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Riemann-Hilbert approach and N -soliton formula for a higher-order Chen-Lee-Liu equation

Juan Hu

*Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou 310023, P.R. China
hujuan@lsec.cc.ac.cn*

Jian Xu

*College of Science, University of Shanghai for Science and Technology, Shanghai 200093, P.R. China
jianxu@usst.edu.cn*

Guo-Fu Yu*

*School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, P.R. China
gfyu@sjtu.edu.cn*

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We consider a higher-order Chen-Lee-Liu (CLL) equation with third order dispersion and quintic nonlinearity terms. In the framework of the Riemann-Hilbert method, we obtain the compact N -soliton formula expressed by determinants. Based on the determinant solution, some properties for single soliton and asymptotic analysis of N -soliton solution are explored. The simple elastic interaction of N solitons is confirmed.

Keywords: Higher-order Chen-Lee-Liu equation; Riemann-Hilbert method; N -soliton.

2010 Mathematics Subject Classification: 35Q55, 35C08, 35C11, 37K40

1. Introduction

The nonlinear Schrödinger equation (NLS) is an important integrable model that governs weakly nonlinear and dispersive wave packets in one-dimensional physical systems. It plays an important role in wide range of physical subjects, such as nonlinear water waves, nonlinear optics and plasma physics. The NLS equation is low order approximation model of nonlinear effects in optical fibers. To get a more accurate approximation of the higher-order nonlinear effects, a natural approach is to introduce additional higher-order nonlinear terms in the model. Hirota equation, Kundu-Eckhaus equation, and Lakshmanan-Porsezian-Daniel equation are all extension of NLS equation with higher-order dispersion and nonlinear terms.

To study the effect of higher-order perturbations, various modifications and generalizations of the NLS equation have been proposed. Among them, there are three celebrated equations with derivative-type nonlinearities, which are called the derivative NLS (DNLS) equation. The first two DNLS equations are analogues of the NLS equation with second order dispersion and cubic nonlinearity and the third DNLS equation possesses second order dispersion and quintic nonlinear term.

*Corresponding author.

The first DNLS equation is the Kaup-Newell (KN) equation [11],

$$iu_t + u_{xx} - i(|u|^2 u)_x = 0, \quad (1.1)$$

which is a canonical dispersive equation derived from the Magneto-hydrodynamic equations in the presence of the Hall effect and usually called DNLS (I). Under the gauge transformation,

$$q(x, t) = u(x, t) \exp\left(\frac{i}{2} \int_{-\infty}^x |u(y, t)|^2 dy\right),$$

the KN equation (1.1) becomes

$$iq_t + q_{xx} + i|q|^2 q_x = 0, \quad (1.2)$$

which appears in optical models of ultrashort pulses and is also referred to as the Chen-Lee-Liu (CLL) equation [2] and called DNLS (II).

The third one takes the form

$$iq_t + q_{xx} - iq^2 q_x^* + \frac{1}{2}|q|^4 q = 0,$$

which is called the Gerdjikov-Ivanov (GI) equation or DNLS (III) [7]. The unified expression of KN, CLL and GI equations was presented in [4].

Like the NLS equation, the DNLS (II) equation is also a real physical model in optics. In 2007, Moses et al [14] proved optical pulse propagation involving self-steepening without self-phase-modulation. This experiment provide the first experimental evidence of the DNLS (II) equation. The importance of the higher-order nonlinear effects in nonlinear optics and other fields motivates us to consider an integrable model that possesses third dispersion and quintic nonlinearity.

In this paper, we consider the higher-order generalized CLL equation with third dispersion and quintic nonlinear term,

$$q_t + q_{xxx} + \frac{3}{2}i|q|^2 q_{xx} - \frac{3}{4}|q|^4 q_x + \frac{3}{2}iq_x^2 q^* = 0. \quad (1.3)$$

This equation can be derived from the generalized KN hierarchy under $n = 2$ and proper parameter. The Liouville integrability and multi-Hamiltonian structure for the higher-order CLL equation (1.3) are investigated in [4]. The higher-order CLL equation is also Lax integrable with the linear spectral problem

$$Y_x = UY = \left(-i\sigma_3\lambda^2 + \sigma_3 Q\lambda + \frac{1}{4}i\sigma_3 Q^2\right)Y, \quad (1.4)$$

$$Y_t = VY = \left(-4i\sigma_3\lambda^6 + 4\sigma_3 Q\lambda^5 + 2i\sigma_3 Q^2\lambda^4 + 2Z_0\lambda^3 + Z_1\lambda^2 + Z_2\lambda + \frac{1}{4}Z_3\right)Y, \quad (1.5)$$

where

$$\begin{aligned} \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \\ Z_0 &= iQ_x - \frac{1}{2}\sigma_3 Q^3, \quad Z_1 = [Q, Q_x] - \frac{1}{2}i\sigma_3 Q^4, \\ Z_2 &= Q_{xx}\sigma_3 + \frac{1}{2}iQQ_xQ - \frac{3}{2}iQ^2Q_x + \frac{1}{4}\sigma_3 Q^5, \\ Z_3 &= \frac{1}{4}i\sigma_3 Q^6 + \frac{3}{2}Q^2[Q_x, Q] - i\sigma_3(Q_{xx}Q + QQ_{xx} - Q_x^2). \end{aligned}$$

Here λ is a spectral parameter. The superscript “*” represents complex conjugate and $[A, B] = AB - BA$, i.e., commutator. When $r = q^*$, the compatibility condition yields the zero-curvature equation, $U_t - V_x - [U, V] = 0$, which generates equation (1.3). The rogue wave solutions of the higher-order CLL equation were studied in [21] by using Darboux transformation method.

The topic of the singular Riemann-Hilbert problem was very well researched between 1979 and 1984. There are numerous papers on this topic especially in connection with integrable systems. First, there are papers by Zakharov and his group including the paper by Zakharov and Mikhailov who introduced this technique to study integrable relativistic models [20]. Then Harnad and his collaborators contributed several papers on this topic, see [9]. All these papers by Zakharov and Mikhailov as well as much more general set up of Harnad *et al.*, focus on Lax pairs with rational dependence on the spectral parameter, rather than on the polynomial case. Hirota’s τ function method and then the breakthrough by the Kyoto school of interpreting the τ function in representation theoretic theorems [3, 10] achieve success in the polynomial case. The NLS hierarchy and many more are covered by the theory of free two-components fermions. The main advantage of the representation theoretic construction is that the soliton τ function is constructed for the whole hierarchy, and the soliton solution field u is known for the whole hierarchy.

In this paper, based on the Riemann-Hilbert method for the higher-order CLL equation (1.3), we will present its N -soliton solutions with vanishing boundary condition in the compact determinant form and give an asymptotic analysis for these solutions. The Riemann-Hilbert method streamlines the inverse scattering transformation (IST) method and could be regarded as simpler version of IST [1, 15, 18]. Recently, the Riemann-Hilbert method have been widely adopted to solve nonlinear integrable models [5, 6, 8, 12, 16, 17, 19, 22].

The paper is organized as follows. In Section 2, we present the construction of Riemann-Hilbert problem for the higher-order CLL equation (1.3). In Section 3, we solve the non-regular and regular Riemann-Hilbert problems. In Section 4, we construct the N -soliton solution to the higher-order CLL equation in the determinant form and discuss the asymptotic behaviour of N -soliton interactions. The Section 5 is devoted to conclusion and discussion.

2. The Riemann-Hilbert problem for higher-order CLL

We first assume the vanishing boundary condition,

$$Q \rightarrow 0, \quad \text{as } x \rightarrow \infty.$$

Note that when $x \rightarrow \infty$, from the spectral problem (1.4), we have asymptotic behaviour $Y \propto e^{-i\lambda^2 \sigma_3 x - 4i\lambda^6 \sigma_3 t}$. Thus it will be convenient to express Y as

$$Y = \Phi e^{-i\lambda^2 \sigma_3 x - 4i\lambda^6 \sigma_3 t}, \quad (2.1)$$

so that the new matrix function Φ is x -independent at infinity. Inserting (2.1) into the Lax pair (1.4)-(1.5), we can rewrite the Lax pair in the form

$$\Phi_x + i\lambda^2 [\sigma_3, \Phi] = \left(\sigma_3 Q \lambda + \frac{1}{4} i \sigma_3 Q^2 \right) \Phi, \quad (2.2)$$

$$\Phi_t + 4i\lambda^6 [\sigma_3, \Phi] = \left(4\sigma_3 Q \lambda^5 + 2i\sigma_3 Q^2 \lambda^4 + 2Z_0 \lambda^3 + Z_1 \lambda^2 + Z_2 \lambda + \frac{1}{4} Z_3 \right) \Phi. \quad (2.3)$$

In order to formulate a Riemann-Hilbert problem for the solution of the inverse spectral problem, we seek solutions of the spectral problem which approach the 2×2 identity matrix as $\lambda \rightarrow \infty$.

Consider a solution of (2.2)-(2.3) of the form

$$\Phi = D + \frac{\Phi_1}{\lambda} + \frac{\Phi_2}{\lambda^2} + \frac{\Phi_3}{\lambda^3} + \mathcal{O}\left(\frac{1}{\lambda^4}\right),$$

where D and $\Phi_k (k = 1, 2, 3)$ are independent of the spectral parameter λ . Substituting the above expansion into the the Lax pair (2.2)-(2.3) and comparing the same order of λ , we find that D is diagonal and satisfies

$$D_x = -\frac{i}{4}qr\sigma_3 D, \quad (2.4)$$

$$D_t = -\frac{1}{4}Z_3 D, \quad (2.5)$$

with $Z_3 = (\frac{i}{4}q^3r^3 + \frac{3}{2}qr(q_xr - qr_x) + iq_xr_x - iq_{xx}r - iqr_{xx})\sigma_3$.

Note that the CLL equation admits the conservation law

$$(qr)_t = \left(q_xr_x - \frac{3}{2}iqr(q_xr - qr_x) + \frac{1}{4}q^3r^3 - qr_{xx} - qr_{xx} \right)_x.$$

Thus, the two eqs. (2.4) and (2.5) for D are consistent and are both satisfied if we define

$$D = \exp\left(-\frac{i}{4}\int_{-\infty}^x q(x',t)r(x',t)dx'\sigma_3\right).$$

We introduce a new function J by $DJ = \Phi$. According to the asymptotic behaviour of Φ , it's easy to see that

$$J = \mathbb{I} + \mathcal{O}\left(\frac{1}{\lambda}\right), \quad \lambda \rightarrow \infty.$$

The Lax pair of eq. (2.2)-(2.3) becomes

$$J_x + i\lambda^2[\sigma_3, J] = \hat{U}J, \quad (2.6)$$

$$J_t + 4i\lambda^6[\sigma_3, J] = \hat{V}J, \quad (2.7)$$

where

$$\hat{U} = \exp\left(\frac{i}{4}\int_{-\infty}^x q(x',t)r(x',t)dx'\hat{\sigma}_3\right)(\lambda\sigma_3Q + \frac{i}{2}\sigma_3Q^2),$$

$$\hat{V} = \exp\left(\frac{i}{4}\int_{-\infty}^x q(x',t)r(x',t)dx'\hat{\sigma}_3\right)(4\lambda^5\sigma_3Q + 2i\lambda^4\sigma_3Q^2 + 2\lambda^3Z_0 + \lambda^2Z_1 + \lambda Z_2 + \frac{Z_3}{2}).$$

Here the notation $\hat{\sigma}_3$ is defined by $\hat{\sigma}_3A = [\sigma_3, A]$ and thus $\exp(\hat{\sigma}_3)A = \exp(\sigma_3)A\exp(-\sigma_3)$.

The Jost solutions J_{\pm} of the spectral equation (2.6) obey the constant asymptotic condition

$$J_{\pm} \rightarrow J_{\pm}^0, \quad x \rightarrow \pm\infty, \quad (2.8)$$

respectively, and they satisfy the following integral equation

$$J_+(x; \lambda) = J_+^0 - \int_x^{\infty} e^{-i\lambda^2(x-y)\sigma_3} \hat{U}(y) J_+(y) e^{i\lambda^2(x-y)\sigma_3} dy,$$

$$J_-(x; \lambda) = J_-^0 + \int_{-\infty}^x e^{-i\lambda^2(x-y)\sigma_3} \hat{U}(y) J_-(y) e^{i\lambda^2(x-y)\sigma_3} dy,$$

where $J_-^0 = \mathbb{I}$ and $J_+^0 = \exp(\frac{i}{4}\int_{-\infty}^{\infty} qrdx\sigma_3)$. From the above Volterra type integral equations, we can prove the existence and uniqueness of the Jost solutions through standard iteration method. We

partition J_{\pm} into columns as $J_{\pm} = (J_{\pm}^{(1)}, J_{\pm}^{(2)})$, then $J_{-}^{(1)}, J_{+}^{(2)}$ are analytic for $\lambda \in \mathbb{C}_{+}$ and continuous for $\lambda \in \mathbb{C}_{+} \cup \mathbb{R} \cup i\mathbb{R}$, while the columns $J_{+}^{(1)}, J_{-}^{(2)}$ are analytic for $\lambda \in \mathbb{C}_{-}$ and continuous for $\lambda \in \mathbb{C}_{-} \cup \mathbb{R} \cup i\mathbb{R}$, where

$$\mathbb{C}_{+} = \left\{ \lambda \mid \arg \lambda \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3}{2}\pi\right) \right\}, \quad \mathbb{C}_{-} = \left\{ \lambda \mid \arg \lambda \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3}{2}\pi, 2\pi\right) \right\}.$$

The quarter \mathbb{C}_{+} and \mathbb{C}_{-} are displayed in Fig. 1.

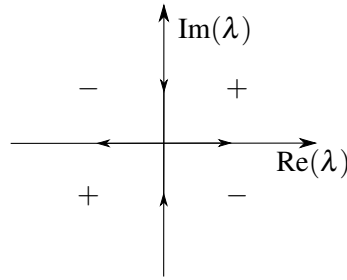


Fig. 1. The jump contour in the complex λ -plane.

We define $E = e^{-i\lambda^2 \sigma_3 x}$, then $J_{+}E$ and $J_{-}E$ are both solutions to linear equation (1.4). They are dependent and linearly related by a scattering matrix $S(\lambda)$ as

$$J_{-}E = J_{+}ES(\lambda), \quad \lambda \in \mathbb{R} \cup i\mathbb{R}. \quad (2.9)$$

From the Abel's identity and since the trace of Q satisfies $\text{tr}(Q) = 0$, the determinants of J_{\pm} are constants for all x . Considering the boundary conditions (2.8), we have

$$\det J_{\pm} = 1.$$

Thus we can derive $\det S(\lambda) = 1$ according to the relation (2.9). Furthermore, from (2.9), we have

$$S(\lambda) = (s_{ij})_{2 \times 2} = \lim_{x \rightarrow +\infty} E^{-1} J_{-} E = I + \int_{-\infty}^{\infty} e^{i\lambda^2 \sigma_3 y} \hat{U} J_{-} e^{-i\lambda^2 \sigma_3 y} dy, \quad \lambda \in \mathbb{R} \cup i\mathbb{R}.$$

Based on the analytic property of J_{-} , s_{11} is analytic extension to \mathbb{C}_{+} , and s_{22} is analytic in \mathbb{C}_{-} . We define a new Jost solution P^{+} as

$$P^{+} = (J_{-}^{(1)}, J_{+}^{(2)}) = J_{-}H_1 + J_{+}H_2 = J_{+}E \begin{pmatrix} s_{11} & 0 \\ s_{21} & 1 \end{pmatrix} E^{-1}, \quad (2.10)$$

with $H_1 = \text{diag}\{1, 0\}$ and $H_2 = \text{diag}\{0, 1\}$. P^{+} is analytic in \mathbb{C}_{+} with the asymptotic behavior

$$P^{+}(x, \lambda) \rightarrow I, \quad \lambda \in \mathbb{C}_{+} \rightarrow \infty. \quad (2.11)$$

In order to obtain the behavior of Jost solution P^{+} for large λ , we use the following expansion

$$P^{+} = I + \frac{P_1^{+}}{\lambda} + \frac{P_2^{+}}{\lambda^2} + \mathcal{O}(\lambda^{-3}).$$

Substituting the expansion into the spectral problem (2.6) and compare the coefficients of λ , we have

$$i[\sigma_3, P_1^+] = \exp\left(\frac{i}{4} \int_{-\infty}^x q r dx' \sigma_3\right) \sigma_3 Q \exp\left(-\frac{i}{4} \int_{-\infty}^x q r dx' \sigma_3\right),$$

which leads to

$$q = 2i(P_1^+)_{12} \exp\left(-\frac{i}{2} \int_{-\infty}^x q r dx'\right), \quad r = 2i(P_1^+)_{21} \exp\left(\frac{i}{2} \int_{-\infty}^x q r dx'\right). \quad (2.12)$$

We consider the adjoint scattering problem of (2.2)

$$K_x + i\lambda^2[\sigma_3, K] = -K\hat{U}.$$

It is easy to verify that J_{\pm}^{-1} satisfy the above adjoint equation together with the boundary condition $J_{\pm}^{-1} \rightarrow I$ when $x \rightarrow \pm\infty$, respectively. Denote the k -th row vector of J_{\pm}^{-1} as $(J_{\pm}^{-1})^{(k)}$ for convenience and define

$$P^- = \begin{pmatrix} (J_-^{-1})^{(1)} \\ (J_+^{-1})^{(2)} \end{pmatrix}.$$

One can check that P^- is analytic in \mathbb{C}_- and tends to identity I when $\lambda \rightarrow \infty$. If we use the notation $R(\lambda) = S^{-1}(\lambda)$, then

$$J_-^{-1} = E R E^{-1} J_+^{-1},$$

and

$$P^- = H_1 J_-^{-1} + H_2 J_+^{-1} = E \begin{pmatrix} s_{22} & -s_{12} \\ 0 & 1 \end{pmatrix} E^{-1} J_+^{-1}. \quad (2.13)$$

P^- is analytic in \mathbb{C}_- with the asymptotic behavior

$$P^-(x, \lambda) \rightarrow I, \quad \lambda \in \mathbb{C}_- \rightarrow \infty. \quad (2.14)$$

Summarizing the above results, we have constructed two matrix functions P^+ and P^- , that are analytic in the complex region \mathbb{C}^+ and \mathbb{C}^- , respectively. Thus we have the Riemann-Hilbert problem by P^+, P^- as

$$P^- P^+ = G(x, \lambda) = E \begin{pmatrix} 1 & -s_{12} \\ s_{21} & 1 \end{pmatrix} E^{-1}, \quad \lambda \in \mathbb{R} \cup i\mathbb{R}. \quad (2.15)$$

We now investigate the evolutions of the scattering coefficients s_{ij} . The relation (2.9) and (2.8) leads to

$$S_t = \lim_{x \rightarrow +\infty} E^{-1} J_{-,t} E.$$

Then according to the evolution property (2.3) and $Q \rightarrow 0$ as $|x| \rightarrow \infty$, we have

$$S_t + 4i\lambda^6[\sigma_3, S] = 0,$$

and the evolutions of the entries of the scattering matrix S satisfy

$$\begin{aligned} s_{11,t} &= s_{22,t} = 0, \\ s_{12,t} + 8i\lambda^6 s_{12} &= 0, \quad s_{21,t} - 8i\lambda^6 s_{12} = 0. \end{aligned}$$

Thus the elements s_{11}, s_{22} are time-independent and

$$s_{12}(\lambda, t) = s_{12}(\lambda, 0)e^{-8i\lambda^6 t}, \quad s_{21}(\lambda, t) = s_{21}(\lambda, 0)e^{8i\lambda^6 t}.$$

3. Solutions for the Riemann-Hilbert problem

Note that the transpose and conjugate of \hat{U} satisfies the relation $\hat{U}^\dagger = -\hat{U}$, we can verify that

$$J^\dagger(x, \lambda^*) = J^{-1}(x, \lambda), \quad \lambda \in \mathbb{R} \cup i\mathbb{R}.$$

Here \dagger denotes the operation of transpose and complex conjugate. From the definition of P^+ and P^- , we have the relation

$$(P^+)^\dagger(\lambda^*) = P^-(\lambda) \quad \lambda \in \mathbb{R} \cup i\mathbb{R}. \quad (3.1)$$

It follows from the relation (2.9) that

$$S^\dagger(\lambda^*) = S^{-1}(\lambda), \quad \lambda \in \mathbb{R} \cup i\mathbb{R}.$$

which implies the following relations

$$s_{11}^*(\lambda^*) = s_{22}(\lambda), \quad s_{22}^*(\lambda^*) = s_{11}(\lambda), \quad s_{12}^*(\lambda^*) = -s_{21}(\lambda), \quad s_{21}^*(\lambda^*) = -s_{12}(\lambda), \quad \lambda \in \mathbb{R} \cup i\mathbb{R}.$$

Furthermore, from the symmetric property $\sigma_3 Q \sigma_3 = -Q$ and $\sigma_3 Q^2 \sigma_3 = Q^2$, we conclude that

$$J(x, t, -\lambda) = \sigma_3 J(x, t, \lambda) \sigma_3.$$

Thus from the definition of $P^\pm(\lambda)$, we have the symmetry relation

$$P^\pm(x, t, -\lambda) = \sigma_3 P^\pm(x, t, \lambda) \sigma_3. \quad (3.2)$$

Applying this reduction to (2.9), then it readily implies

$$S(-\lambda) = \sigma_3 S(\lambda) \sigma_3.$$

Thus $s_{11}(\lambda)$ is an odd function, and each zero λ_k of s_{11} is accompanied with zero $-\lambda_k$. For simplicity, we assume all zeros are simple and then the kernels of $P^+(\lambda_k)$ and $P^-(\bar{\lambda}_k)$ contain only a single

column vector $|v_k\rangle$ and row vector $\langle v_k|$, respectively,

$$P^+(\lambda_k)|v_k\rangle = 0, \quad \langle v_k|P^-(\bar{\lambda}_k) = 0, \quad k = 1, 2, \dots, N. \quad (3.3)$$

Here $|v_k\rangle = \langle v_k|^\dagger$. From the relation (3.1), we have

$$\bar{\lambda}_k = \lambda_k^*. \quad (3.4)$$

Differentiating both sides of the first equation of (3.3) with respect to x and t , and recalling the Lax pair (1.4)-(1.5), we have

$$P^+(\lambda_k; x) \left(\frac{d|v_k\rangle}{dx} + i\lambda_k^2 \sigma_3 |v_k\rangle \right) = 0, \quad P^+(\lambda_k; x) \left(\frac{d|v_k\rangle}{dt} + 4i\lambda_k^6 \sigma_3 |v_k\rangle \right) = 0.$$

It concludes that

$$|v_k\rangle = e^{-i\lambda_k^2 \sigma_3 x - 4i\lambda_k^6 \sigma_3 t} |v_{k,0}\rangle e^{\int_{x_0}^x a_k(s) ds + \int_{t_0}^t b_k(s) ds}, \quad (3.5)$$

where $a_k(x)$ and $b_k(t)$ are two scalar functions.

Based on the above analysis, we have the following theorem for the solution to the non-regular Riemann-Hilbert problem.

Theorem 1. *The solution to a non-regular Riemann-Hilbert problem (2.15) with simple zeros under the canonical normalized condition (2.11) and (2.14) is*

$$P^+(\lambda) = P_+(\lambda)T(\lambda), \quad P^-(\lambda) = T^{-1}(\lambda)P_-(\lambda),$$

where

$$\begin{aligned} T(\lambda) &= \prod_{k=1}^N T_k(\lambda_k) = \prod_{k=1}^N \left(I + \frac{A_k}{\lambda - \lambda_k^*} - \frac{\sigma_3 A_k \sigma_3}{\lambda + \lambda_k^*} \right), \\ T^{-1}(\lambda) &= \prod_{k=1}^N T_k^{-1}(\lambda_k) = \prod_{k=1}^N \left(I + \frac{A_k^\dagger}{\lambda - \lambda_k} - \frac{\sigma_3 A_k^\dagger \sigma_3}{\lambda + \lambda_k} \right), \\ A_k &= \frac{\lambda_k - \lambda_k^*}{2} \begin{pmatrix} \alpha_k & 0 \\ 0 & \alpha_k^* \end{pmatrix} |w_k\rangle \langle w_k|, \quad \lambda_k^{-1} = \langle w_k| \begin{pmatrix} \lambda_k^* & 0 \\ 0 & \lambda_k \end{pmatrix} |w_k\rangle, \end{aligned}$$

$|w_k\rangle$ is a column vector and defined by $|w_k\rangle = T_{k-1}(\lambda_k) \cdots T_1(\lambda_k) |v_k\rangle$ and $|v_k\rangle = \langle v_k|^\dagger$. P_\pm is the unique solution to the regular Riemann-Hilbert problem

$$P_-(\lambda)P_+(\lambda) = T(\lambda)G(\lambda)T^{-1}(\lambda), \quad \lambda \in \mathbb{R} \cup i\mathbb{R}, \quad (3.6)$$

where P_\pm are analytic in \mathbb{C}_\pm , respectively, and $P_\pm \rightarrow I$ as $\lambda \rightarrow \infty$.

Proof. From the symmetry relation (3.2), we can suppose that simple zeros of $\det P^+(\lambda)$ are $\{\pm\lambda_k \in \mathbb{C}_+, 1 \leq k \leq N\}$. The symmetry relation (3.4) implies that $\{\pm\lambda_k^*\}$ are zeros of $\det P^-(\lambda)$. Both $\ker(P^+(\pm\lambda_k))$ and $\ker(P^-(\pm\lambda_k^*))$ are one-dimensional and the kernel space is spanned by a

single column vector $|v_k\rangle$ and single row vector $\langle v_k|$, respectively, i.e.,

$$P^+(\lambda_k)|v_k\rangle = 0, \quad \langle v_k|P^-(\lambda_k) = 0.$$

From the definition of A_1 , we construct a meromorphic matrix function

$$T_1(\lambda) = I + \frac{A_1}{\lambda - \lambda_1^*} - \frac{\sigma_3 A_1 \sigma_3}{\lambda + \lambda_1^*},$$

with simple poles at $\lambda = \pm\lambda_1^* \in \mathbb{C}_-$. One can check through direct computation that

$$T_1^{-1}(\lambda) = I + \frac{A_1^\dagger}{\lambda - \lambda_1} - \frac{\sigma_3 A_1^\dagger \sigma_3}{\lambda + \lambda_1}, \quad \det T_1(\lambda) = \frac{\lambda^2 - \lambda_1^2}{\lambda^2 - \lambda_1^{*2}}.$$

Since $\pm\lambda_1$ are simple zeros of $\det P^\pm(\lambda)$, the matrix $P^+(\lambda)T_1^{-1}(\lambda)$ is non-singular at $\lambda = \pm\lambda_1$, and $T_1(\lambda)P^-(\lambda)$ is non-singular at $\lambda = \pm\lambda^*$. In general, near the point λ_k , we have $\det P^+(\lambda) \sim \lambda - \lambda_k$ and $\det P^+(\lambda) \sim \lambda + \lambda_k^*$ near the point $-\lambda_k$. Similarly, $\det P^-(\lambda) \sim \lambda \pm \lambda_k^*$ near the point $\pm\lambda_k^*$ by the involution relations (3.3) and (3.4). Let $T_k(\lambda)$ be a matrix with determinant

$$\det T_k(\lambda) = \frac{\lambda^2 - \lambda_k^2}{\lambda^2 - \lambda_k^{*2}},$$

then $\det P^+(\lambda)T_k^{-1}(\lambda) \neq 0$ at points $\pm\lambda_k$ and $\det T_k(\lambda)P^-(\lambda) \neq 0$ at points $\pm\lambda_k^*$. We introduce

$$T(\lambda) = T_N(\lambda)T_{N-1}(\lambda)\cdots T_1(\lambda), \quad T^{-1}(\lambda) = T_1^{-1}(\lambda)T_2^{-1}(\lambda)\cdots T_N^{-1}(\lambda),$$

that accumulates all zeros of the Riemann-Hilbert problem. We can cancel all zeros $\pm\lambda_j$ and $\pm\lambda_j^*$, ($j = 1, 2, \dots, N$) of $\det P^\pm(\lambda)$ by

$$P_+(\lambda) = P^+(\lambda)T^{-1}(\lambda), \quad P_-(\lambda) = T(\lambda)P^-(\lambda). \quad (3.7)$$

Substituting $P^\pm(\lambda)$ into (2.15), we obtain a normalized regular Riemann-Hilbert problem (3.6).

From the above properties of $T_k(\lambda)$, we could readily obtain the explicit expression for the matrix $T_k(\lambda)$ (cf. Ref. [13])

$$T_k(\lambda) = I + \frac{A_k}{\lambda - \lambda_k^*} - \frac{\sigma_3 A_k \sigma_3}{\lambda + \lambda_k^*}, \quad T_k^{-1}(\lambda) = I + \frac{A_k^\dagger}{\lambda - \lambda_k} - \frac{\sigma_3 A_k^\dagger \sigma_3}{\lambda + \lambda_k}, \quad k = 1, 2, \dots, N.$$

□

4. Soliton solutions

Based on the above analysis, we are ready to construct N -soliton solutions for the higher-order CLL equation (1.3). From the asymptotic expansion of Jost solutions as $\lambda \rightarrow \infty$, the potential q can be expressed as (2.12). From the expression of $T_k^{-1}(\lambda)$ and A_k , there exists some column vector $|z_k\rangle$

such that

$$\text{Res}_{\lambda=\lambda_k} T^{-1}(\lambda) = \lim_{\lambda \rightarrow \lambda_k} (\lambda - \lambda_k) T^{-1}(\lambda) = |v_k\rangle \langle z_k|, \quad \langle z_k| = |z_k\rangle^\dagger.$$

Meanwhile, considering the symmetry relation $T_k(-\lambda) = \sigma_3 T_k(\lambda) \sigma_3$, $T(\lambda)$ and $T^{-1}(\lambda)$ have compact form

$$T(\lambda) = I + \sum_{k=1}^N \left(\frac{B_k}{\lambda - \lambda_k^*} - \frac{\sigma_3 B_k \sigma_3}{\lambda + \lambda_k^*} \right), \quad (4.1)$$

$$T^{-1}(\lambda) = I + \sum_{k=1}^N \left(\frac{B_k^\dagger}{\lambda - \lambda_k} - \frac{\sigma_3 B_k^\dagger \sigma_3}{\lambda + \lambda_k} \right), \quad (4.2)$$

with $B_k = |z_k\rangle \langle z_k|$. According to the identity $T(\lambda) T^{-1}(\lambda) = T^{-1}(\lambda) T(\lambda) = I$, we have

$$\text{Res}_{\lambda=\lambda_j} T(\lambda) T^{-1}(\lambda) = 0.$$

We arrive at

$$T(\lambda_j) B_j^\dagger = 0,$$

and it yields

$$\left[I + \sum_{k=1}^N \left(\frac{B_k}{\lambda_j - \lambda_k^*} - \frac{\sigma_3 B_k \sigma_3}{\lambda_j + \lambda_k^*} \right) \right] |v_j\rangle = 0, \quad j = 1, 2, \dots, N,$$

or equivalently

$$|v_j\rangle = \sum_{k=1}^N \left(\frac{\sigma_3 |z_k\rangle \langle v_k| \sigma_3 |v_j\rangle}{\lambda_j + \lambda_k^*} - \frac{|z_k\rangle \langle v_k| |v_j\rangle}{\lambda_j - \lambda_k^*} \right), \quad j = 1, 2, \dots, N.$$

Solving these linear algebraic equations, we obtain

$$\begin{aligned} |z_l\rangle_1 &= \sum_{j=1}^N M^{-1}(l, j) |v_j\rangle_1, \\ |z_l\rangle_2 &= \sum_{j=1}^N \hat{M}^{-1}(l, j) |v_j\rangle_2, \end{aligned}$$

where $|z_l\rangle_k$ denotes the k -th element of $|z_l\rangle$, and the entries of matrices M and \hat{M} are defined as

$$M_{jk} = \frac{\langle v_k | \sigma_3 | v_j \rangle}{\lambda_j + \lambda_k^*} - \frac{\langle v_k | v_j \rangle}{\lambda_j - \lambda_k^*}, \quad \hat{M}_{jk} = -\frac{\langle v_k | \sigma_3 | v_j \rangle}{\lambda_j + \lambda_k^*} - \frac{\langle v_k | v_j \rangle}{\lambda_j - \lambda_k^*}, \quad j, k = 1, 2, \dots, N.$$

According to the Plemelj formula, the solution to the Riemann-Hilbert problem (3.6) is

$$\left(P_+(\lambda) \right)^{-1} = I + \frac{1}{2\pi i} \int_{\Gamma} \frac{T(\xi)(I - G)T^{-1}(\xi)(P_+(\xi))^{-1}}{\xi - \lambda} d\xi, \quad \lambda \in \mathbb{C}_+.$$

and thus when $\lambda \rightarrow \infty$, we have

$$P_+(\lambda) \rightarrow I + \frac{1}{2\pi i \lambda} \int_{\Gamma} T(\xi)(I - G)T^{-1}(\xi)(P_+(\xi))^{-1} d\xi.$$

From (4.1), we find $T(\lambda)$ has expansion

$$T(\lambda) \rightarrow I + \frac{1}{\lambda} \sum_{k=1}^N (|z_k\rangle\langle v_k| - \sigma_3 |z_k\rangle\langle v_k| \sigma_3), \quad \lambda \rightarrow \infty$$

Thus, as $\lambda \rightarrow \infty$, $P^+(\lambda) = P_+(\lambda)T(\lambda)$ has the expansion formula

$$P^+ = I + \frac{P_1^+}{\lambda} + \frac{P_2^+}{\lambda^2} + \mathcal{O}\left(\frac{1}{\lambda^3}\right)$$

with

$$P_1^+ = \sum_{k=1}^N (|z_k\rangle\langle v_k| - \sigma_3 |z_k\rangle\langle v_k| \sigma_3) + \frac{1}{2\pi i} \int_{\Gamma} T(\xi)(I - G)T^{-1}(\xi)(P_+(\xi))^{-1} d\xi.$$

In the reflection-less case, that is $G = I$, we obtain N -soliton solutions from (2.12). The formula for N -soliton solution is

$$q = 2i(P_1^+)_{12} \exp\left(2i \int_{-\infty}^x (P_1^+)_{12}(P_1^+)_{21} dx'\right), \quad (4.3)$$

$$r = 2i(P_1^+)_{21} \exp\left(-2i \int_{-\infty}^x (P_1^+)_{12}(P_1^+)_{21} dx'\right), \quad (4.4)$$

$$(P_1^+)_{12} = 2 \sum_{k=1}^N |z_k\rangle_1 \langle v_k|_2 = -2 \frac{\det M_1}{\det M}, \quad (4.5)$$

$$(P_1^+)_{21} = 2 \sum_{k=1}^N |z_k\rangle_2 \langle v_k|_1 = -2 \frac{\det M_2}{\det \hat{M}}, \quad (4.6)$$

where the matrix M_1 and M_2 have the form

$$M_1 = \begin{pmatrix} M_{11} & \cdots & M_{1N} & |v_1\rangle_1 \\ \vdots & \cdots & \vdots & \vdots \\ M_{N1} & \cdots & M_{NN} & |v_N\rangle_1 \\ \langle v_1|_2 & \cdots & \langle v_N|_2 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} \hat{M}_{11} & \cdots & \hat{M}_{1N} & |v_1\rangle_2 \\ \vdots & \cdots & \vdots & \vdots \\ \hat{M}_{N1} & \cdots & \hat{M}_{NN} & |v_N\rangle_2 \\ \langle v_1|_1 & \cdots & \langle v_N|_1 & 0 \end{pmatrix}.$$

To obtain the explicit formulae for N -soliton solutions, we take

$$|v_k\rangle = \begin{pmatrix} c_k e^{-\theta_k} \\ e^{\theta_k} \end{pmatrix},$$

where $\theta_k = i(\lambda_k^2 x + 4\lambda_k^6 t)$, and λ_k, c_k are arbitrary constants. Then the general N -soliton solution to the higher-order CLL equation (1.3) is represented by

$$q = -4i \frac{\det \hat{M}_1}{\det M} \exp\left(8i \int_{-\infty}^x \frac{\det(\hat{M}_1 \hat{M}_2)}{\det(M \hat{M})} dx'\right),$$

with

$$\hat{M}_1 = \begin{pmatrix} M_{11} & \cdots & M_{1N} & c_1 e^{-\theta_1} \\ \vdots & \cdots & \vdots & \vdots \\ M_{N1} & \cdots & M_{NN} & c_N e^{-\theta_N} \\ e^{\theta_1^*} & \cdots & e^{\theta_N^*} & 0 \end{pmatrix}, \quad \hat{M}_2 = \begin{pmatrix} \hat{M}_{11} & \cdots & \hat{M}_{1N} & e^{\theta_1} \\ \vdots & \cdots & \vdots & \vdots \\ \hat{M}_{N1} & \cdots & \hat{M}_{NN} & e^{\theta_N} \\ c_1^* e^{-\theta_1^*} & \cdots & c_N^* e^{-\theta_N^*} & 0 \end{pmatrix},$$

and

$$M_{j,k} = \frac{-2\lambda_k^*}{\lambda_j^2 - \lambda_k^{*2}} c_j c_k^* e^{-\theta_j - \theta_k^*} - \frac{2\lambda_j}{\lambda_j^2 - \lambda_k^{*2}} e^{\theta_j + \theta_k^*}, \quad \hat{M}_{j,k} = \frac{-2\lambda_j}{\lambda_j^2 - \lambda_k^{*2}} c_j c_k^* e^{-\theta_j - \theta_k^*} - \frac{2\lambda_k^*}{\lambda_j^2 - \lambda_k^{*2}} e^{\theta_j + \theta_k^*}.$$

In what follows, we shall investigate the properties of the single, two- and N -soliton solutions.

4.1. One and two-soliton solution

To obtain one-soliton solution, we set $N = 1$, $\theta_k = i\lambda_k^2 x + 4i\lambda_k^6 t$ and $|v_{k,0}\rangle = (c_k, 1)^T$ in formula (3.5). Then according to (4.3), one-soliton reads as

$$q = \frac{-2ic_1(\lambda_1^2 - \lambda_1^{*2})\exp(-\theta_1 + \theta_1^*)}{|c_1|^2\lambda_1^* \exp(-\theta_1 - \theta_1^*) + \lambda_1 \exp(\theta_1 + \theta_1^*)} \exp\left(-32i \int_{-\infty}^x \frac{\lambda_{1,R}^2 \lambda_{1,I}^2}{|c_1|^2\lambda_1^* \exp(-\theta_1 - \theta_1^*) + \lambda_1 \exp(\theta_1 + \theta_1^*)|^2} dx'\right).$$

When we take $c_1 = 1$, one-soliton solution yields the form

$$q = \frac{-2i(\lambda_1^2 - \lambda_1^{*2})\exp(-2i\theta_{1,I})}{\lambda_1^* \exp(-2\theta_{1,R}) + \lambda_1 \exp(2\theta_{1,R})} \exp\left(-32i \int_{-\infty}^x \frac{\lambda_{1,R}^2 \lambda_{1,I}^2}{|\lambda_1^* \exp(-2\theta_{1,R}) + \lambda_1 \exp(2\theta_{1,R})|^2} dx'\right).$$

Here the subscripts R and I denote the real and imaginary part of θ_1 , respectively. We can separate the real and imaginary parts of θ_1 as

$$\begin{aligned} \theta_{1,R} &= -2\lambda_{1,R}\lambda_{1,I}\left(x + 4(3\lambda_{1,R}^4 - 10\lambda_{1,R}^2\lambda_{1,I}^2 + 3\lambda_{1,I}^4)t\right), \\ \theta_{1,I} &= (\lambda_{1,R}^2 - \lambda_{1,I}^2)\left(x + 4(\lambda_{1,R}^2 - 4\lambda_{1,R}\lambda_{1,I} + \lambda_{1,I}^2)(\lambda_{1,R}^2 + 4\lambda_{1,R}\lambda_{1,I} + \lambda_{1,I}^2)t\right). \end{aligned}$$

Thus the velocity for the single soliton is $v_1 = 4(3\lambda_{1,R}^4 - 10\lambda_{1,R}^2\lambda_{1,I}^2 + 3\lambda_{1,I}^4)$ and the center position for $|q|$ locates on the line

$$x + 4(3\lambda_{1,R}^4 - 10\lambda_{1,R}^2\lambda_{1,I}^2 + 3\lambda_{1,I}^4)t = 0,$$

with the amplitude $4|\lambda_{1,I}|$. The profile to one-soliton solution with parameter $\lambda = 1 + \frac{i}{2}$ is displayed in Fig. 2.

The two-soliton solution with parameters $\lambda_1 = 1 - \frac{i}{4}$, $\lambda_2 = 1 + \frac{i}{2}$ and $c_1 = c_2 = 1$ is depicted in Fig. 3. In this case, the two left-travelling solitons with amplitude 1 and 2, velocity 9.6875 and 2.75, respectively, pass through each other and keep each amplitude and velocity.

4.2. Interactions between N -solitons

In order to analyze the N -soliton solution interactions with the variations of corresponding positions and amplitudes, we assume $v_1 < v_2 < \dots < v_N$ and keep $x - v_k t = \text{constant}$ with $v_j = 4(3\lambda_{j,R}^4 - 10\lambda_{j,R}^2\lambda_{j,I}^2 + 3\lambda_{j,I}^4)$, $j = 1, 2, \dots, N$. In the following we study the asymptotic behavior of the N solitons.

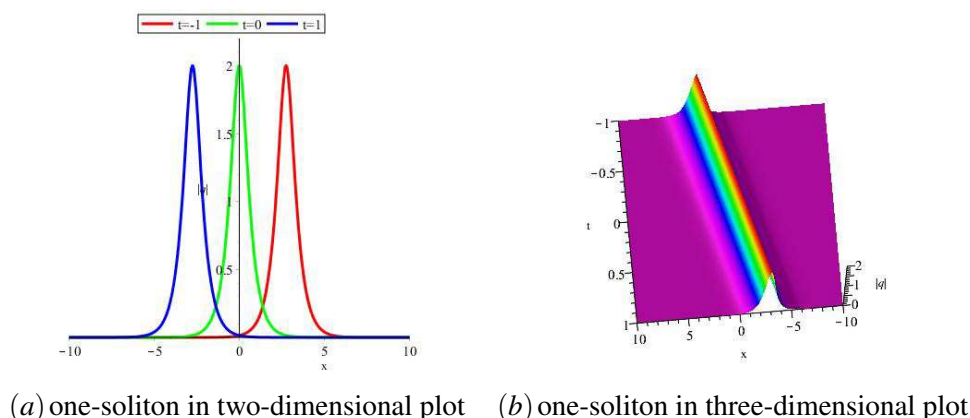


Fig. 2. (Color online) One-soliton solution for $|q|$.

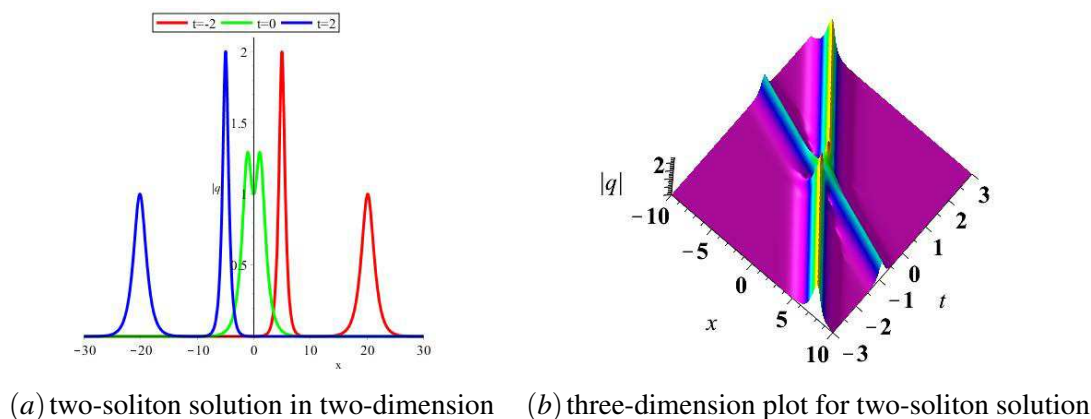


Fig. 3. (Color online) Profile for interaction of two solitons.

Theorem 2. Assuming $\text{Im}(\lambda_k^2) < 0$ ($k = 1, 2, \dots, N$) and $v_1 < v_2 < \dots < v_N$, then N -soliton solution has the asymptotic behavior

$$\hat{q} \equiv -4i \frac{\det \hat{M}_1}{\det M} \sim \sum_{k=1}^N \frac{-2i(\lambda_k^2 - \lambda_k^{*2}) \exp(-2i\theta_{k,I} \mp \arg(\Phi_k) \mp \arg \Psi_k)}{\lambda_k^* \exp(-2\theta_{k,R} \mp \ln |\Phi_k \Psi_k|) + \lambda_k \exp(2\theta_{k,R} \pm \ln |\Psi_k \Phi_k|)}, \quad (4.7)$$

as $t \rightarrow \pm\infty$ with

$$\Phi_k = \prod_{j=1}^{k-1} \frac{\lambda_j^{*2} - \lambda_k^2}{\lambda_k^2 - \lambda_j^2}, \quad \Psi_k = \prod_{j=k+1}^N \frac{\lambda_j^2 - \lambda_k^2}{\lambda_j^{*2} - \lambda_k^2}. \quad (4.8)$$

Proof. When $\text{Im}\lambda_k^2 < 0$, the asymptotic behaviour of $\exp(\theta_{k,R}) = -2\lambda_{k,R}\lambda_{k,I}(x + v_k t)$ is mainly determined by $x + v_k t$. In the vicinity $\Omega_k : x = -v_k t$, when $t \rightarrow +\infty$, we have asymptotic behaviour

$$\begin{aligned} x + v_j t &= (v_j - v_k)t \rightarrow +\infty, & j > k, \\ x + v_j t &= (v_j - v_k)t \rightarrow -\infty, & j < k. \end{aligned}$$

Thus, when $t \rightarrow +\infty$,

$$\exp(-\theta_j) \rightarrow 0, \quad j > k \quad \text{and} \quad \exp(\theta_j) \rightarrow 0, \quad j < k.$$

and the matrix elements m_{jl} have the asymptotic expression

$$\begin{cases} j < k \\ l < k \end{cases} \quad m_{jl} \sim a_{jl} e^{-\theta_j - \theta_l^*}, \quad \begin{cases} j < k \\ l > k \end{cases} \quad m_{jl} \sim e^{-\theta_j + \theta_l^*} (a_{jl} e^{-2\theta_l^*} + b_{jl} e^{2\theta_j}),$$

$$\begin{cases} j > k \\ l > k \end{cases} \quad m_{jl} \sim b_{jl} e^{\theta_j + \theta_l^*}, \quad \begin{cases} j > k \\ l < k \end{cases} \quad m_{jl} \sim e^{\theta_j - \theta_l^*} (a_{jl} e^{-2\theta_j} + b_{jl} e^{2\theta_l^*}),$$

and

$$m_{jk} \sim \begin{cases} a_{jk} e^{-\theta_k^* - \theta_j} & j < k \\ b_{jk} e^{\theta_j + \theta_k^*} & j > k \end{cases}, \quad m_{kl} \sim \begin{cases} a_{kl} e^{-\theta_k - \theta_l^*} & l < k \\ b_{kl} e^{\theta_k + \theta_l^*} & l > k \end{cases},$$

where

$$a_{jl} = \frac{-2\lambda_l^*}{\lambda_j^2 - \lambda_l^{*2}}, \quad b_{jl} = \frac{-2\lambda_j}{\lambda_j^2 - \lambda_l^{*2}}.$$

In the vicinity Ω_k , we have

$$\det M \sim \begin{vmatrix} \frac{-2\lambda_1^*}{\lambda_1^2 - \lambda_1^{*2}} & \cdots & \frac{-2\lambda_{k-1}^*}{\lambda_1^2 - \lambda_{k-1}^{*2}} & \frac{-2\lambda_k^*}{\lambda_1^2 - \lambda_k^{*2}} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{-2\lambda_1^*}{\lambda_{k-1}^2 - \lambda_1^{*2}} & \cdots & \frac{-2\lambda_{k-1}^*}{\lambda_{k-1}^2 - \lambda_{k-1}^{*2}} & \frac{-2\lambda_k^*}{\lambda_{k-1}^2 - \lambda_k^{*2}} & 0 & \cdots & 0 \\ \frac{-2\lambda_1^*}{\lambda_k^2 - \lambda_1^{*2}} & \cdots & \frac{-2\lambda_{k-1}^*}{\lambda_k^2 - \lambda_{k-1}^{*2}} & \frac{-2\lambda_k^* - 2\lambda_k e^{2\theta_k + 2\theta_k^*}}{\lambda_k^2 - \lambda_k^{*2}} & \frac{-2\lambda_k e^{2\theta_k}}{\lambda_k^2 - \lambda_{k+1}^{*2}} & \cdots & \frac{-2\lambda_k e^{2\theta_k}}{\lambda_k^2 - \lambda_N^{*2}} \\ 0 & \cdots & 0 & \frac{-2\lambda_{k+1} e^{2\theta_k^*}}{\lambda_{k+1}^2 - \lambda_k^{*2}} & \frac{-2\lambda_{k+1}}{\lambda_{k+1}^2 - \lambda_{k+1}^{*2}} & \cdots & \frac{-2\lambda_{k+1}}{\lambda_{k+1}^2 - \lambda_N^{*2}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \frac{-2\lambda_N e^{2\theta_k^*}}{\lambda_N^2 - \lambda_k^{*2}} & \frac{-2\lambda_N}{\lambda_N^2 - \lambda_{k+1}^{*2}} & \cdots & \frac{-2\lambda_N}{\lambda_N^2 - \lambda_N^{*2}} \end{vmatrix}$$

$$\times \exp \left(- \sum_{j=1}^k (\theta_j + \theta_j^*) + \sum_{j=k+1}^N (\theta_j + \theta_j^*) \right),$$

and

$$\det M_1 \sim \begin{vmatrix} \frac{-2\lambda_1^*}{\lambda_1^2 - \lambda_1^{*2}} & \cdots & \frac{-2\lambda_{k-1}^*}{\lambda_1^2 - \lambda_{k-1}^{*2}} & \frac{-2\lambda_k^*}{\lambda_1^2 - \lambda_k^{*2}} & 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{-2\lambda_1^*}{\lambda_{k-1}^2 - \lambda_1^{*2}} & \cdots & \frac{-2\lambda_{k-1}^*}{\lambda_{k-1}^2 - \lambda_{k-1}^{*2}} & \frac{-2\lambda_k^*}{\lambda_{k-1}^2 - \lambda_k^{*2}} & 0 & \cdots & 0 & 1 \\ \frac{-2\lambda_1^*}{\lambda_k^2 - \lambda_1^{*2}} & \cdots & \frac{-2\lambda_{k-1}^*}{\lambda_k^2 - \lambda_{k-1}^{*2}} & \frac{-2\lambda_k^* - 2\lambda_k e^{2\theta_k + 2\theta_k^*}}{\lambda_k^2 - \lambda_k^{*2}} & \frac{-2\lambda_k e^{2\theta_k}}{\lambda_k^2 - \lambda_{k+1}^{*2}} & \cdots & \frac{-2\lambda_k e^{2\theta_k}}{\lambda_k^2 - \lambda_N^{*2}} & 1 \\ 0 & \cdots & 0 & \frac{-2\lambda_{k+1} e^{2\theta_k^*}}{\lambda_{k+1}^2 - \lambda_k^{*2}} & \frac{-2\lambda_{k+1}}{\lambda_{k+1}^2 - \lambda_{k+1}^{*2}} & \cdots & \frac{-2\lambda_{k+1}}{\lambda_{k+1}^2 - \lambda_N^{*2}} & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{-2\lambda_N e^{2\theta_k^*}}{\lambda_N^2 - \lambda_k^{*2}} & \frac{-2\lambda_N}{\lambda_N^2 - \lambda_{k+1}^{*2}} & \cdots & \frac{-2\lambda_N}{\lambda_N^2 - \lambda_N^{*2}} & 0 \\ 0 & \cdots & 0 & e^{2\theta_k^*} & 1 & \cdots & 1 & 0 \end{vmatrix} \\ \times \exp \left(- \sum_{j=1}^k (\theta_j + \theta_j^*) + \sum_{j=k+1}^N (\theta_j + \theta_j^*) \right).$$

By use of the Laplace expansion and the determinant formula of Cauchy matrix, we conclude that in the vicinity Ω_k , q has the asymptotic behaviour as

$$\hat{q} = -4i \frac{\det M_1}{\det M} \sim \frac{-2i(\lambda_k^2 - \lambda_k^{*2}) \exp(2i\theta_{k,I} - \arg(\Phi_k) - \arg \Psi_k)}{\lambda_k^* \exp(-2\theta_{k,R} - \ln |\Phi_k \Psi_k|) + \lambda_k \exp(2\theta_{k,R} + \ln |\Psi_k \Phi_k|)}.$$

Similarly, when $t \rightarrow -\infty$, we can prove

$$\hat{q} = -4i \frac{\det M_1}{\det M} \sim \frac{-2i(\lambda_k^2 - \lambda_k^{*2}) \exp(2i\theta_{k,I} + \arg(\Phi_k) + \arg \Psi_k)}{\lambda_k^* \exp(-2\theta_{k,R} + \ln |\Phi_k \Psi_k|) + \lambda_k \exp(2\theta_{k,R} - \ln |\Psi_k \Phi_k|)}.$$

Thus we have the result that on the whole plane \hat{q} has the asymptotic behaviour as (4.7). \square

From the asymptotic behaviour of N -soliton solution, we know that the interaction of N solitons is elastic and only phase shifts and displacements happen.

5. Conclusion and discussion

The inverse scattering method has been applied to the higher-order CLL equation and by considering the associated Riemann-Hilbert problem, we successfully give a simple representation for the N -soliton in the determinant form. In the context we only consider the simple zeros for of the scattering matrix. The much more general case with multiple zeros would lead to more solutions. Here we only consider solutions with vanishing boundary conditions.

Because $\{\pm \lambda_j\}$ appear simultaneously as zeros of $\det P^+$, we can assume that $\det P^+$ has $2N$ simple zeros $\{\lambda_j\}_1^{2N}$ satisfying $\lambda_{N+j} = \lambda_j$, $1 \leq j \leq N$, which all lie in \mathbb{C}^+ . From the symmetry property (3.2), we can choose the particular column vector relation

$$|v_j\rangle = \sigma_3 |v_{j-N}\rangle, \quad N+1 \leq j \leq 2N.$$

In this paper, we give explicit soliton solutions to a higher-order CLL equation in the determinant form. We know that the whole NLS hierarchy could be reduced from the (extended) KP hierarchy

and soliton solutions can be obtained by using the reduction method. It is deserved to compare soliton formulas obtained here with the tau function formalism.

A third problem is how to adjust the analysis and seek the Jost solutions of the spectral problem so that solutions with non-vanishing boundary conditions can be obtained. The global well-posedness, long-time behavior and asymptotic stability of solitons are left for future studies.

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