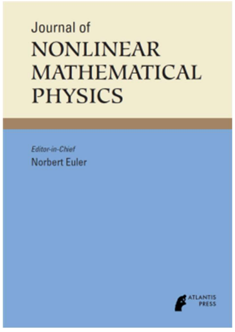

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Nonlocal symmetries of Plebański's second heavenly equation

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We study nonlocal symmetries of Plebański's second heavenly equation in an infinite-dimensional covering associated to a Lax pair with a non-removable spectral parameter. We show that all local symmetries of the equation admit lifts to full-fledged nonlocal symmetries in the infinite-dimensional covering. Also, we find two new infinite hierarchies of commuting nonlocal symmetries in this covering and describe the structure of the Lie algebra of the obtained nonlocal symmetries.

Keywords: Plebański's second heavenly equation; Lax pair; differential covering; nonlocal symmetries

2000 Mathematics Subject Classification: 58H05, 58J70, 35A30

1. Introduction

In this paper we study nonlocal symmetries of Plebański's second heavenly equation, [32],

$$u_{xz} - u_{ty} - u_{xx}\,u_{yy} + u_{xy}^2 = 0. \tag{1.1}$$

This equation attached considerable attention because of its importance in general relativity. In particular, the equation is a reduction of the Einstein equations that govern self-dual gravitation fields, see [12, 26, 30, 35] and references therein. Eq. (1.1) is an example of a nonlinear integrable equation in four independent variables. Here integrability means the existence of a Lax pair, or a differential covering,

$$\begin{cases} q_t = (u_{xy} + \lambda)\,q_x - u_{xx}\,q_y, \\ q_z = u_{yy}\,q_x - (u_{xy} - \lambda)\,q_y, \end{cases} \tag{1.2}$$

with a non-removable parameter λ. Expanding the pseudopotential q into a Taylor series $q = \sum_{k=0}^{\infty} \lambda^k q_k$ yields a new covering (4.1) with pseudopotentials q_k over Eq. (1.1). The goal of this paper is to study nonlocal symmetries for Eq. (1.1) in this covering.

Infinite-dimensional Lie algebras of nonlocal symmetries are well known to play an important role in the theory of nonlinear integrable equations and provide a useful tool to study of the latter, see e.g. [5, 7, 16, 17, 20, 33, 34] and references therein. Eq. (1.1) has the infinite-dimensional Lie algebra $\mathfrak{s}$ of local contact symmetries and, as it was shown in [24], is uniquely defined by this algebra, see also [11, 25] for discussion of geometric properties of Eq. (1.1) and related equations. The algebra $\mathfrak{s}$ is the semi-direct product $\mathfrak{s} = \mathfrak{s}_\infty \rtimes \mathfrak{s}_\diamond$, where $\mathfrak{s}_\infty = [[\mathfrak{s},\mathfrak{s}],[\mathfrak{s},\mathfrak{s}]]$ is an infinite-dimensional ideal and $\mathfrak{s}_\diamond$ is a three-dimensional solvable Lie algebra. We show that all local symmetries of Eq. (1.1) have

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lifts to nonlocal symmetries, that is to symmetries of the system (1.1), (4.1). Note that not every integrable equation allows lifts of local symmetries to nonlocal symmetries, see e.g. [10]. Also, we find two infinite hierarchies $\mathfrak{a}$ and $\mathfrak{b}$ of nonlocal symmetries such that

$$[\mathfrak{a},\mathfrak{a}] = 0, \qquad [\mathfrak{b},\mathfrak{b}] = 0. \tag{1.3}$$

Note that existence of an infinite hierarchy of commuting flows is one of the most important properties of integrability, see [1, 6, 14, 19, 31] and references therein. Also, we find the structure of the Lie algebra[a] $\mathfrak{s} \oplus \mathfrak{a} \oplus \mathfrak{b}$ (the sum as vector spaces). In paricular, we show that

$$[\mathfrak{s}_\infty,\mathfrak{a}] = [\mathfrak{s}_\infty,\mathfrak{b}] = 0, \quad [\mathfrak{s}_\diamond,\mathfrak{a}] = \mathfrak{a}, \quad [\mathfrak{s}_\diamond,\mathfrak{b}] = \mathfrak{b}, \quad [\mathfrak{a},\mathfrak{b}] = \mathfrak{a}. \tag{1.4}$$

There is a great many of works devoted to methods of studying nonlinear partial differential equations (PDEs) that admit nonlocal symmetries. For the case of potential symmetries, that is nonlocal symmetries corresponding to Abelian coverings[b], see [8, Ch. 7], [9], and references therein. In regard to applications of nonlocal symmetries in non-Abelian coverings to studying of exact solutions and other integrability properties of nonlinear PDEs, see [5, 16, 17, 34] and references therein.

2. Preliminaries

All considerations in this paper are local. The presentation in this section closely follows [20, 21], see also [22, 23]. Let $\pi\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\pi\colon (x^1,\dots,x^n,u^1,\dots,u^m) \mapsto (x^1,\dots,x^n)$ be a trivial bundle, and $J^\infty(\pi)$ be the bundle of its jets of the infinite order. The local coordinates on $J^\infty(\pi)$ are $(x^i,u^\alpha,u^\alpha_I)$, where $I = (i_1,\dots,i_n)$ is a multi-index, and for every local section $f\colon \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ of π the corresponding infinite jet $j_\infty(f)$ is a section $j_\infty(f)\colon \mathbb{R}^n \to J^\infty(\pi)$ such that $u^\alpha_I(j_\infty(f)) = \dfrac{\partial^{\#I} f^\alpha}{\partial x^I} = \dfrac{\partial^{i_1+\dots+i_n} f^\alpha}{(\partial x^1)^{i_1}\dots(\partial x^n)^{i_n}}$. We put $u^\alpha = u^\alpha_{(0,\dots,0)}$. Also, in the case of $n = 4$, $m = 1$ we denote $x^1 = t$, $x^2 = x$, $x^3 = y$, $x^4 = z$ and $u^1_{(i,j,k,l)} = u_{t\dots tx\dots xy\dots yz\dots z}$ with i times t, j times x, k times y, and l times z.

The vector fields

$$D_{x^k} = \frac{\partial}{\partial x^k} + \sum_{\#I\ge 0}\sum_{\alpha=1}^{m} u^\alpha_{I+1_k}\frac{\partial}{\partial u^\alpha_I}, \qquad k \in \{1,\dots,n\},$$

$(i_1,\dots,i_k,\dots,i_n) + 1_k = (i_1,\dots,i_k+1,\dots,i_n)$, are called *total derivatives*. They commute everywhere on $J^\infty(\pi)$: $[D_{x^i}, D_{x^j}] = 0$.

The *evolutionary derivation* associated to an arbitrary smooth function $\varphi\colon J^\infty(\pi) \to \mathbb{R}^m$ is the vector field

$$\mathbf{E}_\varphi = \sum_{\#I\ge 0}\sum_{\alpha=1}^{m} D_I(\varphi^\alpha)\frac{\partial}{\partial u^\alpha_I}, \tag{2.1}$$

with $D_I = D_{(i_1,\dots i_n)} = D^{i_1}_{x^1}\circ\dots\circ D^{i_n}_{x^n}$.

A system of PDEs $F_r(x^i,u^\alpha_I) = 0$, $\#I \le s$, $r \in \{1,\dots,R\}$, of the order $s \ge 1$ with $R \ge 1$ defines the submanifold $\mathcal{E} = \{(x^i,u^\alpha_I) \in J^\infty(\pi) \mid D_K(F_r(x^i,u^\alpha_I)) = 0,\ \#K \ge 0\}$ in $J^\infty(\pi)$.

[a]We identify here the Lie algebra $\mathfrak{s}$ of local symmetries with its lift to nonlocal symmetries in the covering (4.1)
[b]See definition of an Abelian covering in Section 2

A function $\varphi\colon J^\infty(\pi) \to \mathbb{R}^m$ is called a *(generator of an infinitesimal) symmetry* of $\mathcal{E}$ when $\mathbf{E}_\varphi(F) = 0$ on $\mathcal{E}$. The symmetry φ is a solution to the *defining system*

$$\ell_{\mathcal{E}}(\varphi) = 0, \tag{2.2}$$

where $\ell_{\mathcal{E}} = \ell_F|_{\mathcal{E}}$ with the matrix differential operator

$$\ell_F = \left(\sum_{\#I \geq 0} \frac{\partial F_r}{\partial u_I^\alpha} D_I \right).$$

Solutions to (2.2) constitute the *Lie algebra* $\mathfrak{Lie}(\mathcal{E})$ *of (infinitesimal) symmetries of equation* $\mathcal{E}$ with respect to the *Jacobi bracket* $\{\varphi, \psi\} = \mathbf{E}_\varphi(\psi) - \mathbf{E}_\psi(\varphi)$. The subalgebra of *contact symmetries* of $\mathcal{E}$ is $\mathfrak{Lie}_0(\mathcal{E}) = \mathfrak{Lie}(\mathcal{E}) \cap C^\infty(J^1(\pi), \mathbb{R}^m)$.

Denote $\mathcal{W} = \mathbb{R}^\infty$ with coordinates w^s, $s \in \mathbb{N} \cup \{0\}$. Locally, an (infinite-dimensional) *differential covering* of $\mathcal{E}$ is a trivial bundle $\tau\colon J^\infty(\pi) \times \mathcal{W} \to J^\infty(\pi)$ equipped with the *extended total derivatives*

$$\tilde{D}_{x^k} = D_{x^k} + \sum_{s=0}^{\infty} T_k^s(x^i, u_I^\alpha, w^j) \frac{\partial}{\partial w^s} \tag{2.3}$$

such that $[\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0$ for all $i \neq j$ whenever $(x^i, u_I^\alpha) \in \mathcal{E}$. We define the partial derivatives of w^s by $w^s_{x^k} = \tilde{D}_{x^k}(w^s)$. This yields the system of *covering equations*

$$w^s_{x^k} = T_k^s(x^i, u_I^\alpha, w^j). \tag{2.4}$$

This over-determined system of PDEs is compatible whenever $(x^i, u_I^\alpha) \in \mathcal{E}$.

A covering is said to be Abelian when system (2.4) can be mapped to the form

$$w^0_{x^k} = T_k^0(x^i, u_I^\alpha), \qquad w^j_{x^k} = T_k^s(x^i, u_I^\alpha, w^0, \ldots, w^{j-1}), \quad j \in \mathbb{N}$$

by a change of variables x^i, u_I^α, w^s. Otherwise the covering is non-Abelian.

Denote by $\tilde{\mathbf{E}}_\varphi$ the result of substitution of $\tilde{D}_{x^k}$ for D_{x^k} in (2.1). A *shadow of nonlocal symmetry* of $\mathcal{E}$ corresponding to the covering τ with the extended total derivatives (2.3), or τ*-shadow*, is a function $\varphi \in C^\infty(\mathcal{E} \times \mathcal{W}, \mathbb{R}^m)$, such that

$$\tilde{\mathbf{E}}_\varphi(F) = 0 \tag{2.5}$$

is a consequence of equations $D_K(F) = 0$ and (2.4). A *nonlocal symmetry* of $\mathcal{E}$ corresponding to the covering τ (or τ*-symmetry*) is the vector field

$$\tilde{\mathbf{E}}_{\varphi,A} = \tilde{\mathbf{E}}_\varphi + \sum_{s=0}^{\infty} A^s \frac{\partial}{\partial w^s}, \tag{2.6}$$

with $A^s \in C^\infty(\mathcal{E} \times \mathcal{W})$ such that φ satisfies (2.5) and

$$\tilde{D}_{x^k}(A^s) = \tilde{\mathbf{E}}_{\varphi,A}(T_k^s) \tag{2.7}$$

for T_k^s from (2.3), see [10, Ch. 6, §3.2].

Remark 2.1. In general, not every τ-shadow corresponds to a τ-symmetry, since equations (2.7) provide an obstruction for existence of A^s in (2.6). But for any τ-shadow φ there exists a covering τ_φ and a nonlocal τ_φ-symmetry whose τ_φ-shadow coincides with φ, see [10, Ch. 6, §5.8].

3. Local symmetries

The local contact symmetries of Eq. (1.1) are solutions $\varphi = \varphi(t,x,y,z,u,u_t,u_x,u_y,u_z)$ to the equation

$$\ell_{\mathcal{E}}(\varphi) = D_x D_z(\varphi) - D_t D_y(\varphi) - u_{xx} D_y^2(\varphi) + 2\,u_{xy} D_x D_y(\varphi) = 0.$$

Direct computations[c] show that the Lie algebra $\mathfrak{s} = \mathfrak{Lie}_0(\mathcal{E})$ is generated by the following functions

$$\begin{aligned}
\varphi_0(A) &= -A_z\,u_t - (x A_{tz} + y A_{zz})\,u_x + (x A_{tt} + y A_{tz})\,u_y + A_t\,u_z \\
&\quad -\tfrac{1}{6}\left(x^3 A_{ttt} + 3x^2 y A_{ttz} + 3xy^2 A_{tzz} + y^3 A_{zzz}\right),\\
\varphi_1(A) &= -A_z\,u_x + A_t\,u_y - \tfrac{1}{2}\left(x^2 A_{tt} + 2xyA_{tz} + y^2 A_{zz}\right),\\
\varphi_2(A) &= -A_t - yA_z,\\
\varphi_3(A) &= -A,\\
\psi_1 &= 3\,u - x\,u_x - y\,u_y,\\
\psi_2 &= -3\,t\,u_t - x\,u_x - y\,u_y - 3\,z\,u_z,\\
\psi_3 &= -t\,u_x - z\,u_y,
\end{aligned}$$

where $A = A(t,z)$ and $B = B(t,z)$ below are arbitrary functions of t and z. The structure of $\mathfrak{s}$ is given by equations

$$\{\varphi_i(A), \varphi_j(B)\} = \begin{cases} \varphi_{i+j}(A_t B_z - A_z B_t), & i+j \le 4,\\ 0, & i+j > 4, \end{cases} \tag{3.1}$$

$$\{\psi_1, \varphi_j(A)\} = j\,\varphi_j(A), \tag{3.2}$$

$$\{\psi_2, \varphi_j(A)\} = \varphi_j(2\,(3-j)A - 3\,(tA_t + zA_z)), \tag{3.3}$$

$$\{\psi_3, \varphi_j(A)\} = \begin{cases} \varphi_{j+1}((2-j)A - tA_t - zA_z), & j \le 2,\\ 0, & j = 3, \end{cases} \tag{3.4}$$

$$\{\psi_1,\psi_2\} = 0, \qquad \{\psi_1,\psi_3\} = \psi_3, \qquad \{\psi_2,\psi_3\} = -2\,\psi_3. \tag{3.5}$$

Remark 3.1. We have $\mathfrak{s} = \mathfrak{s}_\infty \rtimes \mathfrak{s}_\diamond$, where $\mathfrak{s}_\infty$ is generated by $\varphi_i(A)$ and $\mathfrak{s}_\diamond$ is generated by ψ_j. The algebra $\mathfrak{s}_\infty$ admits the following description. Consider the (commutative associative) algebra of truncated polynomials $\mathbb{R}_4[s] = \mathbb{R}[s]/(s^4)$ and the Lie algebra $\mathfrak{h}$ of Hamiltonian vector fields on $\mathbb{R}^2$, [15]. Then $\mathfrak{s}_\infty$ is isomorphic to the Lie algebra $\mathbb{R}_4[s]\otimes\mathfrak{h}$ with the bracket $[f\otimes V, g\otimes W] = fg\otimes[V,W]$ for $f,g\in\mathbb{R}_4[s]$ and $V,W\in\mathfrak{h}$.

4. Infinite-dimensional covering, shadows, and nonlocal symmetries

Substituting

$$q = \sum_{k=0}^{\infty} \lambda^k\, q_k.$$

in the system (1.2) yields new (infinite-dimensional) covering

$$\begin{cases}
q_{0,t} &= u_{xy}\,q_{0,x} - u_{xx}\,q_{0,y},\\
q_{0,z} &= u_{yy}\,q_{0,x} - u_{xy}\,q_{0,y},\\
q_{m,t} &= u_{xy}\,q_{m,x} - u_{xx}\,q_{m,y} + q_{m-1,x},\\
q_{m,z} &= u_{yy}\,q_{m,x} - u_{xy}\,q_{m,y} + q_{m-1,y}, & m \ge 1.
\end{cases} \tag{4.1}$$

[c] We carried out all computations in the *Jets* software [4].

Direct computations prove:

Proposition 4.1. *Function $\upsilon = q$ is a shadow in the covering (1.2).*

Then we have:

Corollary 4.1. *Functions $\upsilon_k = q_k$, $k \geq 0$, are shadows in the covering (4.1).*

A nonlocal symmetry of Eq. (1.1) is an infinite sequence $(\varphi, Q_0, Q_1, \dots, Q_m, \dots)$, where $\varphi = \varphi(x^i, u, u_{x^i}, \dots, q_j, q_{j,x}, q_{j,y}, \dots)$ and $Q_m = Q_m(x^i, u, u_{x^i}, \dots, q_j, q_{j,x}, q_{j,y}, \dots)$, $m \geq 0$, are solutions to the equation

$$\varphi_{xz} - \varphi_{ty} - u_{yy}\,\varphi_{xx} - u_{xx}\,\varphi_{yy} + 2\,u_{xy}\,\varphi_{xy} = 0$$

and the linearization

$$\begin{cases} Q_{0,t} &= u_{xy}\,Q_{0,x} + q_{0,x}\,\varphi_{xy} - u_{xx}\,Q_{0,y} - q_{0,y}\,\varphi_{xx}, \\ Q_{0,z} &= u_{yy}\,Q_{0,x} + q_{0,x}\,\varphi_{yy} - u_{xy}\,Q_{0,y} - q_{0,y}\,\varphi_{xy}, \\ Q_{m,t} &= u_{xy}\,Q_{m,x} + q_{m,x}\,\varphi_{xy} - u_{xx}\,Q_{m,y} - q_{m,y}\,\varphi_{xx} + Q_{m-1,x}, \\ Q_{m,z} &= u_{yy}\,Q_{m,x} + q_{m,x}\,\varphi_{yy} - u_{xy}\,Q_{m,y} - q_{m,y}\,\varphi_{xy} + Q_{m-1,y}, \qquad m \geq 1. \end{cases}$$

of the system (4.1).

Remark 4.1. To simplify notation, here and below we use φ_{xz} for $\tilde{D}_x\tilde{D}_z(\varphi)$, etc.

The nonlocal symmetries of Eq. (1.1) are described by the following theorems:

Theorem 4.1. *The local symmetries $\varphi_0(A)$, ... , $\varphi_3(A)$, ψ_1, ψ_2, ψ_3 have the lifts $\Phi_0(A)$, ..., $\Phi_3(A)$, Ψ_1, Ψ_2, Ψ_3 to the nonlocal symmetries in the covering (4.1) defined as $\Phi_i(A) = (\varphi_i(A), \Phi_{i,0}(A), \Phi_{i,1}(A), ..., \Phi_{i,k}(A), ...)$, $\Psi_j = (\psi_j, \Psi_{j,0}, \Psi_{j,1}, ..., \Psi_{j,k}, ...)$, with*

$$\begin{aligned} \Phi_{0,k}(A) &= -A_z\,q_{k,t} - (x A_{tz} + y A_{zz})\,q_{k,x} + (x A_{tt} + y A_{tz})\,q_{k,y} + A_t\,q_{k,z}, \\ \Phi_{1,k}(A) &= -A_z\,q_{k,x} + A_t\,q_{k,y}, \\ \Phi_{2,k}(A) &= \Phi_{3,k}(A) = 0, \\ \Psi_{1,k} &= -x\,q_{k,x} - y\,q_{k,y} - k\,q_k, \\ \Psi_{2,k} &= -3\,t\,q_{k,t} - x\,q_{k,x} - y\,q_{k,y} - 3\,z\,q_{k,z} + 2\,k\,q_k, \\ \Psi_{3,k} &= -t\,q_{k,x} - z\,q_{k,y} + (k+1)\,q_{k+1}. \end{aligned}$$

The proof is similar to the proof of theorem 4.2 below and is therefore omitted.

Theorem 4.2. *The shadows $\upsilon_k = q_k$, $k \geq 0$, have the lifts Υ_k to the nonlocal symmetries in the covering (4.1) defined as $\Upsilon_k = (q_k, \Upsilon_{k,0}, \Upsilon_{k,1}, ..., \Upsilon_{k,m}, ...)$ with*

$$\Upsilon_{k,m} = \sum_{s=0}^{m} \langle q_s, q_{k+m+1-s} \rangle,$$

where $\langle a, b \rangle = a_x\,b_y - a_y\,b_x$.

Remark 4.2. The bracket $\langle \cdot, \cdot \rangle$ is a Lie bracket. It endows the space $C^\infty(\mathbb{R}^2)$ of smooth functions on $\mathbb{R}^2$ with the structure of a Lie algebra with the noncentral part isomorphic to the algebra of Hamiltonian vector fields on $\mathbb{R}^2$ and the center generated by the constant functions.

Proof. We claim that [d]

$$\Upsilon_{k,m,t} = \langle \Upsilon_{k,m}, u_x \rangle + \langle q_m, q_{k,x} \rangle + \Upsilon_{k,m-1,x}.$$

Indeed,

$$\begin{aligned}
\Upsilon_{k,m,t} &= \sum_{s=0}^{m} \langle q_s, q_{k+m+1-s} \rangle_t = \sum_{s=0}^{m} \langle q_{s,t}, q_{k+m+1-s} \rangle + \sum_{s=0}^{m} \langle q_s, q_{k+m+1-s,t} \rangle = \\
&= \langle \langle q_0, u_x \rangle, q_{k+m+1} \rangle + \sum_{s=1}^{m} \langle \langle q_s, u_x \rangle + q_{s-1,x}, q_{k+m+1-s} \rangle + \\
&\quad + \langle q_0, \langle q_{k+m+1}, u_x \rangle + q_{k+m,x} \rangle + \sum_{s=1}^{m} \langle q_s, \langle q_{k+m+1-s}, u_x \rangle + q_{k+m-s,x} \rangle = \\
&= \langle \langle q_0, u_x \rangle, q_{k+m+1} \rangle + \sum_{s=1}^{m} \langle \langle q_s, u_x \rangle, q_{k+m+1-s} \rangle + \underbrace{\sum_{s=1}^{m} \langle q_{s-1,x}, q_{k+m+1-s} \rangle}_{= \sum_{s=0}^{m-1} \langle q_{s,x}, q_{k+m-s} \rangle} + \\
&\quad + \langle q_0, \langle q_{k+m+1}, u_x \rangle \rangle + \underbrace{\langle q_0, q_{k+m,x} \rangle + \sum_{s=1}^{m} \langle q_s, q_{k+m-s,x} \rangle \rangle}_{= \langle q_m, q_{k,x} \rangle + \sum_{s=0}^{m-1} \langle q_s, q_{k+m-s,x} \rangle} + \sum_{s=1}^{m} \langle q_s, \langle q_{k+m+1-s}, u_x \rangle \rangle = \\
&= \langle q_m, q_{k,x} \rangle + \Upsilon_{k,m-1,x} + \sum_{s=0}^{m} \underbrace{\langle \langle q_s, u_x \rangle, q_{k+m+1-s} \rangle}_{= \langle \langle q_s, q_{k+m+1-s} \rangle, u_x \rangle - \langle \langle q_{k+m+1-s}, u_x \rangle, q_s \rangle} + \langle \langle q_{k+m+1-s}, u_x \rangle, q_s \rangle = \\
&= \langle \Upsilon_{k,m}, u_x \rangle + \langle q_m, q_{k,x} \rangle + \Upsilon_{k,m-1,x}.
\end{aligned}$$

Proof of the equality $\Upsilon_{k,m,z} = \langle \Upsilon_{k,m}, u_y \rangle + \langle q_m, q_{k,y} \rangle + \Upsilon_{k,m-1,y}$ is analogous. □

Remark 4.3. We note that the nonlocal symmetries Υ_k are similar to the nonlocal symmetries for the four-dimensional Martínez Alonso–Shabat equation found in [29], but the Lie brackets on $C^\infty(\mathbb{R}^2)$ here and in the constructions of [29] are different.

Theorem 4.3. *Eq. (1.1) has 'invisible' symmetries (symmetries with the zero shadow) in the covering (4.1) defined as*

$$\Gamma_k = (\underbrace{0, \ldots, 0}_{k}, q_0, q_1, q_2, \ldots, q_m, \ldots), \qquad k \geq 1.$$

The proof is similar to the proof of theorem 4.2 and is therefore omitted.

5. The structure of the algebra of nonlocal symmetries

The structure of the algebra $\tilde{\mathfrak{s}}$ of nonlocal symmetries $\Phi_m(A)$, Ψ_k, Υ_i, Γ_j of Eq. (1.1) in the covering (4.1) is described by the following theorem.

[d] see remark 4.1.

Theorem 5.1. *The Jacobi brackets of lifts of the local symmetries are lifts of the Jacobi brackets of the corresponding local symmetries, that is, the commutators for $\Phi_m(A)$, Ψ_k satisfy equations (3.1)—(3.5) with $\varphi_m(A)$, ψ_k replaced by $\Phi_m(A)$, Ψ_k, respectively. The other Jacobi brackets are*

$$\{\Phi_m(A),\Upsilon_k\} = \{\Phi_m(A),\Gamma_i\} = \{\Upsilon_k,\Upsilon_l\} = \{\Gamma_i,\Gamma_j\} = 0, \qquad 0 \le m \le 3, \quad k,l \ge 0, \quad i,j \ge 1,$$
$$\{\Psi_1,\Upsilon_k\} = (k+3)\,\Upsilon_k, \qquad \{\Psi_2,\Upsilon_k\} = -2k\,\Upsilon_k, \qquad \{\Psi_3,\Upsilon_k\} = -(k+1)\,\Upsilon_{k+1},$$
$$\{\Psi_1,\Gamma_i\} = -(i-1)\,\Gamma_i, \qquad \{\Psi_2,\Gamma_i\} = 2\,(i-1)\,\Gamma_k, \qquad \{\Psi_3,\Gamma_i\} = (i-1)\,\Gamma_{i-1},$$
$$\{\Gamma_i,\Upsilon_k\} = \begin{cases} -\Upsilon_{k-i+1}, & k \ge i-1, \\ 0, & k < i-1. \end{cases}$$

Proof. We will prove that $\{\Phi_0(A),\Upsilon_k\} = 0$. The proof for the other Jacobi brackets is similar and therefore is omitted.

We start with writing down explicitly the formula for computing the Jacobi bracket in case of some infinite-dimensional vectors Θ and Ω. In order to facilitate applying this formula to vectors $\Phi_0(A)$ and Υ_k we enumerate coordinates of the vectors Θ and Ω in the following way: $\Theta = (\Theta_{-1},\Theta_0,\dots,\Theta_m,\dots)$, $\Omega = (\Omega_{-1},\Omega_0,\dots,\Omega_m,\dots)$. Then the Jacobi bracket is of the form $\{\Theta,\Omega\} = (\{\Theta,\Omega\}_{-1},\{\Theta,\Omega\}_0,\{\Theta,\Omega\}_1,\dots)$, where

$$\{\Theta,\Omega\}_j = \sum_I \Bigg(\tilde{D}_I\,(\Theta_{-1})\,\frac{\partial}{\partial u_I}\,(\Omega_j) - \tilde{D}_I\,(\Omega_{-1})\,\frac{\partial}{\partial u_I}\,(\Theta_j) + \\ + \sum_{m=0}^{\infty}\left(\tilde{D}_I\,(\Theta_m)\,\frac{\partial}{\partial q_{m,I}}\,(\Omega_j) - \tilde{D}_I\,(\Omega_m)\,\frac{\partial}{\partial q_{m,I}}\,(\Theta_j)\right)\Bigg), \qquad j \ge -1.$$

As for coordinates $q_{m,I}$, $\Phi_{0,j}(A)$ depends at most on $q_{j,t}$, $q_{j,x}$, $q_{j,y}$, $q_{j,z}$, q_j or q_{j+1} and $\Upsilon_{k,j}$ depends at most on $q_{m,x}$, $q_{k+j+1-m,x}$, $q_{m,y}$, $q_{k+j+1-m,y}$ for $m = 0$, ..., j. The following observations will be used frequently:

$$\sum_I \sum_{m=0}^{\infty} \tilde{D}_I(a_m)\frac{\partial}{\partial q_{m,I}}\,(\Phi_{0,j}) = \sum_{l=t,x,y,z} \tilde{D}_l(a_j)\frac{\partial}{\partial q_{j,l}}\,(\Phi_{0,j}), \qquad j \ge -1, i = 0,1,2,3.$$
$$\begin{aligned}\sum_I \sum_{m=0}^{\infty} \tilde{D}_I(a_m)\frac{\partial}{\partial q_{m,I}}\,(\Upsilon_{k,j}) &= \sum_{m=0}^{j}\big(\tilde{D}_x(a_m)q_{k+j+1-m,y} - \tilde{D}_x(a_{k+j+1-m})q_{m,y} \\ &\quad -\tilde{D}_y(a_m)q_{k+j+1-m,x} + \tilde{D}_y(a_{k+j+1-m})q_{m,x}\big) \\ &= \sum_{m=0}^{j}\langle a_m, q_{k+j+1-m}\rangle - \langle a_{k+j+1-m}, q_m\rangle.\end{aligned}$$

We have

$$\{\Phi_i(A),\Upsilon_k\}_{-1} = \sum_I \tilde{D}_I\,(\phi_i(A))\underbrace{\frac{\partial}{\partial u_I}\,(q_k)}_{=0} - \sum_I \tilde{D}_I\,(q_k)\,\frac{\partial}{\partial u_I}\,(\varphi_i(A)) + \\ + \underbrace{\sum_I \sum_{m=0}^{\infty} \tilde{D}_I\,(\Phi_{i,m}(A))\,\frac{\partial}{\partial q_{m,I}}\,(q_k)}_{=\Phi_{i,k}(A)} - \sum_I \sum_{m=0}^{\infty} \tilde{D}_I\,(\Upsilon_{k,m})\underbrace{\frac{\partial}{\partial q_{m,I}}\,(\varphi_i(A))}_{=0}, \qquad 0 \le i \le 3.$$

It is easy to see, that $\sum\limits_I \tilde{D}_I(q_k)\frac{\partial}{\partial u_I}(\varphi_i(A)) = \Phi_i^k(A)$ for $i = 0,1,2,3$, hence $\{\Phi_i(A),\Upsilon_k\}_{-1} = 0$. Then for $j \geq 0$ we get

$$\{\Phi_i(A),\Upsilon_k\}_j = \sum_I \tilde{D}_I(\varphi_i(A))\underbrace{\frac{\partial}{\partial u_I}(\Upsilon_{k,j})}_{=0} - \sum_I \tilde{D}_I(q_k)\underbrace{\frac{\partial}{\partial u_I}(\Phi_{i,j}(A))}_{=0} +$$
$$+\sum_I \sum_{m=0}^{\infty}\left(\tilde{D}_I(\Phi_{i,m}(A))\frac{\partial}{\partial q_{m,I}}(\Upsilon_{k,j})\right) - \sum_I \sum_{m=0}^{\infty}\left(\tilde{D}_I(\Upsilon_{k,m})\frac{\partial}{\partial q_{m,I}}(\Phi_{i,j}(A))\right).$$

Furthermore, we have

$$\{\Phi_0(A),\Upsilon_k\}_j =$$
$$= \sum_{l=x,y}\sum_{m=0}^{\infty}\left(\tilde{D}_l(\Phi_{0,m}(A))\frac{\partial}{\partial q_{m,l}}(\Upsilon_{k,j})\right) - \sum_{l=t,x,y,z}\sum_{m=0}^{\infty}\left(\tilde{D}_l(\Upsilon_{k,m})\frac{\partial}{\partial q_{m,l}}(\Phi_{0,j}(A))\right) =$$
$$= \sum_{m=0}^{j}\big(\tilde{D}_x(\Phi_{0,m}(A))\,q_{k+j+1-m,y} - \tilde{D}_x\left(\Phi_{0,k+j+1-m}(A)\right)q_{m,y} - \tilde{D}_y(\Phi_{0,m}(A))\,q_{k+j+1-m,x} +$$
$$+\tilde{D}_y\left(\Phi_{0,k+j+1-m}(A)\right)q_{m,x}\big) - \sum_{l=t,x,y,z}\tilde{D}_l\left(\Upsilon_{k,j}\right)\frac{\partial}{\partial q_{j,l}}(\Phi_{0,j}(A)) =$$
$$= \sum_{m=0}^{j}\big(\langle\Phi_{0,m}(A),q_{k+j+1-m}\rangle - \langle\Phi_{0,k+j+1-m}(A),q_m\rangle\big) -$$
$$- \sum_{l=t,x,y,z}\tilde{D}_l\left(\sum_{m=0}^{j}\langle q_m,q_{k+j+1-m}\rangle\right)\frac{\partial}{\partial q_{j,l}}(\Phi_{0,j}(A)) =$$
$$= \sum_{m=0}^{j}\Bigg(\langle\Phi_{0,m}(A),q_{k+j+1-m}\rangle - \langle\Phi_{0,k+j+1-m}(A),q_m\rangle$$
$$- \sum_{l=t,x,y,z}\big(\langle q_{m,l},q_{k+j+1-m}\rangle + \langle q_m,q_{k+j+1-m,l}\rangle\big)\frac{\partial}{\partial q_{j,l}}(\Phi_{0,j}(A))\Bigg) =$$
$$= \sum_{m=0}^{j}\big(\langle\Phi_{0,m}(A),q_{k+j+1-m}\rangle - \langle\Phi_{0,k+j+1-m}(A),q_m\rangle +$$
$$+\langle\Phi_{0,k+j+1-m}(A),q_m\rangle - \langle\Phi_0^m(A),q_{k+j+1-m}\rangle\big) = 0.$$

$\square$

Remark 5.1. Denote by $\mathfrak{a}$ the ideal of the Lie algebra $\tilde{\mathfrak{s}}$ generated by Υ_i and by $\mathfrak{b}$ the subalgebra generated by Γ_j. Identify $\mathfrak{s}_\infty$ and $\mathfrak{s}_\diamond$ with the ideal generated by $\Phi_m(A)$ and the subalgebra generated by Ψ_k, respectively. Then we obtain

$$\tilde{\mathfrak{s}} = (\mathfrak{s}_\infty \oplus (\mathfrak{a} \rtimes \mathfrak{b})) \rtimes \mathfrak{s}_\diamond$$

(here $\oplus$ denotes a direct sum of commuting Lie algebras), in particular we have equations (1.3) and (1.4) together with $[\mathfrak{s}_\infty,\mathfrak{s}_\infty] = \mathfrak{s}_\infty$, $[\mathfrak{s}_\diamond,\mathfrak{s}_\infty] = \mathfrak{s}_\infty$, and $[\mathfrak{s}_\diamond,\mathfrak{s}_\diamond] = \mathfrak{s}_\diamond$.

6. Conclusion

In this paper we have found two infinite hierarchies of commuting nonlocal symmetries for Plebański's second heavenly equation. One of these hierarchies can be employed to construct new solutions using the techniques presented e.g. in [10, 19, 31], see also [5, 16, 17, 34]. Furthermore, we find the lifts of local symmetries and the structure of the Lie algebra of the nonlocal symmetries. We emphasize that finding an explicit form of nonlocal symmetries and the commutator relations for an infinite-dimensional symmetry algebra of nonlocal symmetries (rather than just shadows) is quite rare. We know just a number of similar results in the literature, [1–3, 5, 13, 16–18, 29]. We hope that it would be very interesting to study how the infinite-dimensional Lie algebra of nonlocal symmetries reflects the algebraic structure behind the integrability properties of the considered equation, cf. [27, 28].

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