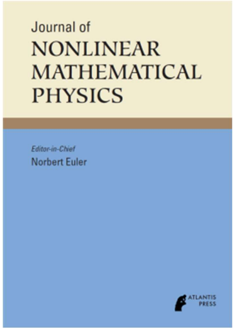

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Symmetry Reduction of Ordinary Differential Equations Using Moving Frames

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The symmetry reduction algorithm for ordinary differential equations due to Sophus Lie is revisited using the method of equivariant moving frames. Using the recurrence formulas provided by the theory of equivariant moving frames, computations are performed symbolically without relying on the coordinate expressions for the canonical variables and the differential invariants occurring in Lie's original procedure.

Keywords: Canonical variables, differential invariants, equivariant moving frames, Lie point symmetries, ordinary differential equations.

1. Introduction

Lie group integration techniques are amongst the most effective methods available for obtaining analytic solutions of nonlinear differential equations. Nowadays, there are many excellent textbooks surveying the subject, [3, 5–8, 10, 15–17, 27, 35–37]. Following Sophus Lie's seminal ideas, most symmetry-based integration methods rely on the infinitesimal symmetry generators of the differential equation under consideration. For example, given an n^{th} order ordinary differential equation with one infinitesimal symmetry generator, there is a well known procedure for reducing the order of the equation by one by introducing canonical variables that rectify the infinitesimal symmetry generator. More generally, if the differential equation admits an r-dimensional solvable symmetry group, then the equation can be reduced to an order $n-r$ differential equation, assuming $n > r$. Given a solution to the reduced equation, the solution to the original differential equation can then be recovered by quadrature.

To find the canonical variables (t,s) that rectify an infinitesimal symmetry generator, a system of first order linear partial differential equations must be solved. Once the canonical variables have been found, the order of the differential equation is reduced by re-expressing the equation in those new coordinates and introducing the new dependent variable $w = s_t$. For r-dimensional solvable symmetry groups, this procedure is repeated r times. At each iteration, one must keep track of the coordinate expressions of the prolonged infinitesimal symmetry generators in the new system of coordinates in order to compute the subsequent set of canonical variables to be introduced in the following iterations. For large symmetry groups, these coordinate dependent computations can become cumbersome and thereby limit the scope of the method. The main goal of the present paper is to present a symbolic implementation of Lie's symmetry reduction procedure for ordinary differential equations that does not rely on the coordinate expressions of the canonical variables and the differential invariants occurring in the standard implementation of the method. The constructions introduced are completely algorithmic and can therefore directly be implemented in symbolic computer packages such as MAPLE, MATHEMATICA, or SAGE.

This paper is part of recent efforts aimed at developing new symbolic computational tools for differential equations based on the method of equivariant moving frames. Recent developments have occurred in the calculus of variations, [19, 38], the computation of conservation laws, [12–14], the theory of geometric curve flows, [4, 23, 24, 29], the classification of differential invariants and their syzygies, [9, 33], the symmetry classification problem of differential equations, [21], the general equivalence problem of differential equations, [25, 41], and the method of group foliation, [22, 39]. For a comprehensive overview of modern applications of the method of equivariant moving frames, we refer the reader to [22, 31].

For one-dimensional symmetry groups, our symbolic implementation of the symmetry reduction algorithm is based on the standard implementation of the equivariant moving frame method, [11,22]. For solvable symmetry groups, we use the inductive/recursive moving frame constructions introduced in [18, 30, 34, 40] as group parameters are recursively normalized each time the order of the differential equation is reduced. As a byproduct, the ideas developed in this paper provide a new and nontrivial application of the inductive and recursive moving frame implementations introduced in [18,30,34,40]. When compared to the group foliation method developed in [22,39], where group parameters are normalized all at once, the recursive approach proposed in this paper offers several benefits. Firstly, as illustrated in [15], it is possible that one of the intermediate equations obtained during the reduction process is easier to solve than the final reduced equation. This situation can occur if, for example, one of the intermediate equations admits (type II) hidden symmetries, [1,20]. In this case, since the group foliation method only computes the final reduced equation, it would miss the intermediate reduced equation that allows to solve the original differential equation more easily. Secondly, in the implementation of the group foliation method, it is not clear which normalizations will produce the simplest reduced equation. On the other hand, when group parameters are normalized recursively, the reduced equation obtained at each stage of the symmetry reduction process can help make educated normalizations that will lead to the simplest reduced differential equation. Finally, in contrast to the group foliation method introduced in [22], our approach does not require the introduction of a "computational variable" when the independent variable is not an invariant of the group action. Avoiding the introduction of computational variables is generally desirable as these tend to lead to more complicated differential equations and also require the introduction of a "companion equation", which leads to the problem of solving a system of differential equations rather than just the original equation.

Given the solution to the fully reduced differential equation, we introduce a recursive reconstruction procedure for recovering the solution to the original differential equation. As in the reduction step, the reconstruction computations are performed symbolically without requiring coordinate expressions for the invariants introduced during the reduction process. To implement the reconstruction procedure, the main data required is the induced action of the various one-parameter groups on the invariants at each step of the reduction process. As it will be shown in Section 5.2, these one-parameter group actions (more precisely their corresponding infinitesimal generators[a]) can be

[a]Given an infinitesimal generator, the corresponding (local) one-parameter group action is obtained by exponentiating the vector field, see (2.4).

deduced symbolically using the recurrence relations provided by the theory of equivariant moving frames, [11, 19, 32].

For completeness, we begin in Section 2 by recalling standard notation and basic definitions pertaining to ordinary differential equations and symmetry groups. Lie's symmetry reduction algorithm for ordinary differential equations is then reviewed in Section 3, and the basic equivariant moving frame constructions are laid out in Section 4.1. In particular, the recurrence formulas for the normalized differential invariants are introduced. These equations provide a complete characterization of the structure of the algebra of differential invariants and their syzygies. An important aspect of these equations is that they can be obtained symbolically without relying on the coordinate formulas of the differential invariants or the moving frame. In Section 4.2, the inductive/recursive implementation of the moving frame method is introduced. Using the method of moving frames, Lie's symmetry reduction algorithm for one-dimensional symmetry groups is revisited in Section 5.1, and the general case for solvable symmetry groups is considered in Section 5.2.

2. Preliminaries

Let $\mathrm{J}^{(n)} = \mathrm{J}^n(\mathbb{R},\mathbb{R})$ denote the n^{th} order jet bundle of smooth maps $u\colon \mathbb{R} \to \mathbb{R}$. In the following, we allow the domain of the function $u(x)$ to be an open subset of $\mathbb{R}$. Local coordinates on $\mathrm{J}^{(n)}$ are given by $(x, u^{(n)})$, where x denotes the independent variable and $u^{(n)} = (u, u_x, u_{xx}, \ldots, u_n)$ collects the derivatives of the dependent variable u, up to order n. A basis of one-forms on the infinite-order jet bundle $\mathrm{J}^{(\infty)}$ is given by the horizontal one-form dx and the basic contact forms

$$\theta_k = du_k - u_{k+1}dx, \qquad k \geq 0. \tag{2.1}$$

This decomposition splits the exterior derivative into horizontal and vertical (or contact) components, and endows the space of differential forms on $\mathrm{J}^{(\infty)}$ with the structure of a variational bicomplex, [2, 19, 38]. The contact forms (2.1) play a fundamental role in the calculus of variations, but for the purpose of this paper these can be omitted. For this reason, all our considerations are done modulo contact forms.

Definition 2.1. An n^{th} *order ordinary differential equation* is the zero locus of a differential function $\Delta\colon \mathrm{J}^{(n)} \to \mathbb{R}$:

$$\Delta(x, u^{(n)}) = 0. \tag{2.2}$$

Definition 2.2. An ordinary differential equation $\Delta(x, u^{(n)}) = 0$ is said to be *regular* if its differential $d\Delta \neq 0$ does not vanish on the domain of definition of the equation. A differential equation is *locally solvable* at the point $(x_0, u_0^{(n)})$ if $\Delta(x_0, u_0^{(n)}) = 0$ and there exists a smooth solution $u = f(x)$, defined in the neighborhood of x_0, such that $u_0^{(n)} = f^{(n)}(x_0)$. An ordinary differential equation which is both regular and locally solvable is said to be *fully regular*.

In the following, we only consider fully regular ordinary differential equations. Now, let

$$\mathbf{v} = \xi(x,u)\frac{\partial}{\partial x} + \varphi(x,u)\frac{\partial}{\partial u} \tag{2.3}$$

be a local vector field on $\mathrm{J}^{(0)} \simeq \mathbb{R}^2$. The flow of the vector field (2.3) induces the one-parameter group of local transformations

$$(X, U) = g_\varepsilon \cdot (x,u) = \exp[\varepsilon \mathbf{v}] \cdot (x,u), \qquad \varepsilon \in \mathbb{R}. \tag{2.4}$$

The corresponding one-parameter Lie group is denoted by G_ε. We observe that G_ε is locally isomorphic to the Lie group $(\mathbb{R},+)$. In turn, the one-parameter group action (2.4) induces an action on the n^{th} order jet bundle $\mathrm{J}^{(n)}$, called the n^{th} order *prolonged action*, [27]. To obtain the coordinate expressions of the prolonged action, we introduce the implicit derivative operator

$$D_X = \frac{1}{D_x X} D_x = \frac{1}{X_x + X_u\, u_x} D_x, \tag{2.5}$$

where

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{k+1} \frac{\partial}{\partial u_k}$$

is the total derivative operator with respect to the independent variable x. The n^{th} order prolonged action is then obtained by successively differentiating $U = g_\varepsilon \cdot u$ with respect to the implicit derivative operator (2.5):

$$U_{X^k} = g_\varepsilon \cdot u_{x^k} = D_X^k(U), \qquad k = 1, \ldots, n. \tag{2.6}$$

At the infinitesimal level, the n^{th} order prolongation of the infinitesimal generator (2.3) is given by the vector field

$$\mathbf{v}^{(n)} = \xi(x,u) + \sum_{k=0}^{n} \varphi^k(x,u^{(k)}) \frac{\partial}{\partial u_k},$$

where the coefficient $\varphi^k(x,u^{(k)})$ is defined recursively by the formula

$$\varphi^{k+1} = D_x \varphi^k - u_{k+1} \cdot D_x \xi.$$

Definition 2.3. A one-parameter Lie group G_ε is said to be a *symmetry group* of the fully regular ordinary differential equation (2.2) if for every $g_\varepsilon \in G_\varepsilon$

$$\Delta(g_\varepsilon \cdot (x,u^{(n)})) = 0 \qquad \text{whenever} \qquad \Delta(x,u^{(n)}) = 0.$$

At the infinitesimal level, equation (2.2) is invariant if for all infinitesimal generator $\mathbf{v}$ in the Lie algebra $\mathfrak{g}$ of G_ε, we have

$$\mathbf{v}^{(n)}(\Delta) = 0 \qquad \text{whenever} \qquad \Delta = 0.$$

Remark 2.1. The above exposition easily extends to systems of ordinary differential equations involving several dependent variables $u = (u^1, \ldots, u^q)$. Also, the one-parameter group G_ε in Definition 2.3 can, in general, be replaced by an r-dimensional Lie group G.

3. Symmetry Reduction

We now review Lie's symmetry reduction algorithm for an ordinary differential equation invariant under a one-parameter symmetry group, [3, 5–8, 10, 15–17, 27, 35–37]. Therefore, let $\Delta(x,u^{(n)}) = 0$ be an ordinary differential equation invariant under the infinitesimal symmetry generator (2.3). The standard symmetry reduction algorithm is based on the introduction of a set of *canonical variables*

$$(t,s) = \Phi(x,u) \tag{3.1}$$

in which the vector field (2.3) takes the rectified form

$$\mathbf{v} = \frac{\partial}{\partial s}. \tag{3.2}$$

The canonical coordinates (t,s) are found by solving the system of linear first order partial differential equations

$$\mathbf{v}(t) = \xi\, t_x + \varphi\, t_u = 0, \qquad \mathbf{v}(s) = \xi\, s_x + \varphi\, s_u = 1, \tag{3.3}$$

using the method of characteristics. Letting s be the dependent variable and t the independent variable, the differential equation $\Delta(x,u^{(n)}) = 0$ can be rewritten in these new coordinates:

$$\widetilde{\Delta}(t,s^{(n)}) = \Delta(x,u^{(n)}) = 0.$$

Since equation $\widetilde{\Delta}(t,s^{(n)}) = 0$ is invariant under the infinitesimal symmetry generator (3.2), the equation must be independent of s:

$$\widetilde{\Delta}(t,s^{(n)}) = \widetilde{\Delta}(t,s_t,s_{tt},\dots,s_{t^n}).$$

Introducing the new dependent variable $w = s_t$, the order of the differential equation is decreased by one:

$$\widetilde{\Delta}(t,w,w_t,\dots,w_{t^{n-1}}) = \widetilde{\Delta}(t,w^{(n-1)}) = 0. \tag{3.4}$$

Assuming $w = w(t)$ is the general solution of the reduced equation (3.4), we have that

$$s(t) = \int w(t)\,dt,$$

and that the solution to the original differential equation $\Delta(x,u^{(n)}) = 0$ is obtained by inverting the change of variables (3.1):

$$(x(t),u(t)) = \Phi^{-1}(t,s(t)).$$

The latter gives a parametrized solution with parameter t. Expressing $t = t(x)$ as a function of x, the solution to the original differential equation $\Delta(x,u^{(n)}) = 0$ is $u(t(x))$.

As it is well known, the above reduction procedure can be iterated for multi-parameter solvable symmetry groups. Since there are different, yet equivalent, definitions of solvable Lie groups, we now introduce the definition used in the paper, which can be found in [27].

Definition 3.1. Let G be an r-dimensional Lie group with Lie algebra $\mathfrak{g}$. The Lie group G is said to be *solvable* if there exists a chain of Lie subgroups

$$\{e\} = G^{(0)} \lhd G^{(1)} \lhd G^{(2)} \lhd \cdots \lhd G^{(r-1)} \lhd G^{(r)} = G \tag{3.5}$$

such that for each $\ell = 1,\dots,r$, $G^{(\ell)}$ is a ℓ-dimensional subgroup of G and $G^{(\ell-1)}$ is a normal subgroup of $G^{(\ell)}$. At the infinitesimal level, the Lie algebra $\mathfrak{g}$ is solvable if there exists a chain of Lie subalgebras

$$\{0\} = \mathfrak{g}^{(0)} \subset \mathfrak{g}^{(1)} \subset \mathfrak{g}^{(2)} \subset \cdots \subset \mathfrak{g}^{(r-1)} \subset \mathfrak{g}^{(r)} = \mathfrak{g}, \tag{3.6}$$

such that for each ℓ, $\dim \mathfrak{g}^{(\ell)} = \ell$ and $\mathfrak{g}^{(\ell-1)}$ is a normal subalgebra[b] of $\mathfrak{g}^{(\ell)}$, which means that

$$[\mathfrak{g}^{(\ell-1)}, \mathfrak{g}^{(\ell)}] \subset \mathfrak{g}^{(\ell-1)}.$$

Assuming $\Delta(x, u^{(n)}) = 0$ is invariant under an r-dimensional solvable symmetry group G, the solvable structure of the symmetry group G will dictate the order in which symmetry reduction should by performed. After the introduction of a convenient basis $\{\mathbf{v}_1, \ldots \mathbf{v}_r\}$ for $\mathfrak{g}$, assume that

$$\mathfrak{g}^{(\ell)} = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}, \qquad \ell = 1, \ldots, r. \tag{3.7}$$

Such a set of vectors is said to form a *canonical basis*, [15]. According to this canonical basis, the differential equation $\Delta(x, u^{(n)}) = 0$ is first reduced using the infinitesimal generator $\mathbf{v}_1$. The resulting equation is then reduced using $\mathbf{v}_2$, $\mathbf{v}_3$, and so on up to $\mathbf{v}_r$. At the ℓ^{th} iteration, the solvable structure of the symmetry group guarantees that the resulting reduced equation is invariant under the infinitesimal symmetry generators $\mathbf{v}_{\ell+1}, \ldots, \mathbf{v}_r$.

The overall process can be computationally demanding as each time the order is reduced, new canonical variables are computed and the prolonged infinitesimal symmetry generators have to be re-expressed in these new coordinates. In Section 5, we will show how these coordinate dependent computations can be avoided using the method of moving frames. But before doing so, we illustrate Lie's symmetry reduction algorithm with a simple example. This will allow us to compare the standard implementation to the moving frame implementation introduced in Section 5.

Example 3.1. To illustrate Lie's symmetry reduction procedure, we consider the second order ordinary differential equation

$$x^2 u_{xx} = F(xu_x - u), \tag{3.8}$$

where, for now, $F\colon \mathbb{R} \to \mathbb{R}$ is an arbitrary function. Equation (3.8) admits a two-dimensional symmetry Lie algebra spanned by the vector fields

$$\mathbf{v}_1 = x\frac{\partial}{\partial u} \qquad \text{and} \qquad \mathbf{v}_2 = x\frac{\partial}{\partial x}. \tag{3.9}$$

The commutator of these two vector fields is

$$[\mathbf{v}_1, \mathbf{v}_2] = -\mathbf{v}_1.$$

The Lie algebra is therefore solvable[c] with the chain of normal subalgebras

$$\{0\} \subset \mathfrak{g}^{(1)} = \text{span}\{\mathbf{v}_1\} \subset \text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \mathfrak{g}.$$

We therefore start implementing the symmetry reduction algorithm using the infinitesimal generator $\mathbf{v}_1$. The canonical variables (t, s) are found by requiring that

$$\mathbf{v}_1(t) = x\frac{\partial t}{\partial u} = 0, \qquad \mathbf{v}_1(s) = x\frac{\partial s}{\partial u} = 1. \tag{3.10}$$

[b] The subalgebra $\mathfrak{g}^{(\ell-1)}$ is also called an *ideal* of $\mathfrak{g}^{(\ell)}$, [15].
[c] Any two-dimensional Lie algebra is solvable.

Using the method of characteristics, a solution is given by

$$t = x, \qquad s = \frac{u}{x}. \tag{3.11}$$

In these new variables, equation (3.8) becomes

$$t^3 s_{tt} + 2t^2 s_t = F(t^2 s_t).$$

Introducing the new dependent variable $w = s_t$, we obtain the first order reduced differential equation

$$t^3 w_t + 2t^2 w = F(t^2 w). \tag{3.12}$$

Expressing the first order prolonged infinitesimal generator

$$\mathbf{v}_2^{(1)} = x\frac{\partial}{\partial x} - u_x \frac{\partial}{\partial u_x}$$

in the variables

$$t = x \qquad s = \frac{u}{x}, \qquad w = s_t = \frac{x u_x - u}{x^2},$$

we find that

$$\mathbf{v}_2^{(1)} = -s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} - 2w\frac{\partial}{\partial w}.$$

Therefore, $\mathbf{v}_2^{(1)}$ has a well-defined restriction to the invariants (t, w):

$$\widehat{\mathbf{v}}_2 = \mathbf{v}_2^{(1)}\big|_{(t,w)} = t\frac{\partial}{\partial t} - 2w\frac{\partial}{\partial w}.$$

Now let (y, z) be new canonical variables that rectify $\widehat{\mathbf{v}}_2$ to $\dfrac{\partial}{\partial z}$. These variables are found by solving the system of linear partial differential equations

$$\widehat{\mathbf{v}}_2(y) = t\frac{\partial y}{\partial t} - 2w\frac{\partial y}{\partial w} = 0, \qquad \widehat{\mathbf{v}}_2(z) = t\frac{\partial z}{\partial t} - 2w\frac{\partial z}{\partial w} = 1.$$

A solution to these equations is given by

$$y = t^2 w, \qquad z = \ln t. \tag{3.13}$$

In those variables

$$z_y = \frac{1}{2t^2 w + t^3 w_t},$$

and the differential equation (3.12) becomes

$$z_y = \frac{1}{F(y)}.$$

To integrate the latter equation explicitly, one has to consider a particular function $F(y)$. For example, if $F(y) = y^2$ then

$$z_y = \frac{1}{y^2} \qquad \text{and} \qquad z(y) = -\frac{1}{y} - \ln C. \tag{3.14}$$

Substituting (3.13) into (3.14) we get

$$\ln t = -\frac{1}{t^2 w} - \ln C, \qquad \text{which implies that} \qquad w = -\frac{1}{t^2 \ln(Ct)}.$$

Recalling that $w = s_t$, we have

$$s(t) = \int w(t)\,dt = C\,\mathrm{E}_1(\ln(Ct)) + K,$$

where $\mathrm{E}_1(x) = -\mathrm{Ei}(-x)$ and $\mathrm{Ei}(x)$ is the exponential integral function, [26]. Substituting (3.11) into the latter result yields

$$u(x) = Cx\,\mathrm{E}_1(\ln(Cx)) + Kx,$$

which is the general solution to the nonlinear differential equation

$$x^2 u_{xx} = (xu_x - u)^2. \tag{3.15}$$

4. Moving Frames

We now introduce the moving frame constructions that will be used in the symbolic implementation of the symmetry reduction process introduced in Section 5. In Section 4.1, we first survey the standard implementation of the equivariant moving frame method as originally introduced in [11]. In Section 4.2, we review the inductive/recursive implementation of the moving frame method developed in [18, 30, 34, 40].

4.1. *Standard implementation*

Let G be an r-dimensional Lie group acting smoothly on the n^{th} order jet space $\mathrm{J}^{(n)}$. A *right moving frame* is a G-equivariant map $\rho : \mathrm{J}^{(n)} \to G$, where G-equivariance means that

$$\rho(g \cdot (x, u^{(n)})) = \rho(x, u^{(n)})\, g^{-1}$$

for all $g \in G$ where the prolonged action and the moving frame are defined. To every right moving frame ρ corresponds a left moving frame $\overline{\rho}$ given by group inversion:

$$\overline{\rho}(x, u^{(n)}) = (\rho(x, u^{(n)}))^{-1}.$$

The main existence theorem states that a moving frame exists in the neighborhood of a point $(x, u^{(n)}) \in \mathrm{J}^{(n)}$ provided the action is *(locally) free* and *regular* on that neighborhood. We recall that the action is free at a point $(x, u^{(n)})$ if the isotropy group

$$G_{(x,u^{(n)})} = \{g \in G \,|\, g \cdot (x, u^{(n)}) = (x, u^{(n)})\} = \{e\}$$

is trivial, and that the action is locally free at a point if the isotropy group is discrete. The action is regular if the group orbits have the same dimension and each point $(x, u^{(n)}) \in \mathrm{J}^{(n)}$ has arbitrary small neighborhoods whose intersection with each orbit is a connected subset thereof. A theorem originally due to Ovsiannikov, [35], and then corrected by Olver, [28], guarantees that any finite-dimensional Lie group acting locally effectively on subsets of $\mathrm{J}^{(0)} \simeq \mathbb{R}^2$ will eventually act locally freely on an open dense subset of $\mathrm{J}^{(n)}$ for a sufficiently large n. Therefore, a moving frame will exist provided the action is prolonged to a sufficiently high order jet space.

The construction of a moving frame is based on the introduction of a cross-section $\mathscr{K} \subset \mathrm{J}^{(n)}$ to the group orbits. By definition, a cross-section $\mathscr{K}$ is a submanifold transverse and of complementary dimension to the group orbits. In general, it is specified by a system of $r = \dim G$ equations

$$\mathscr{K} = \{F_\ell(x, u^{(n)}) = 0 \mid \ell = 1, \ldots, r\}.$$

To simplify the discussion, we assume

$$\mathscr{K} = \{z_1 = c_1, \ldots, z_r = c_r\} \subset \mathrm{J}^{(n)} \tag{4.1}$$

is a coordinate cross-section obtained by setting r coordinates of the n-jet $(x, u^{(n)})$ to constant values. The right moving frame $\rho(x, u^{(n)}) \in G$ at $(x, u^{(n)})$ is then the unique group element that sends $(x, u^{(n)})$ onto the cross-section $\mathscr{K}$:

$$\rho(x, u^{(n)}) \cdot (x, u^{(n)}) \in \mathscr{K}.$$

Coordinate expressions for the moving frame are obtained by solving the *normalization equations*

$$Z_\ell = g \cdot z_\ell = c_\ell, \qquad \ell = 1, \ldots, r,$$

for the group parameters $g = (g_1, \ldots, g_r)$.

Example 4.1. To illustrate the moving frame construction, we consider the two-parameter group action

$$X = e^{\varepsilon_2} x, \qquad U = u + \varepsilon_1 x, \qquad \varepsilon_1, \varepsilon_2 \in \mathbb{R}, \tag{4.2}$$

acting on the right half-plane $\mathbb{R}^+ \times \mathbb{R}$. To compute the induced prolonged action, we introduce the implicit derivative operator

$$D_X = \frac{1}{e^{\varepsilon_2}} D_x. \tag{4.3}$$

Applying (4.3) to $U = u + \varepsilon_1 x$, we obtain, up to order two, the prolonged action

$$U_X = D_X(U) = \frac{u_x + \varepsilon_1}{e^{\varepsilon_2}}, \qquad U_{XX} = D_X(U_X) = \frac{u_{xx}}{e^{2\varepsilon_2}}.$$

Choosing the coordinate cross-section

$$\mathscr{K} = \{x = 1,\ u = 0\},$$

and solving the normalization equations

$$1 = X = e^{\varepsilon_2} x, \qquad 0 = U = u + \varepsilon_1 x, \tag{4.4}$$

for the group parameters $(\varepsilon_1, \varepsilon_2)$, we obtain the right moving frame

$$\rho(x, u)\colon \qquad \varepsilon_1 = -\frac{u}{x}, \qquad \varepsilon_2 = \ln \frac{1}{x}. \tag{4.5}$$

To proceed further, we introduce the n^{th} order *lifted bundle* $\mathscr{B}_G^{(n)} = \mathrm{J}^{(n)} \times G$. The Lie group G acts on $\mathscr{B}_G^{(n)}$ by the *lifted action*

$$g \cdot ((x, u^{(n)}), h) = (g \cdot (x, u^{(n)}), h g^{-1}), \tag{4.6}$$

also known as the *anti-diagonal action* of G on $\mathscr{B}_G^{(n)}$.

Definition 4.1. The *lift* of a differential function $F \colon \mathrm{J}^{(n)} \to \mathbb{R}$ to the lifted bundle $\mathscr{B}_G^{(n)}$ is defined as

$$\boldsymbol{\lambda}[F(x, u^{(n)})] = F(g \cdot (x, u^{(n)})). \tag{4.7}$$

We observe that the lifted function (4.7) is invariant under the lifted action (4.6). We therefore refer to (4.7) as a *lifted invariant*. In particular, lifting the jet coordinates $(x, u^{(n)})$:

$$\boldsymbol{\lambda}(x, u^{(n)}) = g \cdot (x, u^{(n)}) = (X, U^{(n)}),$$

we recover the prolonged action (2.6). The quantities $(X, U^{(n)})$ are called *fundamental lifted invariants*. This terminology is motivated by the fact that any lifted invariant is a function of the fundamental lifted invariants:

$$\boldsymbol{\lambda}[F(x, u^{(n)})] = F(\boldsymbol{\lambda}(x, u^{(n)})) = F(X, U^{(n)}).$$

Inversely, given a lifted invariant $F(X, U^{(n)})$, we introduce the *inverse lift*

$$\boldsymbol{\lambda}^{-1}[F(X, U^{(n)})] = F(x, u^{(n)}),$$

which is defined as the function whose lift is $F(X, U^{(n)})$.

The lift map $\boldsymbol{\lambda}$ extends to differential forms, and we refer to [11] for more detail. For the purpose of this paper, it is enough to know that, modulo the contact forms (2.1), the lift of the horizontal form dx is

$$\omega = \boldsymbol{\lambda}(dx) = D_x(X)\, dx = (X_x + X_u\, u_x)\, dx.$$

Given a right moving frame $\rho \colon \mathrm{J}^{(n)} \to G$, we introduce the *right moving frame section* $\varrho \colon \mathrm{J}^{(n)} \to \mathscr{B}_G^{(n)}$ defined as

$$\varrho\,(x, u^{(n)}) = ((x, u^{(n)}), \rho(x, u^{(n)})).$$

Definition 4.2. Let $\rho \colon \mathrm{J}^{(n)} \to G$ be a right moving frame. The *invariantization map* is defined as

$$\iota = \varrho^* \circ \boldsymbol{\lambda}.$$

Invariantizing a differential function $F \colon \mathrm{J}^{(n)} \to \mathbb{R}$, we obtain the differential invariant

$$\iota(F)(x, u^{(n)}) = F(\rho(x, u^{(n)}) \cdot (x, u^{(n)})). \tag{4.8}$$

The fact that (4.8) is a differential invariant follows from the G-equivariance of the right moving frame $\rho(x, u^{(n)})$. Of particular importance is the invariantization of the jet coordinates $(x, u^{(n)})$. The differential invariants

$$H = \iota(x), \qquad I^{(n)} = \iota(u^{(n)}) \tag{4.9}$$

are called *normalized invariants*. By construction of the right moving frame, the invariantization of the coordinates used to define the cross-section (4.1) are constants, that is

$$\iota(z_\ell) = c_\ell. \tag{4.10}$$

These normalized invariants are called *phantom invariants*. By the *replacement principle*, [11,22], the normalized invariants (4.9) form a complete set of differential invariants of order $\leq n$. This means that any differential invariant of order $\leq n$ can be expressed in terms of the normalized invariants (4.9). Indeed, since the invariantization of an invariant $J(x,u^{(n)})$ is the invariant itself,

$$J(x,u^{(n)}) = \iota(J(x,u^{(n)})) = J(\iota(x,u^{(n)})) = J(H,I^{(n)}).$$

We also note that the normalized invariants (4.9) provide a local parametrization of the cross-section (4.1) used to construct the moving frame ρ:

$$\mathscr{K} = \{(H,I^{(n)})\}.$$

The invariantization of the horizontal form dx is the contact-invariant[d] horizontal one-form

$$\varpi = \iota(dx) = \varrho^* \circ \boldsymbol{\lambda}(dx) = \varrho^*\,(\omega) = \varrho^*\,(X_x + X_u\,u_x)\,dx. \tag{4.11}$$

Example 4.2. Continuing Example 4.1, the invariantization of the jet coordinates u_x and u_{xx} yields the differential invariants

$$I_1 = \iota(u_x) = \left.\frac{u_x+\varepsilon_1}{e^{\varepsilon_2}}\right|_{(4.5)} = xu_x - u, \qquad I_2 = \iota(u_{xx}) = \left.\frac{u_{xx}}{e^{2\varepsilon_2}}\right|_{(4.5)} = x^2 u_{xx}. \tag{4.12}$$

The contact-invariant horizontal one-form (4.11) is given by

$$\varpi = \iota(dx) = \frac{1}{x}dx. \tag{4.13}$$

One of the most important results from the theory of equivariant moving frames is the introduction of the *universal recurrence relations* for the lift map and the invariantization map. These equations can be used to perform many computations symbolically without requiring coordinate expressions for the lifted or normalized invariants. The universal recurrence relations have played a fundamental role in the development of the group foliation method, [22,39], the theory of invariant variational bicomplexes, [19,38], the study of invariant geometric curve flows, [24], the characterization of the algebra of differential invariants, [33], the development of the recursive moving frame construction, [30,34], and many other problems in applied mathematics, [31]. In order to introduce these recurrence formulas, let

$$\mathbf{v}_\ell = \xi_\ell(x,u)\frac{\partial}{\partial x} + \varphi_\ell(x,u)\frac{\partial}{\partial u} \in \mathfrak{g}, \qquad \ell = 1,\ldots,r,$$

be a basis of infinitesimal generators of the Lie group action. Dually, let $\mu^1, \ldots, \mu^r \in \mathfrak{g}^*$, be a basis of Maurer–Cartan forms. Modulo the contact forms (2.1), the universal recurrence formula for the

[d] A differential one-form Ω on $\mathrm{J}^{(n)}$ is *contact-invariant* if and only if, for every $g \in G$, $g^*\Omega = \Omega + \theta_g$ for some contact form θ_g.

lift map is

$$d[\boldsymbol{\lambda}(F)] = \boldsymbol{\lambda}[D_x(F)]\,\omega + \sum_{\ell=1}^{r} \boldsymbol{\lambda}[\mathbf{v}_\ell^{(\infty)}(F)]\,\mu^\ell, \tag{4.14}$$

where $F\colon \mathrm{J}^{(n)} \to \mathbb{R}$ is an arbitrary differential function[e] . In particular, substituting the function F in (4.14) with the jet coordinate functions $(x, u^{(\infty)})$, we obtain the recurrence relations for the fundamental lifted invariants X, $U^{(\infty)}$:

$$\begin{aligned} dX &= \omega + \sum_{\ell=1}^{r} \boldsymbol{\lambda}(\xi_\ell)\,\mu^\ell, \\ dU_k &= U_{k+1}\,\omega + \sum_{\ell=1}^{r} \boldsymbol{\lambda}(\varphi_\ell^k)\,\mu^\ell, \qquad k \geq 0. \end{aligned} \tag{4.15}$$

Pulling back the universal recurrence relation (4.14) by the right moving frame section ϱ, we obtain the universal recurrence relation for the invariantization map

$$d[\iota(F)] = \iota[D_x(F)]\,\varpi + \sum_{\ell=1}^{r} \iota[\mathbf{v}_\ell^{(\infty)}(F)]\,\nu^\ell,$$

where $\nu^\ell = \rho^*(\mu^\ell)$ denotes the moving frame pull-back of the Maurer–Cartan form μ^ℓ. In particular, pulling back (4.15) by the right moving frame section ϱ, we obtain the recurrence relations for the normalized invariants $H = \iota(x)$, $I^{(\infty)} = \iota(u^{(\infty)})$:

$$\begin{aligned} dH &= \varpi + \sum_{\ell=1}^{r} \iota(\xi_\ell)\,\nu^\ell, \\ dI_k &= I_{k+1}\,\varpi + \sum_{\ell=1}^{r} \iota(\varphi_\ell^k)\,\nu^\ell, \qquad k \geq 0. \end{aligned} \tag{4.16}$$

Remark 4.1. The lifted recurrence relations (4.15) can be computed symbolically without relying on the coordinate expressions of the lifted invariants, the lifted horizontal form ω, and the Maurer–Cartan forms $\mu^1, \ldots, \mu^r$. The only data required are the expressions for the infinitesimal generators $\mathbf{v}_1, \ldots, \mathbf{v}_r$, and their prolongation. Similarly, the invariantized recurrence relations (4.16) can be computed symbolically without knowing the coordinate expressions of the right moving frame $\rho(x, u^{(n)})$. Indeed, using the recurrence relations for the phantom invariants (4.10), one can solve for the normalized Maurer–Cartan forms $\nu^1, \ldots, \nu^r$. Substituting these expressions into the remaining recurrence relations provides symbolic expressions for the differential of the non-phantom normalized invariants.

Example 4.3. To illustrate the above considerations, we consider the group action (4.2) with infinitesimal generators (3.9). First, the recurrence relations for the lifted invariants are

$$\begin{aligned} dX &= \omega + X\mu^2, \\ dU &= U_X\,\omega + X\mu^1, \\ dU_X &= U_{XX}\,\omega + \mu^1 - U_X\mu^2, \\ dU_k &= U_{k+1}\,\omega - kU_k\,\mu^2, \qquad k \geq 2. \end{aligned} \tag{4.17}$$

[e] In its most general formulation, the universal recurrence formula is stated for a differential form defined on $\mathrm{J}^{(\infty)}$.

Pulling back (4.17) by the moving frame section induced by (4.5), we obtain the invariantized recurrence relations

$$\begin{aligned} 0 &= \varpi + \nu^2, \\ 0 &= I_1\varpi + \nu^1, \\ dI_1 &= I_2\varpi + \nu^1 - I_1\nu^2, \\ dI_k &= I_{k+1}\varpi - kI_k\nu^2, \qquad k \geq 2. \end{aligned} \tag{4.18}$$

As observed in Remark 4.1, the moving frame expressions (4.5) are not needed to deduce (4.18). Indeed, the recurrence relations (4.18) are obtained symbolically by substituting the normalizations (4.4) into (4.17) and making the substitutions $U_k \to I_k$, $\omega \to \varpi$, and $\mu^\ell \to \nu^\ell$.

Solving for the normalized Maurer–Cartan forms ν^1, ν^2 using the first two equations in (4.18), we find that

$$\nu^1 = -I_1\varpi, \qquad \nu^2 = -\varpi.$$

Substituting these expressions into the remaining equations, we obtain the recurrence relations for the non-phantom normalized invariants:

$$dI_1 = I_2\varpi, \qquad dI_k = (I_{k+1} + kI_k)\varpi, \qquad k \geq 2. \tag{4.19}$$

4.2. *Inductive/recursive implementation*

We now review the inductive/recursive implementation of the moving frame method as introduced in [18, 30, 34, 40]. The idea behind the inductive/recursive implementation is to take into account the existence of a smaller subgroup $N \subset G$ for which a moving frame and its normalized differential invariants have already been constructed, and use this information to streamline the construction of a moving frame for the larger group G and the computation of its normalized differential invariants. To this end, let $N \lhd G$ be an s-dimensional normal Lie subgroup of the r-dimensional Lie group G. In general N does not have to be normal to implement the inductive/recursive moving frame algorithm. We make this assumption here, as this is the geometric setting in which we will apply the constructions in Section 5.

For convenience, let

$$g = (g_1, \dots, g_s, g_{s+1}, \dots, g_r) = (h, \widehat{g}) \tag{4.20}$$

be local coordinates of the Lie group G such that the first s components $h = (g_1, \dots, g_s)$ parametrize the subgroup N. Since N is normal, the quotient group G/N is well-defined and

$$G/N \simeq \{\widehat{g} = (g_{s+1}, \dots, g_r)\}.$$

In the following we will make us of this isomorphism and use $\widehat{g}$ as local coordinates on G/N.

Assuming the prolonged action of N on the n^{th} order jet space $\mathrm{J}^{(n)}$ is free and regular, let $\rho_N : \mathrm{J}^{(n)} \to N$ be the right moving frame induced by the coordinate cross-section

$$\mathscr{K}_N = \{z_1 = c_1, \dots, z_s = c_s\} \subset \mathrm{J}^{(n)}.$$

Also, let ι_N be the induced invariantization map. With the moving frame ρ_N and the normalized invariants $(H, I^{(n)}) = (\iota_N(x), \iota_N(u^{(n)}))$ in hand, we seek to construct the normalized invariants of the larger group G inductively. To guarantee the existence of a moving frame, we assume that the prolonged action of G on $\mathrm{J}^{(n)}$ is also free and regular. Now, for any $h \in N$, we observe that

$$\boldsymbol{\lambda}_G(x,u^{(n)}) = g\cdot(x,u^{(n)}) = (gh^{-1})\cdot h\cdot(x,u^{(n)}) = \widetilde{g}\cdot(X,U^{(n)}) = \boldsymbol{\lambda}_G(X,U^{(n)}), \tag{4.21}$$

where $\widetilde{g} \in G$ and $(X,U^{(n)}) = \boldsymbol{\lambda}_N(x,u^{(n)}) = h\cdot(x,u^{(n)})$ denotes the N-lifted invariants. Equation (4.21) shows that the lift of the jet coordinates $(x,u^{(n)})$ with respect to the group G, which we refer to as the G-lift of $(x,u^{(n)})$, is equal to the G-lift of the N-lifted invariants. Replacing the group parameter $h \in N$ in (4.21) by the right moving frame $\rho_N\colon \mathrm{J}^{(n)} \to N$, we conclude that

$$\boldsymbol{\lambda}_G(x,u^{(n)}) = \boldsymbol{\lambda}_G(H,I^{(n)}). \tag{4.22}$$

That is, the G-lift of the jet coordinates $(x,u^{(n)})$ is equal to the G-lift of the N-normalized invariants $(H,I^{(n)})$. We note that in applications, since the N-phantom invariants $\iota_N(z_1) = c_1,\ \ldots,\ \iota_N(z_s) = c_s$ are typically set equal to 0 or ± 1, the formulas on the right-hand side of (4.22) are usually simpler than those appearing on the left-hand side. The inductive construction of the G-normalized invariants is achieved by first considering the *partial normalizations*

$$\boldsymbol{\lambda}_G[\iota_N(z_\ell)] = \boldsymbol{\lambda}_G(c_\ell) = c_\ell, \qquad \ell = 1,\ldots,s, \tag{4.23}$$

obtained by requiring that the N-phantom invariants remain unchanged under the prolonged action of G. Solving the normalization equations (4.23) for the group parameters $h \in N$, we can write h in terms of $\widehat{g}$ and the N-normalized invariants $(H,I^{(n)})$:

$$h = h(H,I^{(n)},\widehat{g}).$$

This produces the *partial moving frame* $\widehat{\rho}_N\colon \mathrm{J}^{(n)} \times G/N \to G$ defined by

$$\widehat{\rho}_N((x,u^{(n)}),\widehat{g}) = (h(H,I^{(n)},\widehat{g}),\widehat{g}).$$

The partial moving frame is used to construct the *partially normalized invariants*

$$\widehat{H} = \widehat{\rho}_N\cdot H, \qquad \widehat{I}^{(n)} = \widehat{\rho}_N\cdot I^{(n)}. \tag{4.24}$$

Equation (4.24) provides expressions for the induced action of the quotient group G/N on the N-normalized invariants. Said differently, the quantities in (4.24) are equal to the G/N-lift of the N-normalized invariants:

$$\widehat{H} = \boldsymbol{\lambda}_{G/N}(H), \qquad \widehat{I}^{(n)} = \boldsymbol{\lambda}_{G/N}(I^{(n)}).$$

For later use, we also introduce the partially normalized horizontal form

$$\widehat{\omega} = \widehat{\varrho}_N^{\,*}(\boldsymbol{\lambda}_G(\varpi)),$$

where

$$\widehat{\varrho}_N = (\mathrm{id},\widehat{\rho}_N)\colon \mathrm{J}^{(n)} \times G/N \to \mathrm{J}^{(n)} \times G \tag{4.25}$$

and $\varpi = \iota_N(dx)$.

Since G is assume to act freely and regularly on $\mathrm{J}^{(n)}$, the quotient group G/N acts freely and regularly on the cross-section $\mathscr{K}_N$. Using the N-normalized invariants $(H,I^{(n)})$ to parametrize $\mathscr{K}_N$, we conclude that G/N acts freely and regularly on the space of N-normalized invariants

$(H, I^{(n)})$. Implementing the constructions introduced in Section 4.1, we can construct a moving frame $\rho_{G/N} \colon \mathscr{K}_N \to G/N$. We extend $\rho_{G/N}$ to a section $\varrho_{G/H} \colon \mathrm{J}^{(n)} \to \mathrm{J}^{(n)} \times G/N$ by defining

$$\varrho_{G/N}\,(x, u^{(n)}) = ((x, u^{(n)}), \rho_{G/N}(H, I^{(n)})).$$

The G-normalized differential invariants of order $\leq n$ are then given by

$$\iota_G(x) = \iota_{G/N}(H) = \varrho^*_{G/N}\,(\widehat{H}), \qquad \iota_G(u^{(n)}) = \iota_{G/N}(I^{(n)}) = \varrho^*_{G/H}\,(\widehat{I}^{(n)}),$$

where $\iota_{G/N}$ denotes the invariantization map induced by the moving frame $\rho_{G/N}$.

Example 4.4. As a simple example of the above considerations, we reconsider Examples 4.1 and 4.2, and recover the invariants (4.12) inductively. The one-parameter group $N = G_{\varepsilon_1}$ with group action

$$X = x, \qquad U = u + \varepsilon_1 x, \tag{4.26}$$

is a normal Lie subgroup of the 2-parameter group $G = G_{(\varepsilon_1,\varepsilon_2)}$ with group action (4.2). The prolonged action of (4.26) to $\mathrm{J}^{(2)}$ is

$$U_X = u_x + \varepsilon_1, \qquad U_{XX} = u_{xx}.$$

Choosing the cross-section

$$\mathscr{K}_{G_{\varepsilon_1}} = \{u = 0\}, \tag{4.27}$$

we obtain the moving frame

$$\rho_{G_{\varepsilon_1}} = \varepsilon_1 = -\frac{u}{x}. \tag{4.28}$$

Up to order two, the G_{ε_1}-normalized differential invariants are

$$\begin{gathered} H = \iota_{G_{\varepsilon_1}}(x) = x, \qquad I_0 = \iota_{G_{\varepsilon_1}}(u) = 0, \\ I_1 = \iota_{G_{\varepsilon_1}}(u_x) = \frac{x\,u_x - u}{x}, \qquad I_2 = \iota_{G_{\varepsilon_1}}(u_{xx}) = u_{xx}, \end{gathered} \tag{4.29}$$

and the invariant horizontal form is $\varpi = dx$.

Now, the 2-parameter group $G = G_{(\varepsilon_1,\varepsilon_2)}$ acts on the G_{ε_1}-normalized invariants (4.29) according to

$$\lambda_G(H) = e^{\varepsilon_2} H, \qquad \lambda_G(I) = I + \varepsilon_1 H, \qquad \lambda_G(I_1) = \frac{I_1 + \varepsilon_1}{e^{\varepsilon_2}}, \qquad \lambda_G(I_2) = \frac{I_2}{e^{2\varepsilon_2}},$$

where $I = 0$. Imposing the partial normalization

$$0 = \varepsilon_1 H,$$

and solving for the group parameter ε_1, we obtain the partial moving frame

$$\widehat{\rho}_{G_{\varepsilon_1}} = \varepsilon_1 = 0. \tag{4.30}$$

The partially normalized invariants are then given by

$$\begin{aligned}\widehat{H} &= \boldsymbol{\lambda}_{G_{\varepsilon_2}}(H) = \widehat{\rho}_{G_{\varepsilon_1}} \cdot H = e^{\varepsilon_2}H,\\ \widehat{I}_1 &= \boldsymbol{\lambda}_{G_{\varepsilon_2}}(I_1) = \widehat{\rho}_{G_{\varepsilon_1}} \cdot I_1 = \frac{I_1}{e^{\varepsilon_2}},\\ \widehat{I}_2 &= \boldsymbol{\lambda}_{G_{\varepsilon_2}}(I_2) = \widehat{\rho}_{G_{\varepsilon_1}} \cdot I_2 = \frac{I_2}{e^{2\varepsilon_2}},\end{aligned} \tag{4.31}$$

and the partially normalized horizontal form is

$$\widehat{\omega} = \widehat{\varrho}^*_{G_{\varepsilon_1}}(\boldsymbol{\lambda}_G(\varpi)) = \widehat{\varrho}^*_{G_{\varepsilon_1}}(e^{\varepsilon_2}dx) = e^{\varepsilon_2}dx.$$

The expressions in (4.31) provide the induced action of the quotient group $G/G_{\varepsilon_1} \simeq G_{\varepsilon_2}$ on the G_{ε_1}-normalized invariants H, I_1, and I_2. Choosing the cross-section $\mathscr{K}_{G_{\varepsilon_2}} = \{H = 1\} \subset \mathscr{K}_{G_{\varepsilon_1}} = \{(H, 0, I_1, I_2)\}$ we obtain the moving frame

$$\rho_{G_{\varepsilon_2}} = \varepsilon_2 = \ln \frac{1}{H}.$$

Up to order two, the G-normalized invariants are

$$J_1 = \iota_G(u_x) = \varrho^*_{G_{\varepsilon_2}}(\widehat{I}_1) = H I_1, \qquad J_2 = \iota_G(u_{xx}) = \varrho^*_{G_{\varepsilon_2}}(\widehat{I}_2) = H^2 I_2,$$

which, as it should, coincide with the invariants obtained in (4.12). Finally, the contact-invariant horizontal one-form is

$$\iota_G(dx) = \varrho^*_{G_{\varepsilon_2}}(\widehat{\omega}) = \frac{1}{H}dx,$$

which is identical to (4.13).

To perform the computations symbolically, we now apply the above inductive constructions to the recurrence formulas (4.15). Following the decomposition (4.20), let $\mathfrak{g} = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_s, \mathbf{v}_{s+1}, \ldots, \mathbf{v}_r\}$ be a basis of infinitesimal generators such that the first s vector fields provide a basis of infinitesimal generators for the Lie algebra $\mathfrak{n}$ of N, i.e.

$$\mathfrak{n} = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}.$$

Dually, let $\{\mu^1, \ldots, \mu^s, \mu^1, \ldots, \mu^r\}$ be a basis of Maurer–Cartan forms. Introducing the lifted bundle $\mathscr{B}^{(n)}_N = \mathrm{J}^{(n)} \times N$ and the inclusion map $i_N \colon \mathscr{B}^{(n)}_N \hookrightarrow \mathscr{B}^{(n)}_G$ induced by the containment $N \lhd G$, the lifted recurrence relations for the subgroup N are obtained by pulling-back the G-lifted recurrence relations by the inclusion map i_N. In applications, this is achieved by setting $\mu^{s+1} = \cdots = \mu^r = 0$ in (4.15). To simplify the notation, we omit writing the pull-back by the inclusion map. The recurrence relations for the N-normalized invariants are then obtained by following the standard procedure outlined in Section 4.1.

On the other hand, pulling-back the G-lifted recurrence relations by the map (4.25) induced by the partial moving frame $\widehat{\rho}_N$ yields the lifted recurrence relations for the partially normalized invariant (4.24). In applications, these equations are obtained by solving for the partially normalized Maurer–Cartan forms $\widehat{\mu}^1 = \widehat{\rho}^*_N(\mu^1), \ldots, \widehat{\mu}^s = \widehat{\rho}^*_N(\mu^s)$ using the recurrence relations for the N-phantom invariants, which according to (4.23) are kept constant. Substituting the result into the remaining recurrence relations yields the G/N-lifted recurrence formulas. The recurrence relations for the G-normalized invariants are then found by implementing the usual moving frame computations using the G/N-lifted recurrence relations. In other words, one normalizes $r-s$ of the G/N-lifted invariants and uses the recurrence relations for these phantom invariants to solve for the remaining normalized Maurer–Cartan forms. The result is then substituted into the remaining recurrence formulas.

Example 4.5. Continuing Example 4.4, the recurrence relations for the G-lifted invariants are given in (4.17). Pulling-back theses equation by the inclusion map $i_{G_{\varepsilon_1}} : \mathscr{B}^{(\infty)}_{G_{\varepsilon_1}} \hookrightarrow \mathscr{B}^{(\infty)}_{G}$, we obtain the recurrence relations for the G_{ε_1}-lifted invariants. This is achieved by setting $\mu^2 = 0$ in (4.17):

$$dX = \omega, \qquad dU = U_X\omega + X\mu^1, \qquad dU_X = U_{XX}\omega + \mu^1, \qquad dU_k = U_{k+1}\omega, \quad k \geq 2. \tag{4.32}$$

Pulling back the recurrence relations (4.32) by the right moving frame section induced from the moving frame (4.28) we obtain

$$dH = \varpi, \qquad 0 = I_1\varpi + H\nu^1, \qquad dI_1 = I_2\varpi + \nu^1, \qquad dI_k = I_{k+1}\varpi, \quad k \geq 2. \tag{4.33}$$

Once again, we emphasize that the coordinate expressions for the moving frame (4.28) are not needed to obtain (4.33) symbolically. Solving the second equation for the normalized Maurer–Cartan form ν^1, we obtain

$$\nu^1 = -\frac{I_1}{H}\varpi. \tag{4.34}$$

Substituting the result into the other equations yields the recurrence formulas for the G_{ε_1}-normalized invariants:

$$dH = \varpi, \qquad dI_1 = \left(I_2 - \frac{I_1}{H}\right)\varpi, \qquad dI_k = I_{k+1}\varpi, \qquad k \geq 2. \tag{4.35}$$

Now, pulling-back the G-lifted recurrence relations (4.17) by the partial moving frame (4.30) we obtain the recurrence relations

$$\begin{aligned} d\widehat{H} &= \widehat{\omega} + \widehat{H}\widehat{\mu}^2, & 0 &= \widehat{I}_1\widehat{\omega} + \widehat{H}\widehat{\mu}^1, \\ d\widehat{I}_1 &= \widehat{I}_2\widehat{\omega} + \widehat{\mu}^1 - \widehat{I}_1\widehat{\mu}^2, & d\widehat{I}_k &= \widehat{I}_{k+1}\widehat{\omega} - k\widehat{I}_k\widehat{\mu}^2, \qquad k \geq 2. \end{aligned}$$

Solving the second equation for the partially normalized Maurer-Cartan forms $\widehat{\mu}^1$, we obtain $\widehat{\mu}^1 = -\frac{\widehat{I}_1}{\widehat{H}}\widehat{\omega}$. Substituting the result into the remaining equations yields the $G/G_{\varepsilon_1} \simeq G_{\varepsilon_2}$ lifted recurrence relations

$$d\widehat{H} = \widehat{\omega} + \widehat{H}\widehat{\mu}^2, \qquad d\widehat{I}_1 = \left(\widehat{I}_2 - \frac{\widehat{I}_1}{\widehat{H}}\right)\widehat{\omega} - \widehat{I}_1\widehat{\mu}^2, \qquad d\widehat{I}_k = \widehat{I}_{k+1}\widehat{\omega} - k\widehat{I}_k\widehat{\mu}^2, \qquad k \geq 2. \tag{4.36}$$

Setting $\widehat{H} = 0$ in (4.36), we obtain the normalized recurrence relations

$$0 = \widehat{\varpi} + \widehat{\nu}^2, \qquad dJ_1 = (J_2 - J_1)\widehat{\varpi} - J_1\widehat{\nu}^2, \qquad dJ_k = J_{k+1}\widehat{\varpi} - kJ_k\widehat{\nu}^2, \qquad k \geq 2.$$

Solving for $\widehat{\nu}^2$ in the first equation and substituting the result into the remaining equations we obtain

$$dJ_1 = J_2\widehat{\varpi}, \qquad dJ_k = (J_{k+1} + kJ_k)\widehat{\varpi}, \qquad k \geq 2,$$

which are identical to (4.19), up to the relabelling $J_k \leftrightarrow I_k$ and $\widehat{\varpi} \leftrightarrow \varpi$.

The lifted recurrence relations (4.36) in Example 4.5, will play an important role in our symbolic implementation of the symmetry reduction algorithm introduced in Section 5. As previously emphasized, the recurrence formulas (4.36) can be obtained symbolically without requiring the coordinate expressions for the G_{ε_2}-lifted invariants $\widehat{H}$, $\widehat{I}_k$ and the differential forms $\widehat{\omega}$, $\widehat{\mu}^2$. On the other hand, from (4.36) it is possible to recover the expressions of the G_{ε_2}-lifted invariants in terms of the G_{ε_1}-normalized invariants H, I_k. Indeed, by virtue of the general formula for the lifted recurrence relations (4.15), the coefficients multiplying $\widehat{\mu}^2$ in (4.36) must be the G_{ε_2}-lift of the components of the infinitesimal generator

$$\widehat{\mathbf{v}}_2 = \mathbf{v}_2^{(\infty)}\big|_{(H,I^{(\infty)})} = \xi\frac{\partial}{\partial H} + \sum_{k=1}^{\infty}\varphi_k\frac{\partial}{\partial I_k}.$$

Therefore,

$$\widehat{H} = \boldsymbol{\lambda}_{G_{\varepsilon_2}}(\xi) \qquad \text{and} \qquad -k\widehat{I}_k = \boldsymbol{\lambda}_{G_{\varepsilon_2}}(\varphi_k), \qquad k \geq 1. \tag{4.37}$$

Taking the inverse lift of (4.37), we conclude that the vector field components are

$$\xi = \boldsymbol{\lambda}^{-1}_{G_{\varepsilon_2}}(\widehat{H}) = H, \qquad \varphi_k = \boldsymbol{\lambda}^{-1}_{G_{\varepsilon_2}}(-k\widehat{I}_k) = -kI_k,$$

so that

$$\widehat{\mathbf{v}}_2 = H\frac{\partial}{\partial H} - \sum_{k=1}^{\infty} kI_k\frac{\partial}{\partial I_k}. \tag{4.38}$$

Exponentiating (4.38), we recover the G_{ε_2}-action (4.31) without relying on the coordinate expressions of the G_{ε_1}-normalized invariants H, I_k.

5. Symmetry Reduction Using Moving Frames

In this section we revisit the symmetry reduction algorithm presented in Section 3 using the moving frame machinery introduced in Section 4. By taking advantage of the recurrence formulas (4.16) for the normalized invariants, computations are performed symbolically without relying on the coordinate expressions for the canonical variables, the differential invariants, and the moving frame.

5.1. *One-parameter symmetry group*

As in Section 3, our starting point is an n^{th} order ordinary differential equation

$$\Delta(x, u^{(n)}) = 0 \tag{5.1}$$

invariant under the infinitesimal symmetry generator

$$\mathbf{v} = \xi(x,u)\partial_x + \varphi(x,u)\partial_u. \tag{5.2}$$

Let G_ε be the one-parameter Lie group whose local group action is induced by the flow of $\mathbf{v}$:

$$(X,U) = g_\varepsilon \cdot (x,u) = \exp[\varepsilon\mathbf{v}] \cdot (x,u), \qquad g_\varepsilon \in G_\varepsilon. \tag{5.3}$$

Under this assumption, G_ε is locally isomorphic to the one-dimensional abelian Lie group $(\mathbb{R},+)$. For all $(x,u) \in \mathscr{V}^{(0)} \subset \mathrm{J}^{(0)}$ where $\mathbf{v}|_{(x,u)} \neq 0$, the group action (5.3) is free. To simplify the exposition and notation, we work on $\mathrm{J}^{(0)}$ with the understanding that the constructions might have to

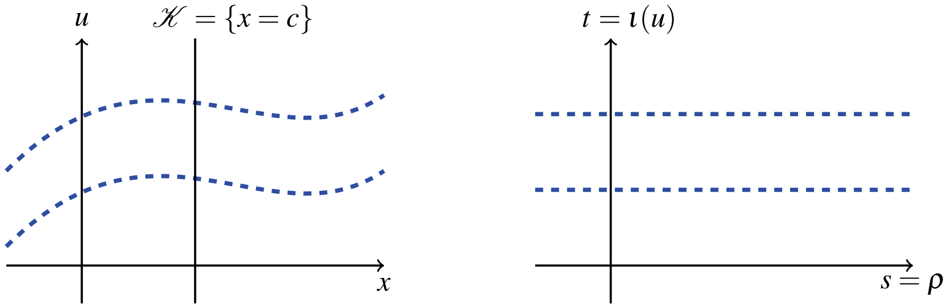

Fig. 1. Rectification of the group orbits.

be restricted to $\mathscr{V}^{(0)}$. Assuming the group action is also regular, a local moving frame can be constructed. For simplicity, assume that $\mathscr{K}$ is a coordinate cross-section obtained by fixing the independent variable x to a convenient constant:

$$\mathscr{K} = \{x = c\}. \tag{5.4}$$

If the independent variable is an invariant of the group action, one can make the hodograph transformation $(x,u) \to (u,x)$ and then choose the cross-section (5.4). Alternatively, the cross-section (5.4) can be replaced by $\mathscr{K} = \{u = c\}$ and the discussion below can easily be adapted to this setting. Let $\rho : \mathrm{J}^{(0)} \to G_\varepsilon$ be the right moving frame induced by the cross-section (5.4).

Given the moving frame ρ, we now introduce the canonical variables (t,s) appearing in Lie's symmetry reduction algorithm. At the group action level, the infinitesimal definition of the canonical variables given in (3.3) implies that t is an invariant of G_ε while G_ε acts on s by translations. In terms of the moving frame ρ, these coordinates can be given by

$$t = I = \iota(u) \qquad \text{and} \qquad s = \rho(x,u). \tag{5.5}$$

Indeed, by definition of the invariantization map, t is an invariant of G_ε while the right-equivariance of ρ implies that

$$g_\varepsilon \cdot s = \rho(g_\varepsilon \cdot (x,u)) = \rho(x,u)\, g_\varepsilon^{-1} = s - \varepsilon, \tag{5.6}$$

since G_ε is locally isomorphic to $(\mathbb{R},+)$. As illustrated in Figure 1, the canonical variables (5.5) rectify the group orbits, foliating the space into horizontal lines. Also, since on the cross-section the moving frame restricts to the identity element, i.e. $\rho\big|_{\mathscr{K}} = 0$, the cross-section (5.4) is mapped onto the t-axis in the space of canonical variables.

Without relying on the coordinate expressions of t and s, it is possible to compute the derivatives s_t, s_{tt}, ..., symbolically. First, we have that

$$ds = d\rho = \nu = F(t)\varpi, \tag{5.7}$$

where $\nu = \rho^*(\mu)$ is the moving frame pull-back of the Maurer–Cartan form μ. As outlined in the previous section, the function $F(t)$ can be found symbolically using the recurrence relation for the phantom invariant $\iota(x) = c$. In fact, $F(t) = -\frac{1}{\xi(c,t)}$. Also, the differential $dt = dI$, is computed symbolically using the recurrence relation for the normalized invariant $I = \iota(u)$:

$$dt = dI = I_1\varpi + \varphi(c,t)\nu = (I_1 + \varphi(c,t)F(t))\varpi, \tag{5.8}$$

where $I_1 = \iota(u_x)$. Combining (5.7) and (5.8), we have that

$$s_t = \frac{ds}{dt} = \frac{\nu}{dt} = \frac{F(t)}{I_1 + \varphi(c,t)F(t)} = \frac{1}{\varphi(c,t) - I_1\xi(c,t)} = \frac{1}{Q(c,t,I_1)} = F_1(t,I_1),$$

where $Q(x,u,u_x) = \varphi - \xi u_x$ is the characteristic of the vector field (5.2). Next, using the recurrence relation for $I_1 = \iota(u_x)$, which is of the form

$$dI_1 = (I_2 + K(t,I_1))\varpi,$$

it is then possible to compute s_{tt} symbolically:

$$s_{tt} = \frac{ds_t}{dt} = \frac{1}{dt}\left(\frac{\partial F_1}{\partial t}dt + \frac{\partial F_1}{\partial I_1}dI_1\right) = F_2(t,I_1,I_2).$$

Similarly, the recurrence relations for the higher order normalized invariants lead to the symbolic expressions

$$s_{t^\ell} = F_\ell(t,I_1,\dots,I_\ell), \qquad \ell \geq 1. \tag{5.9}$$

Inverting (5.9), the normalized invariants I_ℓ can be expressed in terms of t and s_t, …, s_{t^ℓ}:

$$I = t, \qquad I_\ell = \mathscr{I}_\ell(t,s^{(1,\ell)}), \qquad \ell \geq 1,$$

where $s^{(1,\ell)} = (s_t,\dots,s_{t^\ell})$ collects the derivatives of s with respect to t of order $1 \leq k \leq \ell$. Since the differential equation (5.1) is G_ε-invariant, we invariantize the equation to obtain

$$0 = \Delta(x,u^{(n)}) = \iota[\Delta(x,u^{(n)})] = \Delta(c,I^{(n)}) = \Delta(c,t,\mathscr{I}_1(t,s^{(1,1)}),\dots,\mathscr{I}_n(t,s^{(1,n)})) = \widetilde{\Delta}(t,s^{(1,n)}).$$

We observe that the resulting equation $\widetilde{\Delta}(t,s^{(1,n)}) = 0$ is independent of s. This was to be expected as in the canonical variables (t,s) the differential equation is invariant under the group of translations (5.6). Introducing the new dependent variable $w = s_t$, the order of the original differential equation (5.1) is reduced by one:

$$\widetilde{\Delta}(t,w^{(n-1)}) = 0. \tag{5.10}$$

Assuming (5.10) can be solved, we now explain how to recover the solution to the original equation (5.1) without relying on the coordinate expressions of the canonical variables (t,s) (or $w = s_t$). We refer to this procedure as the *reconstruction step*. Let $w = w(t)$ be the general solution to (5.10). Since $s_t = w(t)$, it follows that

$$s(t) = \int w(t)\,dt.$$

Next, by definition of the right moving frame, $\rho \cdot (x,u) = (c,t) \in \mathscr{K}$. Inverting the last equality, it follows that

$$(x,u) = \overline{\rho} \cdot (c,t), \tag{5.11}$$

where $\overline{\rho} = \rho^{-1}$ is the corresponding left moving frame. Since $s = \rho$ and group inversion is given by the additive inverse, equation (5.11) is equivalent to

$$(x,u) = (-s(t)) \cdot (c,t), \tag{5.12}$$

where the right-hand side of (5.12) is obtained by acting on the point $(c,t) \in \mathscr{K}$ with the group element $-s(t) \in G$. The latter provides a parametric solution $(x(t),u(t))$ to the original differential equation (5.1). Inverting the relation $x = x(t)$, to express t as a function of x, we obtain the solution $u(t(x))$ to the original differential equation (5.1).

Before considering an example, let us summarize the steps involved in the symbolic implementation of the symmetry reduction process introduced above. Starting with the ordinary differential equation (5.1):

- Compute a basis of infinitesimal symmetry generators following standard procedures, [3, 5–8, 10, 15–17, 27, 35–37]. Choose one infinitesimal symmetry generator $\mathbf{v}$ with respect to which the equation is to be reduced.
- Let $\mathfrak{g} = \operatorname{span}\{\mathbf{v}\}$ and compute the recurrence relations (4.15) for the lifted invariants of order $\leq n-1$.
- Introduce the cross-section (5.4) and let ρ be the corresponding right moving frame. Note, it is not necessary to compute ρ! We only need to know that it exists.
- Compute the recurrence relations for the normalized invariants $I^{(n)} = \iota(u^{(n)})$ as outlined in Section 4.1.
- Introduce the canonical variables $t = \iota(u)$, $s = \rho$. Invariantizing the differential equation (5.1) and using the recurrence relations for the normalized invariants $I^{(n)}$, rewrite the equation in terms of $t, w = \frac{ds}{dt}, w_t, \ldots, w_{t^{n-1}}$.
- Solve the reduced equation $\widetilde{\Delta}(t, w^{(n-1)}) = 0$ to obtain $w(t)$. The solution to the original equation is, in parametric form,

  $$(x,u) = (-s(t)) \cdot (c,t),$$

  where $s(t) = \int w(t)\,dt$ and the group product is obtained by exponentiating the infinitesimal symmetry generator $\mathbf{v}$.

Example 5.1. To illustrate the above constructions, consider the first order ordinary differential equation

$$2x^4 u u_x + 4x^3 u^2 + 2x = 0. \tag{5.13}$$

This equation is invariant under the infinitesimal symmetry generator

$$\mathbf{v} = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}.$$

The corresponding group action is

$$X = e^{\varepsilon} x, \qquad U = e^{-\varepsilon} u, \qquad \varepsilon \in \mathbb{R}, \tag{5.14}$$

and the order zero lifted recurrence relations are

$$dX = \omega + X\mu, \qquad dU = U_X \omega - U\mu. \tag{5.15}$$

Choosing the cross-section $\mathscr{K} = \{x = 1\}$, the normalized recurrence relations are

$$0 = \varpi + \nu, \qquad dI = I_1 \varpi - I\nu,$$

where $\nu = \rho^*(\mu)$, $\varpi = \varrho^*(\omega)$. The recurrence relation for the phantom invariant gives $\nu = -\varpi$. Substituting ν into the second equation yields the recurrence relation

$$dI = (I_1 + I)\varpi. \tag{5.16}$$

Invariantizing the differential equation (5.13) we get

$$2II_1 + 4I^2 + 2 = 0 \qquad \text{so that} \qquad I_1 = -\frac{2I^2+1}{I}.$$

Substituting the latter equality into the recurrence relation (5.16), we obtain

$$dI = -\frac{I^2+1}{I}\varpi.$$

Introducing the canonical variables

$$t = \iota(u) = I, \qquad s = \rho,$$

we conclude that

$$\frac{ds}{dt} = \frac{\nu}{dI} = \frac{-\varpi}{-\dfrac{I^2+1}{I}\varpi} = \frac{t}{t^2+1}. \tag{5.17}$$

Setting $w = s_t$, equation (5.17) reduces to the algebraic equation

$$w = \frac{t}{t^2+1}.$$

It follows that

$$s(t) = \int w(t)\,dt = \frac{1}{2}\ln(1+t^2) + C,$$

where C is an arbitrary constant of integration. Recalling the group action (5.14), a parametric solution to the original differential equation (5.13) is obtained by acting on the point $(1,t) \in \mathscr{K}$ by the left moving frame $\overline{\rho} = -s(t)$:

$$x(t) = -s(t) \cdot 1 = e^{-s(t)} = \frac{K}{\sqrt{1+t^2}}, \qquad u(t) = -s(t) \cdot t = e^{s(t)} t = \frac{t\sqrt{1+t^2}}{K},$$

where $K = e^C$. Solving the first equation for t, and substituting the result into $u(t)$ yields the solution

$$u(x) = \frac{\sqrt{K^2 - x^2}}{x^2}.$$

5.2. *Solvable symmetry group*

To extend our symbolic implementation of the symmetry reduction algorithm to solvable symmetry groups, we use the inductive/recursive moving frame constructions outlined in Section 4.2. Our starting point is an n^{th} order ordinary differential equation $\Delta(x, u^{(n)}) = 0$ invariant under the prolonged action of an r-dimensional solvable Lie group G with (3.5) as its chain of normal subgroups. Based on this chain of normal subgroups, we introduce the ℓ^{th} lifted subbundle $\mathscr{B}_\ell^{(n)} = \mathrm{J}^{(n)} \times G^{(\ell)}$ and the inclusion map $i_\ell \colon \mathscr{B}_\ell^{(n)} \hookrightarrow \mathscr{B}_G^{(n)}$ induced by the containment $G^{(\ell)} \subset G$. At the infinitesimal level, let $\mathfrak{g}$ be the corresponding r-dimensional solvable Lie algebra spanned by the infinitesimal generators $\mathbf{v}_1, \dots, \mathbf{v}_r$ such that (3.7) holds. Dually, let $\mu^1, \dots, \mu^r \in \mathfrak{g}^*$ be a basis of Maurer–Cartan forms. Finally, for $\ell \in \{1, \dots, r\}$, let G_{ε_ℓ} denote the one-parameter Lie group whose local group action on $\mathrm{J}^{(0)}$ is induced by the flow of $\mathbf{v}_\ell$:

$$g_{\varepsilon_\ell} \cdot (x,u) = \exp[\varepsilon_\ell \mathbf{v}_\ell] \cdot (x,u), \qquad g_{\varepsilon_\ell} \in G_{\varepsilon_\ell}.$$

Prior to the implementation of the symmetry reduction algorithm, we first compute the recurrence relations (4.15) for the G-lifted differential invariants of order $\leq n-1$:

$$\begin{aligned} d[\boldsymbol{\lambda}_G(x)] &= \omega + \sum_{\ell=1}^{r} \boldsymbol{\lambda}_G(\xi_\ell)\, \mu^\ell, \\ d[\boldsymbol{\lambda}_G(u_k)] &= \boldsymbol{\lambda}_G(u_{k+1})\, \omega + \sum_{\ell=1}^{r} \boldsymbol{\lambda}_G(\varphi_\ell^k)\, \mu^\ell, \qquad 0 \leq k \leq n-1. \end{aligned} \tag{5.18}$$

Starting the symmetry reduction process, we first implement the procedure outlined in Section 5.1 for the prolonged action of the 1-parameter group $G_{\varepsilon_1} = G^{(1)}$ on the n^{th} order jet bundle $\mathrm{J}_0^{(n)} = \mathrm{J}^{(n)}$. As in Section 5.1, assume, for simplicity, that x is not an invariant of the G_{ε_1}-action and consider the cross-section

$$\mathscr{K}_1 = \{x = c_1\} \subset \mathrm{J}_0^{(n)}.$$

Let $\rho_1 \colon \mathrm{J}_0^{(n)} \to G_{\varepsilon_1}$ be the corresponding right moving frame and let ι_1 be the induced invariantization map. We introduce the notation

$$I_{;1}^{(n)} = \iota_1(u^{(n)}), \qquad \varpi_1 = \iota_1(dx), \qquad \nu_1^1 = \rho_1^*(\mu^1),$$

to denote the G_{ε_1}-normalized differential invariants, the G_{ε_1}-invariant horizontal form, and the G_{ε_1}-normalized Maurer–Cartan form. To obtain the symbolic expression for ν_1^1, first pull-back the lifted recurrence relations (5.18) by the inclusion map $i_1 \colon \mathscr{B}_1^{(n)} \hookrightarrow \mathscr{B}_G^{(n)}$ to obtain the G_{ε_1}-lifted recurrence relations

$$dX = \omega_1 + \xi_1(X,U)\, \mu^1, \qquad dU_k = U_{k+1}\, \omega_1 + \varphi_1^k(X, U^{(k)})\, \mu^1, \qquad 0 \leq k \leq n-1, \tag{5.19}$$

where $X = \boldsymbol{\lambda}_{G_{\varepsilon_1}}(x)$, $U^{(k)} = \boldsymbol{\lambda}_{G_{\varepsilon_1}}(u^{(k)})$ are the G_{ε_1}-lifted invariants, and $\omega_1 = \boldsymbol{\lambda}_{G_{\varepsilon_1}}(dx)$ is the G_{ε_1}-lifted horizontal form. Recall that in order to simplify the notation, we purposely omit writing the pull-back map i_1^*. Also, note that, symbolically, the equations (5.19) are obtained by setting $\mu^2 = \cdots = \mu^r = 0$ in (5.18) and substituting $\boldsymbol{\lambda}_G(x) \to X$, $\boldsymbol{\lambda}_G(u_k) \to U_k$, and $\omega \to \omega_1$. Taking the pull-back of (5.19) by the right moving frame section $\varrho_1 \colon \mathrm{J}_0^{(n)} \to \mathscr{B}_1^{(n)}$, we obtain the recurrence relations for the G_{ε_1}-normalized differential invariants. The recurrence relation for the phantom invariant $\iota_1(x) = c_1$ implies that

$$\nu_1^1 = H_1(I_{;1})\varpi_1 = -\frac{\varpi_1}{\xi_1(c_1, I_{;1})}. \tag{5.20}$$

The recurrence relations for the G_{ε_1}-normalized differential invariants are then obtained by substituting (5.20) into the remaining recurrence relations, giving

$$dI_{k;1} = F_{k;1}\big(I_{;1}^{(k+1)}\big)\varpi_1 = \left(I_{k+1;1} - \frac{\varphi_1^k(c_1, I_{;1}^{(k)})}{\xi_1(c_1, I_{;1})}\right)\varpi_1, \qquad k = 0,\ldots,n-1.$$

As in the previous section, we introduce the canonical variables

$$t_1 = I_{;1} = \iota_1(u), \qquad\qquad s_1 = \rho_1,$$

and let

$$w_1 = \frac{ds_1}{dt_1} = \frac{\nu_1^1}{dI_{;1}} = \frac{H_1(I_{;1})}{F_{;1}(I_{;1}^{(1)})} = \frac{1}{\varphi_1(c_1, I_{;1}) - \xi_1(c_1, I_{;1})I_{1;1}} = \frac{1}{Q_1(c, I_{;1}, I_{1;1})},$$

where $Q_1(x,u,u_x)$ is the characteristic of the vector field $\mathbf{v}_1$. In the variables (t_1, w_1), the order of the original differential equation (5.1) is reduced by one

$$\Delta(x, u^{(n)}) = \widetilde{\Delta}_1(t_1, w_1^{(n-1)}) = 0,$$

and this completes the first iteration of the symmetry reduction algorithm.

Before proceeding to the second iteration, we recall from Section 5.1 that the normalized invariants $I_{;1}^{(n)}$ can be expressed in terms of $(t_1, w_1^{(n-1)})$:

$$I_{;1} = t_1, \qquad I_{k;1} = \mathscr{I}_k(t_1, w_1^{(k-1)}), \quad 1 \le k \le n-1. \tag{5.21}$$

Therefore, the variables $(t_1, w_1^{(n-1)})$ provide a local system of coordinates for the cross-section $\mathscr{K}_1$,

$$\mathscr{K}_1 = \{(c_1, t_1, w_1^{(n-1)})\} \simeq \mathrm{J}_1^{(n-1)} = \{(t_1, w_1^{(n-1)})\},$$

which is isomorphic to the $(n-1)^{\text{th}}$ order jet space $\mathrm{J}_1^{(n-1)} = \{(t_1, w_1^{(n-1)})\}$.

For the second iteration of the symmetry reduction algorithm, we consider the induced action of the one-parameter group $G_{\varepsilon_2} \simeq G^{(2)}/G^{(1)}$ on $\mathrm{J}_1^{(n-1)}$. To obtain the G_{ε_2}-lifted recurrence relations, we implement the inductive/recursive moving frame constructions introduced in Section 4.2 with $G = G^{(2)}$ and $N = G^{(1)}$. First, we pull-back the lifted recurrence relations (5.18) by the inclusion map $i_2 \colon \mathscr{B}_2^{(n)} \hookrightarrow \mathscr{B}_G^{(n)}$ to obtain the lifted recurrence relations of the $G^{(2)}$-prolonged action on $\mathrm{J}^{(n)}$. Introducing the $G^{(2)}$ lift map $\boldsymbol{\lambda}^{(2)} = \boldsymbol{\lambda}_{G^{(2)}}$, we have

$$\begin{aligned} d[\boldsymbol{\lambda}^{(2)}(x)] &= \omega_2 + \boldsymbol{\lambda}^{(2)}[\xi_1(x,u)]\,\mu^1 + \boldsymbol{\lambda}^{(2)}[\xi_2(x,u)]\,\mu^2,\\ d[\boldsymbol{\lambda}^{(2)}(u_k)] &= \boldsymbol{\lambda}^{(2)}(u_{k+1})\,\omega_2 + \boldsymbol{\lambda}^{(2)}[\varphi_1^k(x,u^{(k)})]\,\mu^1 + \boldsymbol{\lambda}^{(2)}[\varphi_2^k(x,u^{(k)})]\,\mu^2, \quad 0 \le k \le n-1, \end{aligned} \tag{5.22}$$

where $\omega_2 = \boldsymbol{\lambda}^{(2)}(dx)$. In the lifted recurrence relations (5.22), we make the substitution $\boldsymbol{\lambda}^{(2)}(x, u^{(n)}) = \boldsymbol{\lambda}^{(2)}(c_1, I_{;1}^{(n)})$, corresponding to equality (4.22) in the general framework of Section

4.2. This yields the recurrence relations

$$\begin{aligned} d[\boldsymbol{\lambda}^{(2)}(c_1)] &= \omega_2 + \boldsymbol{\lambda}^{(2)}[\xi_1(c_1, I_{;1})]\,\mu^1 + \boldsymbol{\lambda}^{(2)}[\xi_2(c_1, I_{;1})]\,\mu^2, \\ d[\boldsymbol{\lambda}^{(2)}(I_{k;1})] &= \boldsymbol{\lambda}^{(2)}(I_{k+1;1})\,\omega_2 + \boldsymbol{\lambda}^{(2)}[\varphi_1^k(c_1, I_1^{(k)})]\,\mu^1 + \boldsymbol{\lambda}^{(2)}[\varphi_2^k(c_1, I_{;1}^{(k)})]\,\mu^2, \end{aligned} \tag{5.23}$$

where $0 \le k \le n-1$. Introducing the partial normalization

$$\boldsymbol{\lambda}^{(2)}(c_1) = c_1,$$

let $\widehat{\rho}_{G^{(1)}}\colon \mathrm{J}^{(n)} \times G_{\varepsilon_2} \to G^{(2)}$ be the corresponding *partial moving frame*. Pulling-back the recurrence relations (5.23) by $\widehat{\varrho}_{G^{(1)}} = (\mathrm{id}, \widehat{\rho}_{G^{(1)}})\colon \mathrm{J}^{(n)} \times G_{\varepsilon_2} \to \mathrm{J}^{(n)} \times G^{(2)}$, the first equation becomes

$$0 = \widehat{\omega}_2 + \xi_1(c_1, \widehat{I}_{;1})\,\widehat{\mu}^1 + \xi_2(c_1, \widehat{I}_{;1})\,\widehat{\mu}^2,$$

where

$$\widehat{I}_{;1} = \boldsymbol{\lambda}_{G_{\varepsilon_2}}(I_{;1}) = g_{\varepsilon_2} \cdot I_{;1}, \qquad g_{\varepsilon_2} \in G_{\varepsilon_2},$$

$\widehat{\omega}_2 = \widehat{\varrho}^*_{G^{(1)}}(\omega_2)$, and $\widehat{\mu}^\ell = \widehat{\rho}^*_{G^{(1)}}(\mu^\ell)$, $\ell = 1,2$. Solving for $\widehat{\mu}^1$ and substituting the result into the remaining equations of (5.23) gives the lifted recurrence relations for the Lie group G_{ε_2} acting on the G_{ε_1}-normalized invariants of order $\le n-1$:

$$d(\widehat{I}_{k;1}) = \widehat{I}_{k+1;1}\,\widehat{\omega}_2 + \psi_1^k(c_1, \widehat{I}_{;1}^{(k)})\,\widehat{\mu}^2, \qquad 0 \le k \le n-1. \tag{5.24}$$

The change of variables (5.21), implies that

$$\widehat{I}_{;1} = T_1, \qquad \widehat{I}_{k;1} = \mathscr{I}_k(T_1, W_1^{(k-1)}), \tag{5.25}$$

where

$$T_1 = \boldsymbol{\lambda}_{G_{\varepsilon_2}}(t_1) = g_{\varepsilon_2} \cdot t_1 \qquad \text{and} \qquad W_1^{(k-1)} = \boldsymbol{\lambda}_{G_{\varepsilon_2}}(w_1^{(k-1)}) = g_{\varepsilon_2} \cdot w_1^{(k-1)}.$$

Substituting (5.25) into (5.24), we obtain the lifted recurrence relations of the G_{ε_2} action on the jet space $\mathrm{J}_1^{(n-1)} = \{(t_1, w_1^{(n-1)})\}$:

$$\begin{aligned} dT_1 &= \mathscr{T}\big(T_1, W_1\big)\,\widehat{\omega}_2 + \xi(T_1, W_1)\widehat{\mu}^2, \\ dW_{1;T_1^k} &= \mathscr{W}_k\big(T_1, W_1^{(k+1)}\big)\,\widehat{\omega}_2 + \varphi^k(T_1, W_1^{(k)})\widehat{\mu}^2, \qquad k = 0, \ldots, n-1. \end{aligned} \tag{5.26}$$

Taking the inverse lift $\boldsymbol{\lambda}^{-1}_{G_{\varepsilon_2}}$ of the coefficients $\xi(T_1, W_1)$, $\varphi(T_1, W_1)$ multiplying $\widehat{\mu}^2$ in (5.26) we obtain the components of the prolonged vector field $\mathbf{v}_2^{(1)}$ restricted to $\mathrm{J}_1^{(0)}$:

$$\widehat{\mathbf{v}}_2 = \mathbf{v}_2^{(1)}\big|_{(t_1, w_1)} = \xi(t_1, w_1)\frac{\partial}{\partial t_1} + \varphi(t_1, w_1)\frac{\partial}{\partial w_1}. \tag{5.27}$$

Exponentiating the infinitesimal generator (5.27) gives explicit expressions for the G_{ε_2}-action on (t_1, w_1).

With (5.26) in hand, we now implement the second iteration of the symmetry reduction procedure. Again, for simplicity, assume that t_1 is not an invariant of the G_{ε_2}-action, and let $\mathscr{K}_2 = \{t_1 =$

$c_2\} \subset \mathrm{J}_1^{(n-1)}$ be a local cross-section. Let $\rho_2 \colon \mathrm{J}_1^{(n-1)} \to G_{\varepsilon_2}$ be the corresponding right moving frame, and ι_2 the induced invariantization map. We also introduce the notation

$$I_{;2}^{(n-1)} = \iota_2(w_1^{(k-1)}) \qquad \text{and} \qquad \varpi_2 = \varrho_2^* \, (\widehat{\omega}_2).$$

Making the normalization $T_1 = c_2$ in (5.26) and solving for the normalized Maurer–Cartan form $\nu_2^2 = \varrho_2^* \, (\widehat{\mu}^2)$ we obtain

$$\nu_2^2 = H_2(I_{;2})\varpi_2.$$

Substituting the result into the remaining formulas yields the recurrence relations

$$dI_{k;2} = F_{k;2}\big(I_{;2}^{(k+1)}\big)\varpi_2, \qquad k = 1, \ldots, n-2.$$

A new set of canonical variables is introduced by setting

$$t_2 = I_{;2} = \iota_2(w_1) \qquad \text{and} \qquad s_2 = \rho_2.$$

Also, let

$$w_2 = \frac{ds_2}{dt_2} = \frac{\nu_2^2}{dI_{;2}} = \frac{H_2(I_{;2})}{F_{;2}(I_{;2}^{(1)})}.$$

In the new coordinates (t_2, w_2), the order of the original differential equation (5.1) is now reduced by two:

$$\widetilde{\Delta}_2(t_2, w_2^{(n-2)}) = 0.$$

Also, the jet variables $(t_2, w_2^{(n-2)})$ can be used to parametrize the cross-section

$$\mathscr{K}_2 = \{(c_2, t_2, w_2^{(n-2)})\} \simeq \mathrm{J}_2^{(n-2)} = \{(t_2, w_2^{(n-2)})\}.$$

The symmetry reduction procedure continues by considering the action of G_{ε_3} on the submanifold jet bundle $\mathrm{J}_2^{(n-2)} = \{(t_2, w_2^{(n-2)})\}$, and so on. At the ℓ^{th} iteration, the lifted recurrence relations of the G_{ε_ℓ}-prolonged action on the jet space $\mathrm{J}_{\ell-1}^{(n-\ell+1)} = \{(t_{\ell-1}, w_{\ell-1}^{(n-\ell+1)})\}$ is obtained by considering the $G^{(\ell)}$-lifted recurrence relations followed by the normalization of the Maurer–Cartan forms μ^1, …, $\mu^{\ell-1}$. The latter is achieved by pulling back the $G^{(\ell)}$-lifted recurrence relations with respect to the partial moving frame that keeps the partial cross-section

$$\mathscr{K} = \{x = c_1, t_1 = c_2, \ldots, t_{\ell-2} = c_{\ell-1}\}$$

invariant. Assuming that $t_{\ell-1}$ is not an invariant of the G_{ε_ℓ}-action, we introduce the cross-section $\mathscr{K}_\ell = \{t_{\ell-1} = c_\ell\}$ and let $\rho_\ell \colon \mathrm{J}_{\ell-1}^{(n-\ell+1)} \to G_{\varepsilon_\ell}$ be the corresponding right moving frame. The order of the differential equation $\widetilde{\Delta}_{\ell-1}(t_{\ell-1}, w_{\ell-1}^{(n-\ell+1)}) = 0$ is reduced by one by introducing the canonical variables

$$t_\ell = \iota_\ell(w_{\ell-1}), \qquad s_\ell = \rho_\ell, \tag{5.28}$$

and rewriting the equation in terms of the variables t_ℓ, $w_\ell = \frac{ds_\ell}{dt_\ell}$, and the derivatives of w_ℓ with respect to t_ℓ.

At the end of the symmetry reduction algorithm, the fully reduced equation is the order $n-r$ differential equation

$$\widetilde{\Delta}_r(t_r, w_r^{(n-r)}) = 0. \tag{5.29}$$

Let $w_r(t_r)$ be the solution to the fully reduced equation (5.29). We now reconstruct the solution to the original differential equation (5.1). By definition $w_r = \frac{ds_r}{dt_r}$, so that

$$s_r(t_r) = \int w_r\, dt_r.$$

Next, since $\iota_r(t_{r-1}, w_{r-1}) = \rho_r \cdot (t_{r-1}, w_{r-1}) = (c_r, t_r)$ and $\rho_r = s_r$, it follows that

$$t_{r-1} = t_{r-1}(t_r) = \rho_r^{-1} \cdot c_r = (-s_r) \cdot c_r, \tag{5.30a}$$

$$w_{r-1} = w_{r-1}(t_r) = \rho_r^{-1} \cdot t_r = (-s_r) \cdot t_r, \tag{5.30b}$$

where the product is given by the G_{ε_r}-action on (t_{r-1}, w_{r-1}). Inverting (5.30a), we obtain $t_r = t_r(t_{r-1})$. Substituting the result in (5.30b), yields the function $w_{r-1}(t_r(t_{r-1}))$. Repeating the procedure, since $w_{r-1} = \frac{ds_{r-1}}{dt_{r-1}}$ we have

$$s_{r-1}(t_{r-1}) = \int w_{r-1}\, dt_{r-1}$$

and

$$t_{r-2} = (-s_{r-1}) \cdot c_{r-1}, \qquad w_{r-2} = (-s_{r-1}) \cdot t_{r-1},$$

where the product is now given by the $G_{\varepsilon_{r-1}}$-action on (t_{r-2}, w_{r-2}). Continuing in this fashion, at the last iteration we have

$$s_1 = \int w_1\, dt_1$$

and

$$x = x(t_1) = (-s_1) \cdot c_1, \qquad u = u(t_1) = (-s_1) \cdot t_1,$$

where the product is given by the G_{ε_1}-action on (x, u). Inverting the relation $x = x(t_1)$, to obtain $t_1 = t_1(x)$, the solution to the original differential equation (5.1) is $u(t_1(x))$.

Before considering two examples, let us summarize the above considerations into a list of executable steps. Our starting point is an n^{th} order ordinary differential equation $\Delta(x, u^{(n)}) = 0$.

- Compute a basis of infinitesimal symmetry generators using standard techniques, [3, 5–8, 10, 15–17, 27, 35–37].
- Find a sufficiently large solvable Lie subalgebra $\mathfrak{g}$ of dimension r, and let G be the corresponding symmetry group action. Note that we do not have to compute the group action to implement the algorithm. Compute the chain of normal Lie subalgebras (3.6), and consider a canonical basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ such that (3.7) holds.
- Compute the recurrence relations (4.15) for the G-lifted invariants of order $\leq n-1$.

- Reduce the order of the differential equation $\Delta(x,u^{(n)})=0$ iteratively. For $1\le \ell \le r$, the ℓ^{th} iteration consists of reducing the differential equation[f] $\Delta_\ell(t_{\ell-1},\, w_{\ell-1}^{(n-\ell+1)})=0$ with respect to the one-parameter symmetry group $G_{\varepsilon_\ell}\simeq G^{(\ell)}/G^{(\ell-1)}$. To this end,
  - Implement the inductive/recursive moving frame constructions with $N=G^{(\ell-1)}$ and $G=G^{(\ell)}$ to obtain the lifted recurrence relations of the G_{ε_ℓ} action on $\mathrm{J}_{\ell-1}^{(n-\ell+1)}=\{(t_{\ell-1},\, w_{\ell-1}^{(n-\ell+1)})\}$. Recall that the $G^{(\ell)}$-lifted recurrence relations are obtained symbolically by setting $\mu^{\ell+1}=\cdots=\mu^r=0$ in the G-lifted recurrence relations.
  - Choose a cross-section $\mathscr{K}_\ell=\{t_{\ell-1}=c_\ell\}\subset \mathrm{J}_{\ell-1}^{(n-\ell+1)}$ and let ρ_ℓ be the corresponding right moving frame.
  - Introduce the canonical variables t_ℓ, s_ℓ defined in (5.28), and compute $w_\ell=\frac{ds_\ell}{dt_\ell}$, $\frac{dw_\ell}{dt_\ell}$, and its higher derivatives up to order $n-\ell+1$ symbolically using the recurrence relations for the G_{ε_ℓ}-normalized invariants.
  - Invariantize $\widetilde{\Delta}_{\ell-1}(t_{\ell-1},w_{\ell-1}^{(n-\ell+1)})=0$ using the moving frame ρ_ℓ. Writing the result in terms of $(t_\ell,w_\ell^{(n-\ell)})$ yields the reduced equation

$$\widetilde{\Delta}_\ell(t_\ell,w_\ell^{(n-\ell)})=0.$$

- Let $w_r(t_r)$ be the solution to the fully reduced equation $\widetilde{\Delta}_r(t_r,w_r^{(n-r)})=0$. The solution to the original differential equation is obtained by implementing the reconstruction process. For $1\le\ell\le r$, the ℓ^{th} step consists of:
  - Evaluating $s_{r-\ell+1}=\displaystyle\int w_{r-\ell+1}\,dt_{r-\ell+1}$.
  - Computing

$$t_{r-\ell}(t_{r-\ell+1})=(-s_{r-\ell+1})\cdot c_{r-\ell+1},\qquad w_{r-\ell}(t_{r-\ell+1})=(-s_{r-\ell+1})\cdot t_{r-\ell+1},$$

    using the $G_{\varepsilon_{r-\ell+1}}$-action on $(t_{r-\ell},w_{r-\ell})$. This action is obtained symbolically from the $G_{\varepsilon_{r-\ell+1}}$-lifted recurrence relations computed in the $(r-\ell+1)^{\text{th}}$ reduction step as follows. From the lifted recurrence relations

$$dT_{r-\ell}=\cdots+\xi(T_{r-\ell},W_{r-\ell})\widehat{\mu}^{r-\ell+1},\qquad dW_{r-\ell}=\cdots+\varphi(T_{r-\ell},W_{r-\ell})\widehat{\mu}^{r-\ell+1},$$

    extract the coefficients $\xi(T_{r-\ell},W_{r-\ell})$ and $\varphi(T_{r-\ell},W_{r-\ell})$. Their inverse lift yields the coefficients of the vector field

$$\widehat{\mathbf{v}}_{\varepsilon_{r-\ell+1}}=\xi(t_{r-\ell},w_{r-\ell})\frac{\partial}{\partial t_{r-\ell}}+\varphi(t_{r-\ell},w_{r-\ell})\frac{\partial}{\partial w_{t-\ell}}.$$

    Exponentiating $\widehat{\mathbf{v}}_{\varepsilon_{r-\ell+1}}$ gives the formulas for the $G_{\varepsilon_{r-\ell+1}}$-action on $(t_{r-\ell},w_{r-\ell})$.
  - Inverting $t_{r-\ell}=t_{r-\ell}(t_{r-\ell+1})$, and substituting $t_{r-\ell+1}=t_{r-\ell+1}(t_{r-\ell})$ into $w_{r-\ell}(t_{r-\ell+1})$, yields $w_{r-\ell}(t_{r-\ell+1}(t_{r-\ell}))$.
- The output of the reconstruction process is the solution to the original differential equation $\Delta(x,u^{(n)})=0$.

[f] When $\ell=1$, we let $\Delta_0(t_0,w_0^{(n)})=\Delta(x,u^{(n)})=0$.

Example 5.2. We now revisit Example 3.1 using the moving frame machinery. We emphasize the fact that, this time around, computations are performed symbolically without relying on the coordinate expressions of the canonical variables and the differential invariants.

Since the differential equation is of order two, we consider the lifted recurrence relations (4.17) of the full symmetry group (4.2) up to order one. According to the solvable structure of the symmetry group, we first implement the symmetry reduction process using the one-parameter group G_{ε_1}. The recurrence relations for the G_{ε_1}-lifted invariants were obtained in (4.32). Choosing the cross-section (4.27), let ρ_1 be the corresponding right moving frame and let ι_1 be the induced invariantization map. Using the same notation as in (4.29), the recurrence relations for the G_{ε_1}-normalized invariants are given in (4.35).

We now introduce the canonical variables

$$t = \iota_1(x) = H, \qquad s = \rho_1.$$

Using (4.34) and the recurrence relations (4.35), we have that

$$s_t = \frac{ds}{dt} = \frac{\nu^1}{dH} = -\frac{I_1}{H},$$

and

$$s_{tt} = \frac{ds_t}{dt} = \frac{1}{\varpi_1} d\Big(-\frac{I_1}{H}\Big) = \frac{1}{\varpi_1}\Big(-\frac{dI_1}{H} + \frac{I_1}{H^2} dH\Big) = -\frac{I_2}{H} + 2\frac{I_1}{H^2}.$$

Therefore,

$$I_1 = -t s_t \qquad \text{and} \qquad I_2 = -t s_{tt} - 2 s_t.$$

Introducing the variable $w = s_t$,

$$I_1 = -tw \qquad \text{and} \qquad I_2 = -t w_t - 2w. \tag{5.31}$$

Invariantizing the differential equation (3.8), we obtain

$$H^2 I_2 = F(H I_1),$$

which when expressed in the variables $(t, w^{(1)})$, yields the first order differential equation

$$t^3 w_t + 2t^2 w = -F(-t^2 w). \tag{5.32}$$

Now, let G_{ε_2} act on $\mathrm{J}_1^{(1)} = \{(t, w^{(1)})\}$. To obtain the corresponding G_{ε_2}-lifted recurrence relations, recall that the recurrence relations for the G_{ε_2}-lift of the G_{ε_1}-normalized invariants $(H, I_1, I_2, \ldots)$ were obtained in (4.36). Using (5.31), we have that

$$\begin{aligned} \widehat{H} &= g_{\varepsilon_2} \cdot H = g_{\varepsilon_2} \cdot t = T, \\ \widehat{I_1} &= g_{\varepsilon_2} \cdot I_1 = g_{\varepsilon_2} \cdot (-tw) = -TW, \\ \widehat{I_2} &= g_{\varepsilon_2} \cdot I_2 = g_{\varepsilon_2} \cdot (-t w_t - 2w) = -T W_T - 2W. \end{aligned} \tag{5.33}$$

Substituting (5.33) into the first two equations of (4.36), we obtain

$$dT = \widehat{\omega} + T \widehat{\mu}^2, \qquad dW = W_T \widehat{\omega} - 2W \widehat{\mu}^2. \tag{5.34}$$

Assuming $T > 0$, a cross-section to the G_{ε_2}-action is given by

$$\mathscr{K}_2 = \{t = 1\},$$

which is equivalent to making the normalization $T = 1$. Let ρ_2 be the induced right moving frame and ι_2 the corresponding invariantization map. Also, let

$$J = \iota_2(w), \qquad J_1 = \iota_2(w_t), \qquad \nu^2 = \varrho_2^* \, (\widehat{\mu}^2), \qquad \varpi_2 = \varrho_2^* \, (\widehat{\omega}).$$

Then, the normalized recurrence relations are

$$0 = \varpi_2 + \nu^2, \qquad J = J_1 \varpi_2 - 2J\nu^2.$$

The first equation gives

$$\nu^2 = -\varpi_2,$$

which when substituted into the second equation yields

$$dJ = (J_1 + 2J)\varpi_2. \tag{5.35}$$

Invariantizing (5.32) we find that

$$J_1 = -(2J + F(-J)). \tag{5.36}$$

Substituting (5.36) into (5.35) yields

$$dJ = -F(-J)\varpi_2.$$

Introducing the canonical variables

$$y = J, \qquad z = \rho_2,$$

where $z = z(y)$, we have that

$$z_y = \frac{dz}{dy} = \frac{\nu^2}{dJ} = \frac{1}{F(-y)}.$$

As in Example 3.1, assume $F(-y) = y^2$. Then

$$z_y = \frac{1}{y^2} \qquad \text{so that} \qquad z = -\frac{1}{y} + \ln C. \tag{5.37}$$

Given (5.37), we now recover the solution to the original differential equation (3.15). Taking the inverse lift of the coefficients multiplying $\widehat{\mu}^2$ in (5.34), i.e. $\boldsymbol{\lambda}_{\varepsilon_2}^{-1}(T) = t$ and $\boldsymbol{\lambda}_{\varepsilon_2}^{-1}(-2W) = -2w$, we obtain the infinitesimal generator

$$\widehat{\mathbf{v}}_2 = t\frac{\partial}{\partial t} - 2w\frac{\partial}{\partial w}.$$

This vector field induces the one-parameter group action

$$T = e^{\varepsilon_2} t, \qquad W = e^{-2\varepsilon_2} w.$$

Since

$$1 = \iota_2(t) = e^{\rho_2} t, \qquad y = J = \iota_2(w) = e^{-2\rho_2} w,$$

and $\rho_2 = z$, it follows that

$$t = e^{-z} = \frac{1}{C} e^{1/y} \qquad \text{and} \qquad w = e^{2z} y = C^2 y e^{-2/y}.$$

From the first equation $y = \dfrac{1}{\ln(Ct)}$, which when substituted into the expression for w yields

$$w(t) = \frac{1}{t^2 \ln(Ct)}.$$

Since $w = s_t$,

$$s(t) = \int w(t)\, dt = -C\,\mathrm{E}_1(\ln(Ct)) - K.$$

Setting $\varepsilon_2 = 0$ in (4.2) we see that G_{ε_1} acts on (x,u) by Galilean boosts:

$$X = x, \qquad U = u + \varepsilon_1 x.$$

Recalling that

$$t = H = \iota_1(x) = x \qquad \text{and} \qquad 0 = \iota_1(u) = u + \rho_1 x = u + sx,$$

we conclude that the general solution to the original differential equation (3.15) is

$$u = -x\,s(x) = Cx\,\mathrm{E}_1(\ln(Cx)) + Kx.$$

As expected, we recover the same solution as in Example 3.1, with the important distinction that the coordinate expressions for the canonical variables and the differential invariants were not used in the derivation of the solution.

Remark 5.1. The choice of canonical variables is not unique. For example, in Example 3.1 the general solution to the system of differential equations (3.10) is

$$t = f(x), \qquad s = \frac{u}{x} + g(x), \tag{5.38}$$

where $f_x \neq 0$ and g is an arbitrary smooth function. The canonical variables introduced in (3.11) correspond to the simplest possible solution.

This freedom in the choice of the canonical variables also appears in the moving frame approach. First, the choice of the cross-section (4.27) is not unique. If it were replaced by

$$\mathscr{K}_1 = \{u = -x g(x)\},$$

then the solution to the normalization equation $-xg(x) = U = u + \varepsilon_1 x$ would give

$$s = \varepsilon_1 = -\frac{u}{x} - g(x),$$

recovering, up to a sign, the second canonical variable in (5.38). Also, since the function of an invariant is an invariant function, the definition of the canonical variable

$$t = \iota_1(x) = x$$

can be extend to

$$t = f(\iota_1(x)) = f(x),$$

where $f_x \neq 0$. In doing so we recover the first canonical variable in (5.38).

From a purely symbolic and computational standpoint, the symmetry reduction algorithm can be executed using solely the lifted recurrence relations of the the whole symmetry group G. Reviewing the general procedure outlined above, we notice that in the first iteration of the symmetry reduction algorithm, all computations can be carried out using the lifted recurrence relations for G provided the equalities are defined modulo the Maurer–Cartan forms $\mu^2, \ldots, \mu^r$. Similarly, in the second iteration, all computations hold modulo the Maurer–Cartan forms $\mu^3, \ldots, \mu^r$. In general, at the ℓ^{th} iteration, the first $\ell - 1$ Maurer–Cartan forms $\mu^1, \ldots, \mu^{\ell-1}$ will have been (partially) normalized and the symmetry reduction computations can be performed using the recurrence relations of the whole symmetry group provided the equalities are defined modulo the (partially normalized) Maurer–Cartan forms $\widehat{\mu}^{\ell+1}, \ldots, \widehat{\mu}^r$. By working with the recurrence relations of the whole symmetry, the computations and the implementation of the symmetry reduction algorithm becomes more straightforward. This is illustrated in the following example.

Example 5.3. Consider the second order ordinary differential equation

$$u_{xx} = \frac{u_x}{u^2} - \frac{1}{xu}. \tag{5.39}$$

Equation (5.39) admits a two-dimensional symmetry group G with infinitesimal generators

$$\mathbf{v}_1 = x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}, \qquad \mathbf{v}_2 = -x\frac{\partial}{\partial x} - \frac{u}{2}\frac{\partial}{\partial u},$$

and first prolongation

$$\mathbf{v}_1^{(1)} = x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u} + (u - xu_x)\frac{\partial}{\partial u_x}, \qquad \mathbf{v}_2^{(1)} = -x\frac{\partial}{\partial x} - \frac{u}{2}\frac{\partial}{\partial u} + \frac{u_x}{2}\frac{\partial}{\partial u_x}.$$

The lifted recurrence relations of the whole symmetry group G are, up to order 1,

$$\begin{aligned}
dX &= \omega + X^2\mu^1 - X\mu^2,\\
dU &= U_X\omega + XU\mu^1 - \frac{U}{2}\mu^2,\\
dU_X &= U_{XX}\omega + (U - XU_X)\mu^1 + \frac{U_X}{2}\mu^2.
\end{aligned} \tag{5.40}$$

Since

$$[\mathbf{v}_1, \mathbf{v}_2] = \mathbf{v}_1,$$

we first reduce equation (5.39) using the 1-parameter group induced by the infinitesimal generator $\mathbf{v}_1$. Let $\rho_1 \colon \mathrm{J}^{(2)} \to \mathscr{B}^{(2)}_{G_{\varepsilon_1}}$ be the moving frame obtained by normalizing $X = 1$ and let ι_1 be the induced invariantization map. Working with the 2-parameter group G, let $\widehat{\rho}_1$ be the partial moving frame obtained by normalizing $X = 1$. Also, let

$$\widehat{I}_k = \widehat{\rho}_1 \cdot I_k, \qquad \widehat{\omega} = \widehat{\varrho}_1^*(\boldsymbol{\lambda}_G(\varpi_1)), \qquad \widehat{\mu}^1 = \widehat{\rho}_1^*(\mu^1), \qquad \widehat{\mu}^2 = \widehat{\rho}_1^*(\mu^2),$$

where $I_k = \iota_1(u_k)$ and $\varpi_1 = \iota_1(dx)$. Pulling back the lifted recurrence relations (5.40) by the section $\widehat{\varrho}_1$ induced by the partial moving frame $\widehat{\rho}_1$, we obtain

$$0 = \widehat{\omega} + \widehat{\mu}^1 - \widehat{\mu}^2, \qquad d\widehat{I} = \widehat{I}_1\widehat{\omega} + \widehat{I}\widehat{\mu}^1 - \frac{\widehat{I}}{2}\widehat{\mu}^2, \qquad d\widehat{I}_1 = \widehat{I}_2\widehat{\omega} + (\widehat{I} - \widehat{I}_1)\widehat{\mu}^1 + \frac{\widehat{I}_1}{2}\widehat{\mu}^2.$$

Since we are performing symmetry reduction with respect to $\mathbf{v}_1$, we use the first equation to solve for $\widehat{\mu}^1$:

$$\widehat{\mu}^1 = -\widehat{\omega} + \widehat{\mu}^2. \tag{5.41}$$

Substituting the result into the remaining equations yields

$$d\widehat{I} = (\widehat{I}_1 - \widehat{I})\widehat{\omega} + \frac{\widehat{I}}{2}\widehat{\mu}^2, \qquad d\widehat{I}_1 = (\widehat{I}_2 + \widehat{I}_1 - \widehat{I})\widehat{\omega} + \left(\widehat{I} - \frac{\widehat{I}_1}{2}\right)\widehat{\mu}^2. \tag{5.42}$$

Modulo $\widehat{\mu}^2$, we obtain

$$d\widehat{I} \equiv (\widehat{I}_1 - \widehat{I})\widehat{\omega}, \qquad d\widehat{I}_1 \equiv (\widehat{I}_2 + \widehat{I}_1 - \widehat{I})\widehat{\omega}. \tag{5.43}$$

The recurrence relations for the G_{ε_1}-normalized invariants $I = \iota_1(u)$, $I_1 = \iota_1(u_x)$ are symbolically obtained by dropping the hat notation over the invariants in (5.43) and substituting $\widehat{\omega} \to \varpi_1 = \iota_1(dx)$:

$$dI = (I_1 - I)\varpi_1, \qquad dI_1 = (I_2 + I_1 - I)\varpi_1.$$

Also, modulo $\widehat{\mu}^2$, equation (5.41) becomes $\widehat{\mu}^1 \equiv -\widehat{\omega}$, from which we conclude that $\nu^1 = \rho_1^*(\mu^1) = \varpi_1$. Introducing the canonical variables $t = I$, $s = \rho_1$, and setting $w = s_t$, we have that

$$w = s_t = \frac{\nu^1}{dI} \equiv -\frac{1}{I_1 - I} \qquad \text{and} \qquad w_t = s_{tt} = \frac{I_2}{(I_1 - I)^3}. \tag{5.44}$$

Invariantizing (5.39) we obtain the reduced differential equation

$$w_t = \frac{w^2}{t^2}. \tag{5.45}$$

We now reduce (5.45) using the 1-parameter group G_{ε_2} induced by the infinitesimal generator $\mathbf{v}_2$. From (5.44), we have that

$$I = t, \qquad I_1 = t - \frac{1}{w}, \qquad I_2 = -\frac{w_t}{w^3}.$$

Making the substitutions

$$\widehat{I} \to T, \qquad \widehat{I}_1 \to T - \frac{1}{W}, \qquad \widehat{I}_2 \to -\frac{W_T}{W^3},$$

into (5.42), we obtain the order zero G_{ε_2}-lifted recurrence relations

$$dT = -\frac{1}{W}\widehat{\omega} + \frac{T}{2}\widehat{\mu}^2, \qquad dW = -\frac{W_T}{W}\widehat{\omega} + \frac{W}{2}\widehat{\mu}^2. \tag{5.46}$$

Let $\rho_2 \colon \mathrm{J}_1^{(1)} = \{(t,w,w_t)\} \to G_{\varepsilon_2}$ be the moving frame obtained by normalizing $T = 1$, and let ι_2 be the corresponding invariantization map. Introducing the notation

$$J = \iota_2(w), \qquad J_1 = \iota_2(w_t), \qquad \varpi_2 = \varrho_2^*\,(\widehat{\omega}), \qquad \nu^2 = \varrho_2^*\,(\widehat{\mu}^2),$$

and pulling back (5.46) by the moving frame section ϱ_2 we obtain

$$0 = -\frac{1}{J}\varpi_2 + \frac{1}{2}\nu^2, \qquad dJ = -\frac{J_1}{J}\varpi_2 + \frac{J}{2}\nu^2.$$

The first equation implies that

$$\nu^2 = \frac{2}{J}\varpi_2. \tag{5.47}$$

Substituting (5.47) into the second equation yields the recurrence relation

$$dJ = \left(1 - \frac{J_1}{J}\right)\varpi_2. \tag{5.48}$$

Invariantizing (5.45) we obtain the syzygy $J_1 = J^2$, which when substituted in (5.48) yields

$$dJ = (1-J)\varpi_2.$$

Introducing the canonical variables $y = J$ and $z = \rho_2$, with $z = z(y)$, we have that

$$z_y = \frac{dz}{dy} = \frac{\nu^2}{dJ} = \frac{2}{y(1-y)}.$$

Integrating the latter equation gives

$$z(y) = \ln\left(\frac{Cy}{1-y}\right)^2.$$

We now recover the solution to the original differential equation (5.39) by implementing the reconstruction process. Un-lifting the coefficients multiplying $\widehat{\mu}^2$ in (5.46), we conclude that

$$\widehat{\mathbf{v}}_2 = \mathbf{v}_2^{(1)}\big|_{(t,w)} = \frac{t}{2}\frac{\partial}{\partial t} + \frac{w}{2}\frac{\partial}{\partial w}.$$

Therefore, G_{ε_2} acts on (t,w) by dilation:

$$T = e^{\frac{\varepsilon_2}{2}}t, \qquad W = e^{\frac{\varepsilon_2}{2}}w.$$

Since $\iota_2(t,w) = \rho_2 \cdot (t,w) = (1,y)$, we have

$$t = e^{-\frac{\rho_2}{2}} \cdot 1 = e^{-\frac{z}{2}} = \frac{1-y}{Cy} \qquad \text{so that} \qquad y = \frac{1}{Ct+1},$$

and

$$w = e^{-\frac{\rho_2}{2}} y = e^{-\frac{z}{2}} y = \frac{1-y}{C} = \frac{t}{Ct+1}.$$

Hence,

$$s_t = w = \frac{t}{Ct+1} \qquad \text{and} \qquad s(t) = \frac{t}{C} - \frac{1}{C^2}\ln(Ct+1) + K.$$

Finally, since the action of G_{ε_1} on (x,u) is given by

$$X = \frac{x}{1-\varepsilon_1 x}, \qquad U = \frac{u}{1-\varepsilon_1 x},$$

acting on $\iota_1(x,u) = (1,t)$ by the left moving frame $\overline{\rho}_1 = -s(t)$ yields the parametric solution

$$x = \frac{1}{1+s} = \frac{C^2}{Ct+B-\ln(Ct+1)}, \qquad u = \frac{t}{1+s} = \frac{C^2 t}{Ct+B-\ln(Ct+1)},$$

to the differential equation (5.39).

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