-1} - \Phi_{11} s_{1j} s_{2\times 2}^{-1} e^{-4i\lambda^2 t}) d\lambda$$
$$= -i\pi \Phi_{1j}^{(1)}(t) + 2 \int_{\partial D_3} \Phi_{11}(-\lambda) s_{1j}(-\lambda) s_{2\times 2}^{-1}(-\lambda) e^{-4i\lambda^2 t} d\lambda. \tag{A.17}$$

Equations (A.7a), (A.16) and (A.17) yields (A.10).

$\square$

Acknowledgements

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