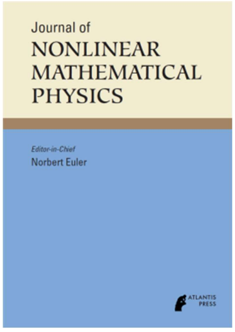

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Pseudo-Hermitian Reduction of a Generalized Heisenberg Ferromagnet Equation. II. Special Solutions

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This paper is a continuation of our previous work in which we studied a $\mathrm{sl}(3,\mathbb{C})$ Zakharov-Shabat type auxiliary linear problem with reductions of Mikhailov type and the corresponding integrable hierarchy of nonlinear evolution equations. Now, we shall demonstrate how one can construct special solutions over constant background through Zakharov-Shabat's dressing technique. That approach will be illustrated on the example of the generalized Heisenberg ferromagnet equation related to the linear problem for $\mathrm{sl}(3,\mathbb{C})$. In doing this, we shall discuss the differences between the Hermitian and pseudo-Hermitian cases.

Keywords: generalized HF equation; dressing method; soliton solutions; quasi-rational solutions.

2010 Mathematics Subject Classification: 35C05, 35C08, 35G50, 37K15, 37K35

1. Introduction

In [7], the system of completely integrable equations

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} + (uu_x^* + vv_x^*)u_x + (uu_x^* + vv_x^*)_x u = 0, \qquad \mathrm{i} = \sqrt{-1}, \\ &\mathrm{i}v_t + v_{xx} + (uu_x^* + vv_x^*)v_x + (uu_x^* + vv_x^*)_x v = 0 \end{aligned} \tag{1.1}$$

and the corresponding auxiliary spectral problem were introduced and studied. Above, the subscripts mean partial differentiation, $*$ denotes complex conjugation and $u, v : \mathbb{R}^2 \to \mathbb{C}$ are smooth functions subject to the condition $|u|^2 + |v|^2 = 1$ ($|z| = \sqrt{zz^*}$, $z \in \mathbb{C}$). System (1.1) is of particular interest since it is an integrable generalization of the classical integrable Heisenberg ferromagnet equation (HF)

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}.$$

HF describes the dynamics of a spin chain with unit spin vector $\mathbf{S} = (S^1, S^2, S^3)$, see [4, 6, 21] for more details.

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A vector system generalizing (1.1) was introduced quite independently by Golubchik and Sokolov [11]. Motivated by studies of another classical model from theoretical physics, the Landau-Lifshitz equation, the authors managed to find a Lax representation for the vector system, discussed the algebraic meaning of the Lax pair and developed its Hamiltonian formulation.

System (1.1) kept attracting some further attention and extensive studies. In [8, 9], the authors described the hierarchy of nonlinear evolution equations (NLEEs) associated with (1.1), the hierarchies of conservation laws as well as the hierarchies of Hamiltonian structures in the case when $u(x) \to \mathrm{e}^{\mathrm{i}\delta_\pm}$ and $v(x) \to 0$ sufficiently fast as $x \to \pm\infty$. Moreover, the generalized Fourier transform interpretation of the inverse scattering transform for (1.1) was established and special soliton solutions to (1.1) were constructed via dressing method. A deep study of the properties of the recursion operators for (1.1) and the geometry linked to those was carried out in [22, 24].

In [23], the authors of the current paper considered a slightly more general system of NLEEs having the form:

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} + (\varepsilon u u_x^* + v v_x^*)u_x + (\varepsilon u u_x^* + v v_x^*)_x u = 0, \qquad \varepsilon^2 = 1, \\ &\mathrm{i}v_t + v_{xx} + (\varepsilon u u_x^* + v v_x^*)v_x + (\varepsilon u u_x^* + v v_x^*)_x v = 0 \end{aligned} \tag{1.2}$$

where u and v satisfy the constraint:

$$\varepsilon |u|^2 + |v|^2 = 1. \tag{1.3}$$

Obviously, (1.2) coincides with (1.1) when $\varepsilon = 1$ and, much like it, (1.2) has a zero curvature representation $[L(\lambda), A(\lambda)] = 0$ with Lax operators given by:

$$L(\lambda) = \mathrm{i}\partial_x - \lambda S, \qquad \lambda \in \mathbb{C}, \qquad S = \begin{pmatrix} 0 & u & v \\ \varepsilon u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix}, \tag{1.4}$$

$$A(\lambda) = \mathrm{i}\partial_t + \lambda A_1 + \lambda^2 A_2, \qquad A_2 = \begin{pmatrix} -1/3 & 0 & 0 \\ 0 & 2/3 - \varepsilon |u|^2 & -\varepsilon u^* v \\ 0 & -v^* u & 2/3 - |v|^2 \end{pmatrix}, \tag{1.5}$$

$$A_1 = \begin{pmatrix} 0 & a & b \\ \varepsilon a^* & 0 & 0 \\ b^* & 0 & 0 \end{pmatrix}, \qquad \begin{aligned} a &= -\mathrm{i}u_x - \mathrm{i}\,(\varepsilon u u_x^* + v v_x^*)\, u \\ b &= -\mathrm{i}v_x - \mathrm{i}\,(\varepsilon u u_x^* + v v_x^*)\, v \end{aligned}. \tag{1.6}$$

The form of (1.4)–(1.6) is rather similar to the form of the Lax pair of the original HF [8, 21, 23]. This is why we call (1.2) generalized Heisenberg ferromagnet equations. Following the convention in [23], the case when $\varepsilon = 1$ will be referred to as Hermitian reduction while the case when $\varepsilon = -1$ — pseudo-Hermitian. As we shall discuss in more detail at the end of this section, the pseudo-Hermitian reduction allows for different classes of solutions compared to the Hermitian one. This observation initially motivated us to extensively study the pseudo-Hermitian reduction.

In [23], we described the integrable hierarchy associated with (1.2) in terms of recursion operators and derived completeness relations of their eigenfunctions. Instead of building the theory *ab initio*, our analysis was based on the gauge equivalence between the auxiliary spectral problem with scattering operator in the form (1.4) and that one for a nonlinear Schrödinger equation, i.e. a generalized Zakharov-Shabat system. This allowed us to obtain all our results for arbitrary constant asymptotic values of the potential functions appearing in the auxiliary linear problems.

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Another issue of fundamental importance concerns the solutions of (1.2). In the present paper, we intend to show how one can derive particular solutions to (1.2) in the simplest case of trivial (constant) background. In doing this, we shall not make use of the gauge equivalence between (1.4)–(1.6) and a generalized Zakharov-Shabat's system in canonical gauge. Our approach will be based on Zakharov-Shabat's dressing method that seems to be suitable for the system of equations we are interested in.

The paper itself is organized as follows. Next section contains our main results and is divided into four subsections. The first subsection is preliminary— its purpose is to give the reader some basic idea of Zakharov-Shabat's dressing method and how it could be applied to linear bundles (1.4). In doing this, we are going to use dressing factors which are meromorphic functions of the spectral parameter. Our analysis depends on the poles of the dressing factor. It turns out there exist three different cases:

- Generic case, when the poles of the dressing factor are complex numbers in general position, i.e. those are not real or imaginary numbers;
- The case, when the poles of the dressing factor are imaginary;
- Degenerate case, when the poles are real.

Each of these cases is discussed in a separate subsection. Most of the considerations in these subsections are general in the sense that they allow one to construct special solutions to any NLEE belonging to the integrable hierarchy of (1.4) not just to the generalized HF. In order to specify the NLEE within the hierarchy, one needs to substitute its dispersion law into the general equations we are going to derive.

The first two of the cases mentioned above lead to soliton type solutions while the degenerate one leads to quasi-rational ones. In the context of the classification of solutions, it becomes quite obvious why the pseudo-Hermitian reduction deserves to be studied. There are essential differences between the properties of the solutions obtained in the Hermitian and pseudo-Hermitian cases. For example, some soliton type solutions develop singularities in the pseudo-Hermitian case while their counterparts are non-singular in the Hermitian one. Moreover, it turns out that quasi-rational solutions are possible in the pseudo-Hermitian case only. This situation is similar to the case of the classical (scalar) nonlinear Schrödinger equation $\mathrm{i}q_t + q_{xx} + \kappa|q|^2 q = 0$, $\kappa = \pm 1$ for $q : \mathbb{R}^2 \to \mathbb{C}$. As it is well known, the defocusing nonlinear Schrödinger equation ($\kappa = -1$) for $\lim_{|x|\to\infty} q = 0$ does not possess soliton solutions unlike the self-focusing case ($\kappa = 1$), see [6]. On the other hand, the defocusing nonlinear Schrödinger equation admit quasi-rational solutions for the afore-mentioned asymptotic condition while the self-focusing case does not.

Last section contains some concluding remarks.

2. Special Solutions

In this section, we shall present our main results: construction of special solutions to (1.2) over constant background. More specifically, we shall assume its solutions obey the following boundary condition:

$$\lim_{x\to\pm\infty} u(x,t) = 0, \qquad \lim_{x\to\pm\infty} v(x,t) = 1 \tag{2.1}$$

that is easily seen to be compatible with (1.3). Our approach to obtain particular solutions will be based on Zakharov-Shabat's dressing method [25, 26]. Since we aim at providing a self-contained exposition, we shall remind the reader its basics as applied to linear bundles in pole gauge.

2.1. *Dressing method and linear bundles in pole gauge*

The dressing method represents an indirect way to solve completely integrable equations, i.e. one constructs new solutions to a given NLEE from a known (seed) solution. In doing this, one essentially uses the existence and the form of Lax representation.

In order to see how the method works, let us consider an arbitrary NLEE from the integrable hierarchy related to (1.4) and denote by (u_0, v_0) a known solution subject to (1.3) and (2.1). Then, as it is shown in [23], the NLEEs has a Lax representation

$$[L_0(\lambda), A_0(\lambda)] = 0 \tag{2.2}$$

where L_0 and A_0 are given by:

$$L_0(\lambda) = \mathrm{i}\partial_x - \lambda S^{(0)}, \qquad S^{(0)} = \begin{pmatrix} 0 & u_0 & v_0 \\ \varepsilon u_0^* & 0 & 0 \\ v_0^* & 0 & 0 \end{pmatrix}, \tag{2.3}$$

$$A_0(\lambda) = \mathrm{i}\partial_t + \sum_{k=1}^{N} \lambda^k A_k^{(0)}, \qquad N \geq 2, \qquad \lambda \in \mathbb{C}. \tag{2.4}$$

The coefficients $A_k^{(0)}$, $k = 1, \ldots, N$ can be uniquely expressed through (u_0, v_0) and their x-derivatives of order up to $N - k$. Here we shall not specify the form of the coefficients of the second Lax operator because only their asymptotic behavior as $x \to \pm\infty$ is essential for our further considerations. We refer the reader who is interested in this issue to our previous paper [23] where they can find more detailed explanations.

Similarly to (1.4)–(1.6), the above Lax operators fulfill the following symmetry conditions:

$$HL_0(-\lambda)H = L_0(\lambda), \quad HA_0(-\lambda)H = A_0(\lambda), \quad H = \mathrm{diag}\,(-1, 1, 1), \tag{2.5}$$

$$Q_\varepsilon \left(S^{(0)}\right)^\dagger Q_\varepsilon = S^{(0)}, \qquad Q_\varepsilon \left(A_k^{(0)}\right)^\dagger Q_\varepsilon = A_k^{(0)}, \qquad Q_\varepsilon = \mathrm{diag}\,(1, \varepsilon, 1) \tag{2.6}$$

where $\dagger$ stands for Hermitian conjugation and $\varepsilon = \pm 1$.

Let ψ_0 be an arbitrary fundamental solution to the auxiliary linear problem:

$$L_0(\lambda)\psi_0(x,t,\lambda) = 0. \tag{2.7}$$

Due to (2.2), ψ_0 also fulfills the linear problem:

$$A_0(\lambda)\psi_0(x,t,\lambda) = \psi_0(x,t,\lambda) f(\lambda) \tag{2.8}$$

where

$$f(\lambda) := \lim_{x \to \pm\infty} \sum_{k=1}^{N} \lambda^k [g_0(x,t)]^{-1} A_k^{(0)}(x,t) g_0(x,t) \tag{2.9}$$

is the dispersion law of the NLEE. Above,

$$g_0 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ \varepsilon u_0^* & \sqrt{2} v_0 & \varepsilon u_0^* \\ v_0^* & -\sqrt{2} u_0 & v_0^* \end{pmatrix}$$

is the gauge transform diagonalizing $S^{(0)}$, see [23] for more explanations. The dispersion law of a NLEE is an essential feature encoding the time dependence of its solutions and that way it labels the NLEE within the integrable hierarchy. The dispersion law of (1.2) is

$$f(\lambda) = -\lambda^2 \mathrm{diag}\,(1,-2,1)/3. \tag{2.10}$$

Linear problems (2.7) and (2.8) will be referred to as bare (seed) linear problems and their fundamental solutions will be called bare (seed) fundamental solutions. We shall denote the set of all bare fundamental solutions by $\mathscr{F}_0$.

As discussed in [23], (2.5) and (2.6) are due to the action of the reduction group $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the set of bare fundamental solutions, see [13, 14] for more explanations. Indeed, in our case the following $\mathbb{Z}_2 \times \mathbb{Z}_2$-action

$$\psi_0(x,t,\lambda) \;\longrightarrow\; H\psi_0(x,t,-\lambda)H, \tag{2.11}$$

$$\psi_0(x,t,\lambda) \;\longrightarrow\; Q_\varepsilon \left[\psi_0^\dagger(x,t,\lambda^*)\right]^{-1} Q_\varepsilon \tag{2.12}$$

leads to (2.5) and (2.6) respectively.

Now, let us apply the gauge (dressing) transform

$$\mathscr{G} : \mathscr{F}_0 \to \mathscr{F}_1 = \mathscr{G}\mathscr{F}_0 := \{\mathscr{G}\psi_0 | \psi_0 \in \mathscr{F}_0\}$$

where $\mathscr{G}(x,t,\lambda)$ is a 3×3-matrix with unit determinant. Since the Lax operators are transformed through:

$$L_0 \;\longrightarrow\; L_1 := \mathscr{G} L_0 \mathscr{G}^{-1}, \qquad A_0 \;\longrightarrow\; A_1 := \mathscr{G} A_0 \mathscr{G}^{-1}, \tag{2.13}$$

the operators L_1 and A_1 commute as well. We shall require that the auxiliary linear problems remain covariant under the dressing transform, i.e. we have

$$L_1(\lambda)\psi_1(x,t,\lambda) = 0, \qquad A_1(\lambda)\psi_1(x,t,\lambda) = \psi_1(x,t,\lambda) f(\lambda), \qquad \psi_1 \in \mathscr{F}_1 \tag{2.14}$$

for L_1 and A_1 being of the same form as L_0 and A_0. Thus we can write

$$L_1(\lambda) = \mathrm{i}\partial_x - \lambda S^{(1)}, \qquad S^{(1)} = \begin{pmatrix} 0 & u_1 & v_1 \\ \varepsilon u_1^* & 0 & 0 \\ v_1^* & 0 & 0 \end{pmatrix}, \tag{2.15}$$

$$A_1(\lambda) = \mathrm{i}\partial_t + \sum_{k=1}^{N} \lambda^k A_k^{(1)} \tag{2.16}$$

where u_1 and v_1 are some new (unknown) functions solving the same NLEE and subject to (1.3) and (2.1). The coefficients $A_k^{(1)}$, $k = 1,\dots,N$ depend on (u_1, v_1) and their x-derivatives of order up to $N-k$ in the same way as $A_k^{(0)}$ depend on (u_0, v_0) and their x-derivatives. Therefore, the dressed Lax pair has the same asymptotic behavior as the bare one.

Finding (u_1, v_1) means that we have obtained a new solution to the same NLEE and this is the main idea underlying the dressing method. Symbolically, it could be presented as the following sequence of steps:

$$(u_0, v_0) \longrightarrow (L_0, A_0) \longrightarrow \psi_0 \xrightarrow{\mathscr{G}} \psi_1 \longrightarrow (L_1, A_1) \longrightarrow (u_1, v_1).$$

Though the covariance of the linear problems is a strong condition, it does not determine the dressing factor $\mathscr{G}$ uniquely. Indeed, after comparing (2.7) and (2.8) with (2.14), we see that $\mathscr{G}$ must solve the system of linear partial differential equations:

$$\mathrm{i}\partial_x \mathscr{G} - \lambda \left(S^{(1)}\mathscr{G} - \mathscr{G}S^{(0)} \right) = 0, \tag{2.17}$$

$$\mathrm{i}\partial_t \mathscr{G} + \sum_{k=1}^{N} \lambda^k \left(A_k^{(1)}\mathscr{G} - \mathscr{G}A_k^{(0)} \right) = 0. \tag{2.18}$$

Equations (2.17) and (2.18) tell us nothing about the λ-dependence of $\mathscr{G}$. This is why we need to make a few assumptions about its behavior with respect to λ in order to obtain more specific results. Let us assume the dressing factor and its derivatives in x and t are defined in the neighbourhood of $\lambda = 0$. After setting $\lambda = 0$ in (2.17) and (2.18), we immediately see that

$$\partial_x \mathscr{G}|_{\lambda=0} = \partial_t \mathscr{G}|_{\lambda=0} = 0.$$

These relations imply that $\mathscr{G}$ should depend non-trivially on λ otherwise it will be merely a constant. We shall pick up $\mathscr{G}|_{\lambda=0} = \mathbb{1}$ since it does not lead to any loss of generality. In fact, this normalization corresponds to the normalization of the fundamental analytic solutions to the auxiliary linear problems at $\lambda = 0$, see [23].

Equation (2.17) allows one to find u_1 and v_1 in terms of the seed solution and the dressing factor. At this point, we require that $\mathscr{G}$ as well as its derivatives in x and t are regular when $|\lambda| \to \infty$. After dividing (2.17) by λ and setting $|\lambda| \to \infty$, we derive the following interrelation:

$$S^{(1)} = \mathscr{G}_\infty S^{(0)} \mathscr{G}_\infty^{-1}, \qquad \mathscr{G}_\infty(x,t) := \lim_{|\lambda|\to\infty} \mathscr{G}(x,t,\lambda). \tag{2.19}$$

Since $S^{(0)}$ is determined by (u_0, v_0) and $S^{(1)}$ is determined by (u_1, v_1), the above relation allows us to construct another solution of our system starting from a known one.

The form of the dressed operators and the zero curvature condition imply L_1 and A_1 obey (2.5) and (2.6) too. Therefore, the set of the dressed fundamental solutions is subject to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action

$$\psi_1(x,t,\lambda) \longrightarrow H\psi_1(x,t,-\lambda)H, \qquad \psi_1 \in \mathscr{F}_1, \tag{2.20}$$

$$\psi_1(x,t,\lambda) \longrightarrow Q_\varepsilon \left[\psi_1^\dagger(x,t,\lambda^*) \right]^{-1} Q_\varepsilon\,. \tag{2.21}$$

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on $\mathscr{F}_0$ and $\mathscr{F}_1$ implies that the dressing factor is not entirely arbitrary but obeys the symmetry relations:

$$H\mathscr{G}(x,t,-\lambda)H = \mathscr{G}(x,t,\lambda), \tag{2.22}$$

$$Q_\varepsilon \mathscr{G}^\dagger(x,t,\lambda^*) Q_\varepsilon = [\mathscr{G}(x,t,\lambda)]^{-1}. \tag{2.23}$$

A simple ansatz for dressing factor meeting all the requirements discussed so far is given by:

$$\mathscr{G}(x,t,\lambda) = \mathbb{1} + \lambda \sum_j \left[\frac{B_j(x,t)}{\mu_j(\lambda - \mu_j)} + \frac{HB_j(x,t)H}{\mu_j(\lambda + \mu_j)} \right], \qquad \mu_j \in \mathbb{C}\backslash\{0\}. \tag{2.24}$$

Generally speaking, the dressing procedure **does not** preserve the spectrum of scattering operator, see [6]. Indeed, let us assume for simplicity that L_0 has no discrete eigenvalues, i.e. the spectrum of the bare scattering operator coincides with the real line, see [23]. Denote the resolvent of L_0 by $R_0(\lambda)$. According to (2.13), the dressing transform maps the bare resolvent onto

$$R_1(\lambda) = \mathscr{G} R_0(\lambda) \mathscr{G}^{-1}. \tag{2.25}$$

Equation (2.25) implies that the singularities of the dressing factor and its inverse "produce" singularities of the dressed resolvent operator $R_1(\lambda)$. This way the dressing procedure adds new discrete eigenvalues to the bare scattering operator.

In order to find L_1 and A_1 through (2.19), we need to know $B_j(x,t) = \mathrm{Res}\,(\mathscr{G}(x,t,\lambda);\mu_j)$. The algorithm to find the residues of (2.24) consists in two steps. In the first step, one considers the identity $\mathscr{G}\mathscr{G}^{-1} = \mathbb{1}$ which gives rise to a set of algebraic relations for B_j. The form of those relations crucially depends on the location of μ_j— whether the poles of the dressing factor are arbitrary complex numbers with nonzero real and imaginary parts (generic case) or the poles are either imaginary or real numbers. If the poles are complex numbers in generic position or purely imaginary then we obtain soliton type solutions. For poles lying on the real line, we have degeneracy in the spectrum of the scattering operator. As a result, we obtain quasi-rational solutions. Due to all these differences, we shall consider the three cases in separate subsections.

Though the algebraic relations may be different, they always imply that the residues of $\mathscr{G}$ are some singular (degenerate) matrices which could be decomposed as follows:

$$B_j(x,t) = X_j(x,t) F_j^T(x,t) \tag{2.26}$$

where $X_j(x,t)$ and $F_j(x,t)$ are rectangular matrices of certain rank and the superscript T stands for matrix transposition. After substituting (2.26) into the algebraic relations, we are able to express X_j through F_j.

The factors F_j are determined in the second step. For this to be done, we consider (2.17) and (2.18). As for the algebraic relations discussed above, the calculations here depend on the location of the poles of $\mathscr{G}$. However, (2.17) always leads to the following result:

$$F_j^T(x,t) = F_{j,0}^T(t)\,[\psi_0(x,t,\mu_j)]^{-1}$$

allowing one to construct F_j through a bare fundamental solution defined in the neighbourhood of μ_j and some arbitrary x-independent matrices $F_{j,0}$. The matrices $F_{j,0}$ depend on t in a way governed by (2.18). Regardless of the location of μ_j, we derive exponential t-dependence. Thus in order to recover the time dependence in all formulas, we may use the rule below:

$$F_{j,0}^T(t) \quad \longrightarrow \quad F_{j,0}^T(t)\mathrm{e}^{-\mathrm{i}f(\mu_j)t}$$

where $f(\lambda)$ is the dispersion law of the NLEE, see (2.9). We shall demonstrate in more detail that procedure in the subsections to follow.

2.2. *Soliton type solutions I. Generic case*

Let us start with the case when the poles of (2.24) are complex numbers in generic position, i.e. $\mu_j^2 \notin \mathbb{R}$ for all j. From the symmetry condition (2.23), we immediately deduce that

$$[\mathcal{G}(x,t,\lambda)]^{-1} = \mathbb{1} + \lambda \sum_i \left[\frac{Q_\varepsilon B_i^\dagger(x,t) Q_\varepsilon}{\mu_i^*(\lambda - \mu_i^*)} + \frac{Q_\varepsilon H B_i^\dagger(x,t) H Q_\varepsilon}{\mu_i^*(\lambda + \mu_i^*)} \right]. \tag{2.27}$$

Thus the dressing factor and its inverse have poles located at different points.

Let us consider the identity $\mathcal{G}(x,t,\lambda)\,[\mathcal{G}(x,t,\lambda)]^{-1} = \mathbb{1}$. Since it holds identically in λ, the residues of $\mathcal{G}\mathcal{G}^{-1}$ should vanish. After evaluating the residue of $\mathcal{G}\mathcal{G}^{-1}$ at μ_i^* we easily obtain the following algebraic relation:

$$\left[\mathbb{1} + \mu_i^* \sum_j \left(\frac{B_j}{\mu_j(\mu_i^* - \mu_j)} + \frac{H B_j H}{\mu_j(\mu_i^* + \mu_j)} \right) \right] Q_\varepsilon B_i^\dagger Q_\varepsilon = 0. \tag{2.28}$$

Evaluation of the residues at $\pm\mu_i$ and $-\mu_i^*$ leads to equations that can easily be reduced to (2.28), thus giving us no new constraints.

It is seen from (2.28) that each B_i should be a degenerate matrix, hence it can be factored

$$B_i(x,t) = X_i(x,t) F_i^T(x,t) \tag{2.29}$$

where X_i and F_i are two rectangular matrices. After substituting (2.29) into (2.28), one obtains the linear system

$$Q_\varepsilon F_i^* = \mu_i^* \sum_j \left(X_j \frac{F_j^T Q_\varepsilon F_i^*}{\mu_j(\mu_j - \mu_i^*)} - H X_j \frac{F_j^T H Q_\varepsilon F_i^*}{\mu_j(\mu_i^* + \mu_j)} \right) \tag{2.30}$$

for the factors X_j. Solving it, allows one to express X_j through F_j. This is easier when the dressing factor has just a single pair of poles: μ and $-\mu$. Assuming X and F are column-vectors, we get:

$$X = \left(\frac{\mu^* F^T Q_\varepsilon F^*}{\mu(\mu - \mu^*)} - \frac{\mu^* F^T H Q_\varepsilon F^*}{\mu(\mu + \mu^*)} H \right)^{-1} Q_\varepsilon F^*. \tag{2.31}$$

Let us return now to the general case. The factors F_j can be found from (2.17). For that purpose we rewrite (2.17) in the following way

$$\lambda S^{(1)} = \mathrm{i}\partial_x \mathcal{G}\mathcal{G}^{-1} + \lambda \mathcal{G} S^{(0)} \mathcal{G}^{-1}$$

and calculate the residues of its right-hand side at $\lambda = \mu_i$. After taking into account (2.30), we obtain the linear differential equation

$$\mathrm{i}\partial_x F_i^T + \mu_i F_i^T S^{(0)} = 0 \tag{2.32}$$

which is immediately solved to give

$$F_i^T(x,t) = F_{i,0}^T(t)\,[\psi_0(x,t,\mu_i)]^{-1}. \tag{2.33}$$

Above, $F_{i,0}$ are "integration constant" matrices which depend on t however. Evaluation of the residues at $-\mu_i$ or $\pm\mu_i^*$ gives equations that are easily reduced to (2.32) after taking into account the symmetries of $S_1^{(0)}$.

The t-dependence of $F_{i,0}$ is determined from (2.18) after it is rewritten as follows:

$$\sum_{k=1}^{N} \lambda^k A_k^{(1)} = -\mathrm{i}\partial_t \mathscr{G}\mathscr{G}^{-1} + \mathscr{G} \sum_{k=1}^{N} \lambda^k A_k^{(0)} \mathscr{G}^{-1}. \tag{2.34}$$

Evaluation of the residues of both hand sides of (2.34) for $\lambda = \mu_i$ gives

$$\mathrm{i}\partial_t F_i^T - F_i^T \sum_{k=1}^{N} \mu_i^k A_k^{(0)} = 0. \tag{2.35}$$

Yet again, we do not need to consider the residues for $\lambda = \pm\mu_i^*$ and $\lambda = -\mu_i$ due to the symmetries of the coefficients of $A(\lambda)$.

After substituting (2.33) into (2.35), we derive a linear differential equation for $F_{i,0}$, namely

$$\mathrm{i}\partial_t F_{i,0}^T - F_{i,0}^T f(\mu_i) = 0 \tag{2.36}$$

where $f(\lambda)$ is the dispersion law of the NLEE. Equation (2.36) is easily solved and allows us to state the rule: in order to recover the t-dependence in all formulas, one has to make the following substitution

$$F_{i,0}^T \quad \longrightarrow \quad F_{i,0}^T \mathrm{e}^{-\mathrm{i}f(\mu_i)t}. \tag{2.37}$$

Now let us apply the general results obtained above to some seed solution. An obvious choice for seed solution satisfying (1.3) and (2.1) is

$$S^{(0)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.38}$$

One can prove that the bare scattering operator in this case has no discrete eigenvalues.

For a bare fundamental solution we shall choose

$$\psi_0(x,\lambda) = \begin{pmatrix} \cos(\lambda x) & 0 & -\mathrm{i}\sin(\lambda x) \\ 0 & 1 & 0 \\ -\mathrm{i}\sin(\lambda x) & 0 & \cos(\lambda x) \end{pmatrix}. \tag{2.39}$$

This fundamental solution is invariant with respect to (2.11) and (2.12) which makes it rather convenient for the calculations to follow.

From this point on, we shall focus on the simplest case when the dressing factor has a single pair of poles and X and F are column-vectors. After substituting (2.29) and (2.31) into (2.24) and (2.27), equation (2.19) can be written in components as follows:

$$u_1 = -\frac{\left[\mu|F^1|^2 + \mu^*(\varepsilon|F^2|^2 + |F^3|^2)\right]\left(\mu^2 - (\mu^*)^2\right) F^2 \left(F^3\right)^*}{\mu^*\left[\mu^*|F^1|^2 + \mu(\varepsilon|F^2|^2 + |F^3|^2)\right]^2}, \tag{2.40}$$

$$v_1 = \frac{\left[\mu|F^1|^2 + \mu^*(\varepsilon|F^2|^2 + |F^3|^2)\right]}{\mu^*\left[\mu^*|F^1|^2 + \mu(\varepsilon|F^2|^2 + |F^3|^2)\right]^2} \times \left\{\mu^*\left[\mu|F^1|^2 + \mu^*(\varepsilon|F^2|^2 + |F^3|^2)\right] + \left(\mu^2 - (\mu^*)^2\right)\varepsilon|F^2|^2\right\} \tag{2.41}$$

where $F^T = (F^1, F^2, F^3)$. It is not hard to check that (2.40) and (2.41) satisfy (1.3) for any F.

On the other hand, (2.33) leads to the following expression for F:

$$F(x)=\begin{pmatrix} F_0^1\cos(\mu x)+\mathrm{i}F_0^3\sin(\mu x)\\ F_0^2\\ F_0^3\cos(\mu x)+\mathrm{i}F_0^1\sin(\mu x)\end{pmatrix},\qquad F_0=\begin{pmatrix}F_0^1\\ F_0^2\\ F_0^3\end{pmatrix}. \tag{2.42}$$

In order to evaluate X, we shall need the following quadratic expressions as well:

$$F^TQ_\varepsilon F^* = \left(|F_0^1|^2+|F_0^3|^2\right)\cosh(2\kappa x)-2|F_0^1F_0^3|\cos\varphi\,\sinh(2\kappa x)+\varepsilon|F_0^2|^2, \tag{2.43}$$

$$F^TQ_\varepsilon HF^* = \left[\left(|F_0^3|^2-|F_0^1|^2\right)\cos(2\omega x)-2|F_0^1F_0^3|\sin\varphi\,\sin(2\omega x)+\varepsilon|F_0^2|^2\right] \tag{2.44}$$

where $\omega=\operatorname{Re}\mu>0$, $\kappa=\operatorname{Im}\mu>0$ and $\varphi=\arg F_0^1-\arg F_0^3$. Taking into account the structure of (2.39)–(2.42), it is natural to consider in detail the following three elementary cases, see [9].

(1) First, let us assume $F_0^2=0$ and $F_0^1\neq\pm F_0^3$. Then for F we have

$$F(x)=\begin{pmatrix} F_0^1\cos(\mu x)+\mathrm{i}F_0^3\sin(\mu x)\\ 0\\ F_0^3\cos(\mu x)+\mathrm{i}F_0^1\sin(\mu x)\end{pmatrix} \tag{2.45}$$

and (2.43), (2.44) now could be simplified to

$$F^TQ_\varepsilon F^*=A\cosh(2\kappa x+\xi_0), \tag{2.46}$$

$$F^TQ_\varepsilon HF^*=-A\cos(2\omega x+\delta_0) \tag{2.47}$$

where

$$\cosh\xi_0=\frac{|F_0^1|^2+|F_0^3|^2}{A},\qquad \sinh\xi_0=-\frac{2|F_0^1F_0^3|\cos\varphi}{A}, \tag{2.48}$$

$$A=\sqrt{\left(|F_0^1|^2+|F_0^3|^2\right)^2-4|F_0^1F_0^3|^2\cos^2\varphi}, \tag{2.49}$$

$$\cos\delta_0=\frac{|F_0^1|^2-|F_0^3|^2}{A},\qquad \sin\delta_0=-\frac{2|F_0^1F_0^3|\sin\varphi}{A}. \tag{2.50}$$

After substituting (2.45)–(2.50) into (2.40) and (2.41), we obtain the dressed solution to (1.2) at a fixed moment of time:

$$u_1(x)=0, \tag{2.51}$$

$$v_1(x)=\exp\left[4\mathrm{i}\arctan\frac{\kappa\cos(2\omega x+\delta_0)}{\omega\cosh(2\kappa x+\xi_0)}\right]. \tag{2.52}$$

In order to recover the time evolution in (2.51) and (2.52), one needs to make the substitution:

$$\varphi\to\varphi,\qquad \xi_0\to\xi_0,\qquad \delta_0\to\delta_0,\qquad A\to A\exp\left(\frac{-4\omega\kappa t}{3}\right) \tag{2.53}$$

that follows directly from (2.10) and (2.37). It is seen that (2.51) and (2.52) remain invariant under transformation (2.53), so the dressed solution is stationary and non-singular. The solution just derived coincides with that found in [9] (see formula (3.26)) for the generalized HF with Hermitian reduction.

(2) Let us assume $F_0^2 \neq 0$. Without any loss of generality we could just set $F_0^2 = 1$. There exist two elementary options: $F_0^1 = F_0^3$ and $F_0^1 = -F_0^3$. Let us consider the former one. After recovering the time evolution, the vector $F(x)$ is given by:

$$F(x,t) = \begin{pmatrix} F_0^1 \mathrm{e}^{\mathrm{i}\mu x + \frac{\mathrm{i}\mu^2 t}{3}} \\ \mathrm{e}^{-\frac{2\mathrm{i}\mu^2 t}{3}} \\ F_0^1 \mathrm{e}^{\mathrm{i}\mu x + \frac{\mathrm{i}\mu^2 t}{3}} \end{pmatrix}.$$

The solution in this case reads:

$$u_1(x,t) = \frac{4\omega\kappa\left[2\omega \mathrm{e}^{\vartheta(x,t)} + \varepsilon(\omega - \mathrm{i}\kappa)\mathrm{e}^{-\vartheta(x,t)}\right]\mathrm{e}^{-\mathrm{i}[\omega x + (\omega^2-\kappa^2)t + \delta]}}{(\omega - \mathrm{i}\kappa)\left[2\omega \mathrm{e}^{\vartheta(x,t)} + \varepsilon(\omega + \mathrm{i}\kappa)\mathrm{e}^{-\vartheta(x,t)}\right]^2}, \tag{2.54}$$

$$v_1(x,t) = 1 - \frac{8\varepsilon\omega\kappa^2}{(\omega - \mathrm{i}\kappa)\left[2\omega \mathrm{e}^{\vartheta(x,t)} + \varepsilon(\omega + \mathrm{i}\kappa)\mathrm{e}^{-\vartheta(x,t)}\right]^2} \tag{2.55}$$

where

$$\vartheta(x,t) = -\kappa(x + 2\omega t) + \ln|F_{0,1}|, \qquad \delta = \left(\frac{\pi}{2} + \arg F_{0,1}\right).$$

Solution (2.54), (2.55) has no singularities and goes into the one obtained in [9] (see equation (3.27)) when $\varepsilon = 1$.

(3) Assume now $F_0^2 = 1$ and $F_0^1 = -F_0^3$. In this case $F(x,t)$ is given by:

$$F(x,t) = \begin{pmatrix} F_0^1 \mathrm{e}^{-\mathrm{i}\mu x + \frac{\mathrm{i}\mu^2 t}{3}} \\ \mathrm{e}^{-\frac{2\mathrm{i}\mu^2 t}{3}} \\ -F_0^1 \mathrm{e}^{-\mathrm{i}\mu x + \frac{\mathrm{i}\mu^2 t}{3}} \end{pmatrix}.$$

The corresponding dressed solution looks as follows:

$$u_1(x,t) = \frac{4\omega\kappa\left[2\omega \mathrm{e}^{\tilde{\vartheta}(x,t)} + \varepsilon(\omega - \mathrm{i}\kappa)\mathrm{e}^{-\tilde{\vartheta}(x,t)}\right]\mathrm{e}^{\mathrm{i}[\omega x + (\kappa^2-\omega^2)t + \tilde{\delta}]}}{(\omega - \mathrm{i}\kappa)\left[2\omega \mathrm{e}^{\tilde{\vartheta}(x,t)} + \varepsilon(\omega + \mathrm{i}\kappa)\mathrm{e}^{-\tilde{\vartheta}(x,t)}\right]^2}, \tag{2.56}$$

$$v_1(x,t) = 1 - \frac{8\varepsilon\omega\kappa^2}{(\omega - \mathrm{i}\kappa)\left[2\omega \mathrm{e}^{\tilde{\vartheta}(x,t)} + \varepsilon(\omega + \mathrm{i}\kappa)\mathrm{e}^{-\tilde{\vartheta}(x,t)}\right]^2} \tag{2.57}$$

where

$$\tilde{\vartheta}(x,t) = \kappa(x - 2\omega t) + \ln|F_{0,1}|, \qquad \tilde{\delta} = \frac{\pi}{2} - \arg F_{0,1}.$$

Formally, this solution could be derived from (2.54), (2.55) by applying the transform $x \to -x$ and $F_{0,1} \to F_{0,3}$.

All the solutions constructed explicitly in this subsection are connected with four distinct poles (quadruples) of $\mathcal{G}$ and $\mathcal{G}^{-1}$, i.e. four discrete eigenvalues of the dressed scattering operator. This is why we call such solutions "quadruplet" solutions.

2.3. *Soliton type solutions II. Imaginary poles*

Let us assume now that all the poles of $\mathscr{G}$ are purely imaginary, i.e. we have $\mu_j = \mathrm{i}\kappa_j$ for some real numbers $\kappa_j \neq 0$. Thus we could write

$$\mathscr{G}(x,t,\lambda) = \mathbb{1} + \lambda \sum_i \left[\frac{B_i(x,t)}{\mathrm{i}\kappa_i(\lambda - \mathrm{i}\kappa_i)} + \frac{HB_i(x,t)H}{\mathrm{i}\kappa_i(\lambda + \mathrm{i}\kappa_i)} \right] \tag{2.58}$$

for the dressing factor and

$$[\mathscr{G}(x,t,\lambda)]^{-1} = \mathbb{1} - \lambda \sum_i \left[\frac{Q_\varepsilon B_i^\dagger(x,t) Q_\varepsilon}{\mathrm{i}\kappa_i(\lambda + \mathrm{i}\kappa_i)} + \frac{Q_\varepsilon H B_i^\dagger(x,t) H Q_\varepsilon}{\mathrm{i}\kappa_i(\lambda - \mathrm{i}\kappa_i)} \right] \tag{2.59}$$

for its inverse. Since $\mathscr{G}$ and $\mathscr{G}^{-1}$ have the same poles, the identity $\mathscr{G}\mathscr{G}^{-1} = \mathbb{1}$ gives rise to algebraic relations that do not follow from (2.28). It is easily checked that the following equations hold true:

$$\lim_{\lambda \to \mathrm{i}\kappa_i} (\lambda - \mathrm{i}\kappa_i)^2 \left(\mathscr{G}\mathscr{G}^{-1} = \mathbb{1} \right) \quad \Rightarrow \quad B_i Q_\varepsilon H B_i^\dagger = 0\,, \tag{2.60}$$

$$\lim_{\lambda \to \mathrm{i}\kappa_i} \partial_\lambda (\lambda - \mathrm{i}\kappa_i)^2 \left(\mathscr{G}\mathscr{G}^{-1} = \mathbb{1} \right) \quad \Rightarrow \quad B_i Q_\varepsilon H \Omega_i^\dagger - \Omega_i Q_\varepsilon H B_i^\dagger = 0 \tag{2.61}$$

where

$$\Omega_i = \mathbb{1} + \frac{B_i}{\mathrm{i}\kappa_i} + \sum_{j \neq i} \frac{\kappa_i B_j}{\mathrm{i}\kappa_j(\kappa_i - \kappa_j)} + \sum_j \frac{\kappa_i H B_j H}{\mathrm{i}\kappa_j(\kappa_i + \kappa_j)}\,.$$

There is no need to perform similar calculations for $\lambda = -\mathrm{i}\kappa_i$ since all algebraic equations can be reduced to (2.60) and (2.61).

Relation (2.60) implies that $B_i(x,t)$ is degenerate, so (2.29) holds again for some rectangular matrices X_i and F_i (we shall assume that these matrices are simply column vectors). Due to (2.60) the factors F_i satisfy the quadratic relation

$$F_i^T Q_\varepsilon H F_i^* = 0. \tag{2.62}$$

After substituting (2.29) into (2.61), we see that there exist functions α_i such that:

$$\Omega_i Q_\varepsilon H F_i^* = -X_i \alpha_i, \qquad \alpha_i^* = \alpha_i\,. \tag{2.63}$$

These relations allow one to find all vectors X_i in terms of F_i and α_i. In the simplest case when the dressing factor has a single pair of poles $\pm\mathrm{i}\kappa$ (2.63) is easily solvable to give:

$$X = -\left(\alpha + \frac{F^T Q_\varepsilon F^*}{2\mathrm{i}\kappa} H \right)^{-1} Q_\varepsilon H F^*\,. \tag{2.64}$$

After taking into account (2.29), (2.58), (2.59) and (2.64), equation (2.19) could be written down in components as follows:

$$u_1(x,t) = \frac{2(|F^1(x,t)|^2 + \mathrm{i}\kappa\alpha(x,t))F^2(x,t)\left(F^3(x,t)\right)^*}{(|F^1(x,t)|^2 - \mathrm{i}\kappa\alpha(x,t))^2}\,, \tag{2.65}$$

$$v_1(x,t) = 1 + \frac{2\left(2\mathrm{i}\kappa\alpha(x,t) - \varepsilon|F^2(x,t)|^2\right)}{|F^1(x,t)|^2 - \mathrm{i}\kappa\alpha(x,t)} + \frac{4\mathrm{i}\kappa\alpha(x,t)\left(\mathrm{i}\kappa\alpha(x,t) - \varepsilon|F^2(x,t)|^2\right)}{(|F^1(x,t)|^2 - \mathrm{i}\kappa\alpha(x,t))^2}. \tag{2.66}$$

It is easy to check that (2.65) and (2.66) are interrelated through $|v|^2 + \varepsilon|u|^2 = 1$ for any real-valued α and F obeying (2.62).

Like in the generic case (see the previous subsection) we can find F_i and α_i by analyzing (2.17) and (2.18). Considerations similar to those in the generic case lead to the following linear differential equations:

$$\lim_{\lambda\to \mathrm{i}\kappa_i}(\lambda-\mathrm{i}\kappa_i)^2\left(\mathrm{i}\partial_x\mathscr{G}\mathscr{G}^{-1}+\lambda\mathscr{G}S^{(0)}\mathscr{G}^{-1}=\lambda S^{(1)}\right)\quad\Rightarrow\quad \mathrm{i}\partial_x F_i^T+\mu_i F_i^T S^{(0)}=0,\quad (2.67)$$

$$\lim_{\lambda\to \mathrm{i}\kappa_i}\partial_\lambda(\lambda-\mathrm{i}\kappa_i)^2\left(\mathrm{i}\partial_x\mathscr{G}\mathscr{G}^{-1}+\lambda\mathscr{G}S^{(0)}\mathscr{G}^{-1}=\lambda S^{(1)}\right)\quad\Rightarrow\quad \mathrm{i}\partial_x\alpha_i=F_i^T S^{(0)}Q_\varepsilon HF_i^*\,. (2.68)$$

Equation (2.67) shows that (2.33) holds true for the vectors F_i while (2.68) implies that the scalar functions α_i can be expressed in terms of the seed fundamental solution as given by:

$$\alpha_i(x)=\alpha_{i,0}+F_{i,0}^T\left[\psi_0(x,\mathrm{i}\kappa_i)\right]^{-1}\frac{\partial\psi_0(x,\mathrm{i}\kappa_i)}{\partial\lambda}K_{\psi_0}Q_\varepsilon HF_{i,0}^*. \qquad (2.69)$$

Above, $\alpha_{i,0}$ is an integration constant and $K_\psi(\lambda)=[\psi(\lambda)]^{-1}HQ_\varepsilon\left[\psi^\dagger(-\lambda^*)\right]^{-1}Q_\varepsilon H$ measures the "deviation" of the solution ψ from invariant solutions, i.e. if ψ is invariant with respect to (2.11) and (2.12), then $K_\psi=\mathbb{1}$.

Finally, we need to recover the time dependence in all formulas. For this to be done we consider (2.18) as we did in the previous subsection. As a result, we derive a set of equations for F_i and α_i, namely:

$$\lim_{\lambda\to \mathrm{i}\kappa_i}(\lambda-\mathrm{i}\kappa_i)^2\left[-\mathrm{i}\partial_t\mathscr{G}\mathscr{G}^{-1}+\mathscr{G}\sum_k\lambda^k A_k^{(0)}\mathscr{G}^{-1}\right]=0\qquad\Rightarrow$$

$$\mathrm{i}\partial_t F_i^T-F_j^T\sum_k(\mathrm{i}\kappa_i)^k A_k^{(0)}=0, \qquad (2.70)$$

$$\lim_{\lambda\to \mathrm{i}\kappa_i}\partial_\lambda\left\{(\lambda-\mathrm{i}\kappa_i)^2\left[-\mathrm{i}\partial_t\mathscr{G}\mathscr{G}^{-1}+\mathscr{G}\sum_k\lambda^k A_k^{(0)}\mathscr{G}^{-1}\right]\right\}=0\qquad\Rightarrow$$

$$\mathrm{i}\partial_t\alpha_i+F_i^T\sum_k k(\mathrm{i}\kappa_i)^{k-1}A_k^{(0)}Q_\varepsilon HF_i^*=0. \qquad (2.71)$$

Equation (2.70) coincides with (2.35). Thus after substituting (2.33) into (2.70), we derive (2.36) and (2.37) holds again. On the other hand, (2.71) is reduced to

$$\mathrm{i}\partial_t\alpha_{i,0}+F_{i,0}^T\frac{\mathrm{d}f(\mathrm{i}\kappa_i)}{\mathrm{d}\lambda}K_{\psi_0}HQ_\varepsilon F_{i,0}^*=0. \qquad (2.72)$$

If the bare fundamental solution is invariant with respect to both reductions, i.e. $K_{\psi_0}=\mathbb{1}$, then (2.72) is simplified to

$$\mathrm{i}\partial_t\alpha_{i,0}+F_{i,0}^T\frac{\mathrm{d}f(\mathrm{i}\kappa_i)}{\mathrm{d}\lambda}HQ_\varepsilon F_{i,0}^*=0. \qquad (2.73)$$

Let us consider the case when the seed solution $S^{(0)}$ is picked up as in (2.38) and the corresponding fundamental solution is given by (2.39). As we discussed in the previous subsection, for (2.39) we have $K_{\psi_0}=\mathbb{1}$. From this point on, we shall assume that $\mathscr{G}$ has a single pair of imaginary poles.

In this case the 3-vector F reads:

$$F(x) = \begin{pmatrix} \cosh(\kappa x)F_0^1 - \sinh(\kappa x)F_0^3 \\ F_0^2 \\ \cosh(\kappa x)F_0^3 - \sinh(\kappa x)F_0^1 \end{pmatrix}, \qquad F_0 = \begin{pmatrix} F_0^1 \\ F_0^2 \\ F_0^3 \end{pmatrix}. \tag{2.74}$$

Our further considerations depend on whether or not F_0^2 is equal to 0.

(1) Let us first assume $F_0^2 = 0$. Then we may set $|F_0^1| = |F_0^3| = 1$ and (2.74) and (2.69) now give

$$|F^1(x)|^2 = \cosh(2\kappa x) - \sinh(2\kappa x)\cos(2\varphi), \quad 2\varphi = \arg F_0^1 - \arg F_0^3 \tag{2.75}$$

$$\alpha(x) = \alpha_0 + 2x\sin(2\varphi) \tag{2.76}$$

where α_0 is t-independent in this case. After substituting (2.75) and (2.76) into (2.65) and (2.66), then taking into account (2.74), we get the following stationary solution to (1.2):

$$u_1(x) = 0, \tag{2.77}$$

$$v_1(x) = \left[\frac{\cosh(2\kappa x) - \sinh(2\kappa x)\cos(2\varphi) + \mathrm{i}\kappa(\alpha_0 + 2x\sin(2\varphi))}{\cosh(2\kappa x) - \sinh(2\kappa x)\cos(2\varphi) - \mathrm{i}\kappa(\alpha_0 + 2x\sin(2\varphi))}\right]^2. \tag{2.78}$$

The dressed solution just derived does not "feel" the presence of ε, i.e. whether the reduction is Hermitian or pseudo-Hermitian, hence it formally coincides with the one obtained in [9]. The right-hand side of (2.78) could be rewritten as:

$$v_1(x) = \exp\left[4\mathrm{i}\arctan\left(\frac{\kappa(\alpha_0 + 2x\sin(2\varphi))}{\cosh(2\kappa x) - \sinh(2\kappa x)\cos(2\varphi)}\right)\right].$$

When we have $F_{0,1} = \pm F_{0,3}$, it simplifies to

$$v_1(x) = \left[\frac{\mathrm{e}^{\mp 2\kappa x} + \mathrm{i}\kappa\alpha_0}{\mathrm{e}^{\mp 2\kappa x} - \mathrm{i}\kappa\alpha_0}\right]^2.$$

It is seen that v_1 is nontrivial only if $\alpha_0 \neq 0$.

(2) Assume now that $F_0^2 \neq 0$. Thus we may simply set $F_0^2 = 1$. Our further analysis depends on the value of ε. Let us pick up $\varepsilon = 1$ first. Due to (2.62) and the invariance of (2.39) under the reductions, we have

$$F_0^T Q_\varepsilon H F_0^* = 0 \qquad \Rightarrow \qquad |F_0^1|^2 - |F_0^3|^2 = 1. \tag{2.79}$$

A natural parametrization for F_0^1 and F_0^3 is the following one:

$$F_0^1 = \cosh(\theta)\,\mathrm{e}^{\mathrm{i}(\delta+\varphi)}, \qquad F_0^3 = \sinh(\theta)\,\mathrm{e}^{\mathrm{i}(\delta-\varphi)}, \qquad \theta > 0. \tag{2.80}$$

Then using (2.80), we have

$$|F^1(x,t)|^2 = \cosh^2(\kappa x - \theta)\cos^2\varphi + \cosh^2(\kappa x + \theta)\sin^2\varphi,$$

$$\alpha(x,t) = x\sinh(2\theta)\sin(2\varphi) - 2\kappa t.$$

In the end we derive the following dressed solution:

$$u_1(x,t) = \frac{2\Delta_p^*}{\Delta_p^2} e^{i(\kappa^2 t-\delta)} \left[-\sinh(\kappa x-\theta)\cos\varphi + i\sinh(\kappa x+\theta)\sin\varphi\right],$$

$$v_1(x,t) = 1 + \frac{2(2i\kappa\alpha - 1)}{\Delta_p} + \frac{4i\kappa\alpha(i\kappa\alpha-1)}{\Delta_p^2},$$

$$\Delta_p = \cosh^2(\kappa x-\theta)\cos^2\varphi + \cosh^2(\kappa x+\theta)\sin^2\varphi - i\kappa(x\sinh 2\theta\sin 2\varphi - 2\kappa t).$$

The result we obtained coincides with the one found in [9].

Let us consider now the case when $\varepsilon = -1$. Relation (2.79) is rewritten as

$$|F_0^3|^2 - |F_0^1|^2 = 1$$

hinting at the following parametrization:

$$F_0^1 = \sinh(\theta)\, e^{i(\delta+\varphi)}, \qquad F_0^3 = \cosh(\theta)\, e^{i(\delta-\varphi)}, \quad \theta > 0.$$

Then we have

$$|F^1(x,t)|^2 = \sinh^2(\kappa x-\theta)\cos^2\varphi + \sinh^2(\kappa x+\theta)\sin^2\varphi,$$
$$\alpha(x,t) = x\sinh(2\theta)\sin(2\varphi) + 2\kappa t$$

and substituting into (2.65) and (2.66) we obtain:

$$u_1(x,t) = \frac{2\Delta_m^*}{\Delta_m^2} e^{i(\kappa^2 t-\delta)} \left[\cosh(\kappa x-\theta)\cos\varphi + i\cosh(\kappa x+\theta)\sin\varphi\right], \tag{2.81}$$

$$v_1(x,t) = 1 + \frac{2(2i\kappa\alpha + 1)}{\Delta_m} + \frac{4i\kappa\alpha(i\kappa\alpha+1)}{\Delta_m^2}, \tag{2.82}$$

$$\Delta_m = \sinh^2(\kappa x-\theta)\cos^2\varphi + \sinh^2(\kappa x+\theta)\sin^2\varphi - i\kappa[x\sinh(2\theta)\sin(2\varphi) + 2\kappa t].$$

It is not hard to see that Δ_m could vanish for particular values of the parameters and thus we obtain singular solutions.

Unlike the quadruplet solutions in the previous subsection, the soliton-like solutions we derived here are related to a pair of discrete eigenvalues (doublet) of the scattering operator. This is the reason we call such solutions "doublet" solutions.

2.4. *Degenerate case*

Hereafter, we shall consider the case when the poles of the dressing factor are all real. Like in doublet case (see the previous subsection), the dressing factor

$$\mathscr{G}(x,t,\lambda) = \mathbb{1} + \lambda \sum_{i=1} \left[\frac{B_i(x,t)}{\mu_i(\lambda-\mu_i)} + \frac{HB_i(x,t)H}{\mu_i(\lambda+\mu_i)}\right], \qquad \mu_i \in \mathbb{R}\backslash\{0\}$$

and its inverse

$$[\mathscr{G}(x,t,\lambda)]^{-1} = \mathbb{1} + \lambda \sum_{i=1} \left[\frac{Q_\varepsilon B_i^\dagger(x,t) Q_\varepsilon}{\mu_i(\lambda-\mu_i)} + \frac{Q_\varepsilon H B_i^\dagger(x,t) H Q_\varepsilon}{\mu_i(\lambda+\mu_i)}\right]$$

have the same poles. Then the identity $\mathscr{G}(\lambda)[\mathscr{G}(\lambda)]^{-1} = \mathbb{1}$ gives rise to the following algebraic relations:

$$B_i Q_\varepsilon B_i^\dagger = 0\,, \tag{2.83}$$

$$\tilde{\Omega}_i Q_\varepsilon B_i^\dagger + B_i Q_\varepsilon \tilde{\Omega}_i^\dagger = 0 \tag{2.84}$$

where

$$\tilde{\Omega}_i = \mathbb{1} + \frac{B_i}{\mu_i} + \sum_{j\neq i} \frac{\mu_i B_j}{\mu_j(\mu_i - \mu_j)} + \sum_j \frac{\mu_i H B_j H}{\mu_j(\mu_i + \mu_j)}\,. \tag{2.85}$$

Like in the previous case, relation (2.83) could be reduced to

$$F_i^T Q_\varepsilon F_i^* = 0 \tag{2.86}$$

for the vector F_i involved in (2.29). It is clear that for $Q_\varepsilon = \mathbb{1}$ (2.86) leads to $F_i = 0$ and $\mathscr{G}$ becomes equal to $\mathbb{1}$. Therefore a nontrivial result is possible only for pseudo-Hermitian reduction, i.e. $Q_\varepsilon = \operatorname{diag}(1,-1,1)$.

From this point on, we shall be interested in the case when X_i and F_i are just 3-vectors. Relation (2.84) implies there exists a scalar function β_i such that

$$\tilde{\Omega}_i Q_\varepsilon F_i^* = X_i \beta_i, \qquad \beta_i^* = -\beta_i\,. \tag{2.87}$$

Relation (2.87) is a linear system of equations for X_i. This system could easily be solved when $\mathscr{G}$ has a single pair of real poles. In this case for X we have:

$$X = \left(\beta - \frac{F^T H Q_\varepsilon F^*}{2\mu} H\right)^{-1} Q_\varepsilon F^*\,. \tag{2.88}$$

In order to obtain F_i and β_i, one considers the differential equations satisfied by the dressing factor, see (2.17) and (2.18). Analysis rather similar to that in the previous subsections shows that F_i depend on bare fundamental solution according to (2.33) while β_i is determined by:

$$\beta_i = \beta_{i,0} F_{i,0}^T [\psi_0(x,\mu_i)]^{-1} \partial_\lambda \psi_0(x,\mu_i) \tilde{K}_{\psi_0}(\mu_i) Q_\varepsilon F_{i,0}^*$$

where $\tilde{K}_\psi(\lambda) = [\psi(\lambda)]^{-1} Q_\varepsilon \left[\psi^\dagger(\lambda^*)\right]^{-1} Q_\varepsilon$ measures the "deviation from invariance" of the solution ψ and $\beta_{i,0}$ is an imaginary x-independent scalar function. Like in the previous cases, (2.18) leads to (2.37) recovering the time dependence of F_i. The time dependence of β_i is obtained through the following substitution rule:

$$\beta_{i,0} \to \beta_{i,0} - \mathrm{i} F_{i,0}^T \left.\frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=\mu_i} \tilde{K}_{\psi_0}(\mu_i) Q_\varepsilon F_{i,0}^* t \tag{2.89}$$

where $f(\lambda)$ is the dispersion law.

We shall focus now on the simplest case when the dressing factor has one pair of real poles. Assume the seed solution $S^{(0)}$ and the corresponding fundamental solution ψ_0 are given by (2.38) and (2.39) respectively. Due to (2.86), for the components of the 3-vector $F_0 = (F_0^1, F_0^2, F_0^3)$ we have

$$|F_0^1|^2 + |F_0^3|^2 = |F_0^2|^2 . \tag{2.90}$$

Without any loss of generality we could set $F_0^2 = 1$. Then the form of (2.90) suggests the parametrization:

$$F_0^1(t) = \cos(\theta)\, \mathrm{e}^{\mathrm{i}\left(\frac{\mu^2 t}{3}+\delta+\varphi\right)}, \qquad F_0^3(t) = \sin(\theta)\, \mathrm{e}^{\mathrm{i}\left(\frac{\mu^2 t}{3}+\delta-\varphi\right)}, \qquad \theta \in [0, \pi/2].$$

Then F could be written down as:

$$F(x,t) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\left(\frac{\mu^2 t}{3}+\delta\right)} \left[\mathrm{e}^{\mathrm{i}\varphi} \cos(\mu x)\cos\theta + \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi} \sin(\mu x)\sin\theta\right] \\ \mathrm{e}^{-\frac{2\mathrm{i}\mu^2 t}{3}} \\ \mathrm{e}^{\mathrm{i}\left(\frac{\mu^2 t}{3}+\delta\right)} \left[\mathrm{e}^{-\mathrm{i}\varphi} \cos(\mu x)\sin\theta + \mathrm{i}\,\mathrm{e}^{\mathrm{i}\varphi} \sin(\mu x)\cos\theta\right] \end{pmatrix}. \tag{2.91}$$

A relatively simple computation shows that β depends linearly on x and t as follows:

$$\beta = \mathrm{i}[2\mu t + x\sin(2\theta)\cos(2\varphi)] . \tag{2.92}$$

Thus taking into account (2.19), the dressed solution to (1.2) acquires the form:

$$u_1(x,t) = -\frac{2\left(\mu\beta + |F^1|^2\right) F^2 \left(F^3\right)^*}{\left(\mu\beta - |F^1|^2\right)^2}, \tag{2.93}$$

$$v_1(x,t) = \frac{\left(\mu\beta + |F^1|^2\right)\left(\mu\beta - 1 - |F^3|^2\right)}{\left(\mu\beta - |F^1|^2\right)^2}. \tag{2.94}$$

It could easily be checked that (2.93) and (2.94) satisfy (1.3) provided β and F obey (2.86) but are otherwise arbitrary.

After substituting (2.91) and (2.92) into (2.93) and (2.94) and making some further simplifications, we derive

$$u_1(x,t) = -\frac{2\Delta_d^*}{\Delta_d^2} \mathrm{e}^{-\left(\mathrm{i}\mu^2 t + \delta + \frac{\pi}{4}\right)} \left[\cos\left(\varphi + \frac{\pi}{4}\right)\sin(\mu x + \theta) - \mathrm{i}\,\sin\left(\varphi + \frac{\pi}{4}\right)\sin(\mu x - \theta)\right], \tag{2.95}$$

$$v_1(x,t) = \frac{\Delta_d^*(\Delta_d^* - 2)}{\Delta_d^2} \tag{2.96}$$

where

$$\Delta_d = \cos^2(\mu x + \theta)\cos^2\left(\varphi + \frac{\pi}{4}\right) + \cos^2(\mu x - \theta)\sin^2\left(\varphi + \frac{\pi}{4}\right) - \mathrm{i}\mu[2\mu t + x\sin(2\theta)\cos(2\varphi)].$$

Solutions (2.95) and (2.96) contain terms that are linear in x and t as well as terms that are bounded oscillating functions. This is why we call such solutions quasi-rational. It is easy to see that the denominator Δ_d has zeros for particular values of the parameters. For example, when $\theta = \pi/2$ it is zero at $x = 0$ and $t = 0$.

3. Conclusion

In this paper, we have shown how Zakharov-Shabat's dressing method can be applied to linear bundles in pole gauge whose potential functions are subject to (2.1). Following the general algorithm described in Section 2.1, we have constructed special solutions to a generalized Heisenberg ferromagnet equation, see (1.2). The simplest class of such solutions corresponds to a dressing factor with simple poles, see (2.24). We have seen that the location of the poles in λ-plane affects the form of the solutions. There exist three "pure" cases: the poles are complex numbers in generic position, the poles are imaginary and the poles are real. The first two options lead to soliton-like solutions of quadruplet and doublet type respectively, see formulas (2.51), (2.52), (2.54), (2.55), (2.56), (2.57), (2.77), (2.78), (2.81) and (2.82). All the quadruplet solutions we have derived are non-singular while the doublet solutions could have singularities in the pseudo-Hermitian case.

The case of real poles differs much from the others. It introduces certain degeneracy in the spectrum of the scattering operator and leads to quasi-rational solutions, see (2.95) and (2.96). We have seen such degeneracy is possible only if we have a pseudo-Hermitian reduction. This is a rather essential difference between the Hermitian and pseudo-Hermitian reductions. The quasi-rational solutions (2.95) and (2.96) could be non-singular for particular values of the parameters.

Apart from the "pure" cases we have discussed in the main text, there exists a situation when some poles of the dressing factor are generic complex numbers while the rest are either real or imaginary numbers. Clearly, the analysis of this "mixed" case can be reduced to the considerations we have already demonstrated.

Though we have discussed in detail special solutions to (1.2), similar procedures could be used to derive explicit solutions to any NLEE belonging to the integrable hierarchy of (1.4). The solutions of such a NLEE will have the same x-dependence as those of (1.2) but a different t-dependence.

In the present paper, we have focused on the simplest class of solutions obeying asymptotic behavior (2.1). A possible way of extending our results is by looking for solutions having more complicated behavior, e.g. non-trivial background solutions or periodic solutions. In the last few decades, nontrivial background solutions have become a topic of increased interest due to the connection to phenomena like rogue (freak) waves, see [1, 3, 17, 18]. Such solutions were obtained for classical integrable equations like the (scalar) nonlinear Schrödinger equation [1, 2, 12, 18], the 3-wave equation [3, 5] and for the scalar derivative nonlinear Schrödinger equation [19, 20]. This is why it is interesting to explicitly construct such type of solutions for the generalized Heisenberg equation and find out how the properties of (1.2) and its spectral problem will change if u and v have a different behavior.

Another direction to extend our results is by studying auxiliary spectral problems and the corresponding integrable hierarchies of NLEEs for other symmetric spaces and/or other reductions. This includes the study of multicomponent versions of (1.2) connected to the symmetric spaces $SU(m+n)/S(U(m)\times U(n))$ and rational bundles like the following one:

$$L(\lambda) = \mathrm{i}\partial_x - \lambda S_1 - \frac{1}{\lambda}S_{-1}, \qquad \lambda \in \mathbb{C}\backslash\{0\},$$

where

$$S_{\pm 1} = \begin{pmatrix} 0 & \mathbf{u}_{\pm} \\ \mathbf{v}_{\pm} & 0 \end{pmatrix}$$

for $\mathbf{u}_{\pm}$ and $\mathbf{v}_{\pm}$ being some vector functions. The above rational bundle can be viewed as a deformation of the linear bundle we have considered here and it has much more complicated properties, see [7, 10] for some discussion of the Hermitian case. The dressing method has already been successfully applied to such complicated spectral problems, see [15,16] for some interesting examples. We intend to discuss all these issues in more detail elsewhere.

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