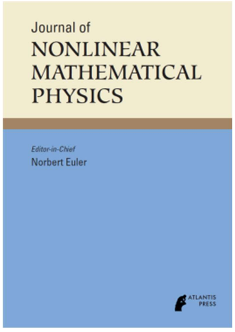

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Free-field realizations of the $W_{\mathscr{A}_n,N}$-algebra

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In this paper, we will construct free-field realizations of the $\mathrm{W}_{\mathscr{A}_n,N}$ algebra associated to an $\mathscr{A}_n$-valued differential operator

$$\mathscr{L} = \mathrm{I}_n\partial^N + U_{N-1}\partial^{N-1} + U_{N-2}\partial^{N-2} + \cdots U_0,$$

where $\mathscr{A}_n$ is a Frobenius algebra with the uint I_n.

Keywords: $\mathrm{W}_{\mathscr{A}_n,N}$ algebra; free-field realization.

2000 Mathematics Subject Classification: 17B63, 35Q53,37K10

1. Introduction

The source of the concept of W-algebras is the conformal field theory (CFT briefly) [2, 11, 23]. The main problem of the CFT is a description of fields having conformal symmetries. Only in the two-dimensional case is the group of conformal diffeomorphisms rich enough to build a meaningful theory on this base. All diffeomorphisms of a circle represent the core of the theory. Its related Lie algebra is a centerless Virasoro algebra, whose extension is the well-known Virasoro algebra. In the study of Virasoro algebra, there were various representations in terms of free fields, based on bosons, fermions and ghosts. In particular, the free-boson representation, including vertex operators, proved to be useful for particular calculations, especially for the evaluation of correlation functions.

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The CFT requires the extension of the Virasoro algebra as far as possible. The work by Zamolodchikov pioneered the concept of CFT. He gave an extension of the Virasoro algebra called the W_3 algebra. In that terminology, the Virasoro algebra was W_2. At the end of the 1980's, it was found that the mathematical framework for further extension of Virasoro algebra already existed as the theory of integrable systems. In other words, the classical realization of the W-algebra [23] appears naturally as the second Poisson bracket of KdV-type hierarchies. For example, the Virasoro algebra W_2 is realized as the Magri bracket for the KdV hierarchy [12, 18], and the Zamolodchikov-Fateev-Lukyanov W_m-algebra as the second Adler-Gelfand-Dickey (AGD briefly) bracket for the m^{th}-order Gelfand-Dickey (GD_m) hierarchy [1, 10, 17, 19]. Free-field relations of W-algebras have also been obtained by constructing the related Miura maps, please see e.g. [7–9, 14] and references therein for details.

In [5], A. Bilal proposed a non-local matrix generalization of the well-known W_m-algebra, called the $V_{n,m}$-algebra, by constructing the second AGD bracket associated with a matrix differential operator of order m

$$\begin{aligned}\mathfrak{L} &= -\mathrm{I}_n\partial^m + U_1\partial^{m-1} + U_2\partial^{m-2} + \cdots + U_m\\ &= -(\mathrm{I}_n\partial - P_1)\cdots(\mathrm{I}_n\partial - P_m), \qquad \partial = \frac{\partial}{\partial x}, \quad P_j, \quad U_j \in gl(n,\mathbb{C}),\end{aligned}$$

where I_n is the n^{th}-order identity matrix. Upon reducing to $U_1 = 0$, the non-commutativity of matrices implies the presence of non-local terms in the $V_{n,m}$-algebra. A Miura transformation relates these Poisson brackets of the U_j to much simpler ones of a set of $P_i \in gl(n,\mathbb{C})$, i.e., the Kupershmidt-Wilson (KW briefly) type theorem. Contrary to the scalar case, generally P_i are not free fields. It is difficult to give such a free-field realization because of the non-local terms except some special cases [3, 4].

Recently motivated by the work in [6, 13, 16, 25], Strachan and Zuo began to study the Frobenius algebra-valued integrable systems [21, 22, 24, 26]. In [21] they introduced an $\mathfrak{F}$-valued KP hierarchy associated with an $\mathfrak{F}$-valued pseudo-differential operator (ΨDO in brief)

$$L = \mathbf{1}_{\mathfrak{F}}\partial + U_1\partial^{-1} + U_2\partial^{-2} + \cdots$$

and constructed infinite series of bi-Hamiltonian structures, where $\mathbf{1}_{\mathfrak{F}}$ is the unit of the Frobenius algebra $\mathfrak{F}$. Via the properties of the second Hamiltonian structures, they have obtained a local matrix generalization of W-type algebras. Because the Frobenius algebra is commutative, upon reducing to the $U_1 = 0$, the second Hamiltonian structure is still local, which gives a chance to construct free-field realizations.

The aim of this paper is to construct free-field realizations of the $W_{\mathscr{A}_n,N}$ algebra associated to a concrete $\mathscr{A}_n$-valued differential operator

$$\mathscr{L} = \mathrm{I}_n\partial^N + U_{N-1}\partial^{N-1} + U_{N-2}\partial^{N-2} + \cdots U_0$$

and organized as follows. Firstly, we recall the definition of $W_{\mathfrak{F}_n,N}$-algebra and then show a KW-type theorem. Afterwards, with the help of the KW-type theorem we will construct the free-field realizations of the $W_{\mathscr{A}_n,N}$ algebra. Finally we give two examples to illustrate our method.

2. The $W_{\mathfrak{F}_n,N}$-algebra and the KW-type theorem

2.1. *Local matrix generalizations of the classical W algebras*

To be self-contained, below we recall some known facts, see [21, 22] for details. Let us begin with some basic definitions.

Definition 2.1. The Frobenius algebra $\mathfrak{F} := \{\mathfrak{F}, \mathrm{tr}_{\mathfrak{F}}, \mathbf{1}_{\mathfrak{F}}, \circ\}$ over $\mathbb{K}$ is a free $\mathbb{K}$-module $\mathfrak{F}$ of finite rank n, equipped with a commutative and associative multiplication $\circ$ and the unit $\mathbf{1}_{\mathfrak{F}}$, and a $\mathbb{K}$-linear form $\mathrm{tr}_{\mathfrak{F}} : \mathfrak{F} \to \mathbb{K}$ whose kernel contains no nontrivial ideas, where $\mathbb{K}$ is $\mathbb{C}$ or $\mathbb{R}$.

Let

$$\mathscr{L} = \mathbf{1}_{\mathfrak{F}}\partial^N + U_{N-1}\partial^{N-1} + U_{N-2}\partial^{N-2} + \cdots U_0 \tag{2.1}$$

be an $\mathfrak{F}$-valued differential operator of order N. The *$\mathfrak{F}$-valued Gefland-Dickey (GD in brief) hierarchy* is defined as

$$\frac{\partial \mathscr{L}}{\partial t_r} = B_r \circ \mathscr{L} - \mathscr{L} \circ B_r, \quad r = 1, 2, \ldots, \tag{2.2}$$

where $B_r = \mathscr{L}_+^{\frac{r}{N}}$ is the pure differential part of the operator $L^{\frac{r}{N}}$. As discussed in [21], the $\mathfrak{F}$-valued GD hierarchy has bi-hamiltonian structures with the second Poisson bracket as

$$\{\tilde{f}, \tilde{g}\}^{(N)} = \mathrm{tr}_{\mathfrak{F}} \int \mathrm{res}\left(\left(\mathscr{L} \circ \frac{\delta f}{\delta \mathscr{L}}\right)_+ \circ \mathscr{L} - \mathscr{L} \circ \left(\frac{\delta f}{\delta \mathscr{L}} \circ \mathscr{L}\right)_+\right) \circ \frac{\delta g}{\delta \mathscr{L}} dx, \tag{2.3}$$

where the variational derivative[a] $\frac{\delta f}{\delta \mathscr{L}}$ is defined by the formula $\frac{\delta f}{\delta \mathscr{L}} = \sum_{i=0}^{N-1} \partial^{-i-1} \frac{\delta f}{\delta U_i}$. Upon reducing to the $U_{N-1} = 0$, the Poisson bracket $\{\ ,\ \}^{(N)}$ is reducible if and only if

$$\mathrm{res}\,[\mathscr{L}, \frac{\delta f}{\delta \mathscr{L}}] = 0. \tag{2.4}$$

We denote the reduced bracket by $\{\ ,\ \}_D^{(N)}$, which provides a local matrix generalization of the classical W_N-algebra ([21]). We would like to call the $W_{\mathfrak{F}_n,N}$-algebra. Especially when one takes $\varphi(x) = \mathrm{tr}_{\mathfrak{F}} U_{N-2}$, with the use of (2.3) and (2.4) the reduced Poisson bracket is given by

$$\{\varphi(x), \varphi(y)\}_D^{(N)} = -\left(\frac{N^3 - N}{12}\partial^3 + \varphi\,\partial + \partial\,\varphi\right)\delta(x-y).$$

This means that the $W_{\mathfrak{F}_n,N}$-algebra contains the Virasoro algebra as its subalgebra.

---

[a]The variational derivative with respect to an algebra-valued field has been discussed in [20]. In the present context, let $\tilde{f} = \int \mathrm{tr}_{\mathfrak{F}} F(V) dx$ for $V = \sum_{q=1}^{n} v_q e_q \in \mathfrak{F}$, the variational derivative $\frac{\delta F}{\delta V}$ is defined by

$$\tilde{f}(v + \delta v) - \tilde{f}(v) = \int \mathrm{tr}_{\mathfrak{F}} \left(\frac{\delta F}{\delta V} \circ \delta V + o(\delta V)\right) dx = \int \sum_{q=1}^{n} \left(\frac{\delta f}{\delta v_q}\,\delta v_q + o(\delta v)\right) dx,$$

where $f(v) = \mathrm{tr}_{\mathfrak{F}} F(V)$, $\delta V = \sum_{q=1}^{n} \delta v_q e_q \in \mathfrak{F}$, $\frac{\delta f}{\delta v_q} = \sum_{j=0}^{\infty} (-\partial)^j \frac{\partial f}{\partial v_q^{(j)}}$ and δv is a small parameter. Without confusion, we use the notation $\frac{\delta f}{\delta V}$ instead of $\frac{\delta F}{\delta V}$.

2.2. *Modifying the second Hamiltonian structure*

In order to construct free-field realizations of the $W_{\mathfrak{F}_n,N}$-algebra, we want to study the transformation of the second Hamiltonian structure $\{\ ,\ \}$ by the factorization

$$\mathscr{L} = \mathscr{L}_r \circ \mathscr{L}_{r-1} \circ \cdots \circ \mathscr{L}_1, \tag{2.5}$$

where $\mathscr{L}_j = \mathbf{1}_{\mathfrak{F}}\partial^{N_j} + V_{j,N_j-1}\partial^{N_j-1} + \cdots$ are $\mathfrak{F}$-valued ΨDOs and $\sum_{j=1}^{r} N_r = N$.

Theorem 2.1. *Assume that the factorization* (2.5) *exists, then the second Poisson bracket for L is a direct sum of those for* $\mathscr{L}_1, \ldots, \mathscr{L}_r$*, that is to say,*

$$\{\tilde{f},\tilde{g}\}^{(N)} = \sum_{j=1}^{r} \{\tilde{f},\tilde{g}\}^{(N_j)}. \tag{2.6}$$

Moreover, the constraint condition $U_{N-1} = 0$ *is equivalent to*

$$\mathrm{res}\left[\mathscr{L}, \frac{\delta f}{\delta \mathscr{L}}\right] = \sum_{j=1}^{r} \mathrm{res}\left[\mathscr{L}_j, \frac{\delta f}{\delta \mathscr{L}_j}\right] = 0. \tag{2.7}$$

When $\mathfrak{F} = \mathbb{R}$*, this result is the so-called KW theorem in* [15].

Proof. Observe that

$$\begin{aligned}
\delta \tilde{f} &= \mathrm{tr}_{\mathfrak{F}} \int \mathrm{res}\, \frac{\delta f}{\delta \mathscr{L}} \circ \delta \mathscr{L} dx = \sum_{j=1}^{r} \mathrm{tr}_{\mathfrak{F}} \int \mathrm{res}\, \frac{\delta f}{\delta \mathscr{L}_j} \circ \delta \mathscr{L}_j dx \\
&= \sum_{j=1}^{r} \mathrm{tr}_{\mathfrak{F}} \int \mathrm{res}\, \frac{\delta f}{\delta \mathscr{L}} \circ \mathscr{L}_r \circ \cdots \circ \mathscr{L}_{j+1} \circ \delta \mathscr{L}_j \circ \mathscr{L}_{j-1} \circ \cdots \circ \mathscr{L}_1 dx \\
&= \sum_{j=1}^{r} \mathrm{tr}_{\mathfrak{F}} \int \mathrm{res}\, \mathscr{L}_{j-1} \circ \cdots \circ \mathscr{L}_1 \circ \frac{\delta f}{\delta \mathscr{L}} \circ \mathscr{L}_r \circ \cdots \circ \mathscr{L}_{j+1} \circ \delta \mathscr{L}_j dx.
\end{aligned}$$

This expression implies

$$\frac{\delta f}{\delta \mathscr{L}_j} = \mathscr{L}_{j-1} \circ \cdots \circ \mathscr{L}_1 \circ \frac{\delta f}{\delta \mathscr{L}} \circ \mathscr{L}_r \circ \cdots \circ \mathscr{L}_{j+1} \mod R(-\infty, -m_j - 1). \tag{2.8}$$

Here $R(-\infty,-k)$ contains all of the $\mathfrak{F}$-valued operators of the form $\sum_{j=-\infty}^{-k} A_j\partial^j$. With the use of (2.8), we get

$$\mathscr{L}_j \circ \frac{\delta f}{\delta \mathscr{L}_j} = \frac{\delta f}{\delta \mathscr{L}_{j+1}} \circ \mathscr{L}_{j+1} = \mathscr{L}_j \circ \cdots \circ \mathscr{L}_1 \circ \frac{\delta f}{\delta \mathscr{L}} \circ \mathscr{L}_r \circ \cdots \circ \mathscr{L}_{j+1} \mod R(-\infty,-1) \tag{2.9}$$

and

$$\sum_{j=1}^{r} \mathrm{res}\,[\mathscr{L}_j, \frac{\delta f}{\delta \mathscr{L}_j}] = \mathrm{res}\,(\mathscr{L}_r \circ \frac{\delta f}{\delta \mathscr{L}_r} - \frac{\delta f}{\delta \mathscr{L}_1} \circ \mathscr{L}_1) = \mathrm{res}\,[\mathscr{L}, \frac{\delta f}{\delta \mathscr{L}}]. \tag{2.10}$$

Obviously, (2.7) follows from (2.4) and (2.10). With the help of (2.9), the right side of (2.6) is

$$\begin{aligned}
&\sum_{j=1}^{r}\{\tilde{f},\tilde{g}\}^{(N_j)}\\
&=\sum_{j=1}^{r}\mathrm{tr}_{\mathfrak{F}}\int \mathrm{res}\left(\left(\mathscr{L}_j\circ\frac{\delta f}{\delta\mathscr{L}_j}\right)_+\circ\mathscr{L}_j-\mathscr{L}_j\circ\left(\frac{\delta f}{\delta\mathscr{L}_j}\circ\mathscr{L}_j\right)_+\right)\circ\frac{\delta g}{\delta\mathscr{L}_j}\,dx\\
&=\sum_{j=1}^{r}\mathrm{tr}_{\mathfrak{F}}\int \mathrm{res}\left(\mathscr{L}_j\circ\left(\frac{\delta f}{\delta\mathscr{L}_j}\circ\mathscr{L}_j\right)_- -\left(\mathscr{L}_j\circ\frac{\delta f}{\delta\mathscr{L}_j}\right)_-\circ\mathscr{L}_j\right)\circ\frac{\delta g}{\delta\mathscr{L}_j}\,dx\\
&=\sum_{j=1}^{r}\mathrm{tr}_{\mathfrak{F}}\int \mathrm{res}\left(\frac{\delta f}{\delta\mathscr{L}_j}\circ\mathscr{L}_j\right)_-\circ\left(\frac{\delta g}{\delta\mathscr{L}_j}\circ\mathscr{L}_j\right)_+\,dx\\
&-\sum_{j=1}^{r}\mathrm{tr}_{\mathfrak{F}}\int \mathrm{res}\left(\mathscr{L}_j\circ\frac{\delta f}{\delta\mathscr{L}_j}\right)_-\circ\left(\mathscr{L}_j\circ\frac{\delta g}{\delta\mathscr{L}_j}\right)_+\,dx\\
&=\sum_{j=1}^{r}\mathrm{tr}_{\mathfrak{F}}\int \mathrm{res}\left(\frac{\delta f}{\delta\mathscr{L}_j}\circ\mathscr{L}_j\right)_-\circ\left(\frac{\delta g}{\delta\mathscr{L}_j}\circ\mathscr{L}_j\right)_+\,dx\\
&-\sum_{j=1}^{r-1}\mathrm{tr}_{\mathfrak{F}}\int \mathrm{res}\left(\frac{\delta f}{\delta\mathscr{L}_{j+1}}\circ\mathscr{L}_{j+1}\right)_-\circ\left(\frac{\delta g}{\delta\mathscr{L}_{j+1}}\circ\mathscr{L}_{j+1}\right)_+\,dx\\
&-\mathrm{tr}_{\mathfrak{F}}\int \mathrm{res}\left(\mathscr{L}_r\circ\frac{\delta f}{\delta\mathscr{L}_r}\right)_-\circ\left(\mathscr{L}_r\circ\frac{\delta g}{\delta\mathscr{L}_r}\right)_+\,dx\\
&=\mathrm{tr}_{\mathfrak{F}}\int \mathrm{res}\left[\left(\frac{\delta f}{\delta\mathscr{L}_1}\circ\mathscr{L}_1\right)_-\circ\left(\frac{\delta g}{\delta\mathscr{L}_1}\circ\mathscr{L}_1\right)_+-\left(\mathscr{L}_r\circ\frac{\delta f}{\delta\mathscr{L}_r}\right)_-\circ\left(\mathscr{L}_r\circ\frac{\delta g}{\delta\mathscr{L}_r}\right)_+\right]dx\\
&=\{\tilde{f},\tilde{g}\}^{(N)}.
\end{aligned}$$

We thus complete the proof of this theorem. □

The above theorem implies that it is possible to simplify the construction of the free-field realization for the $W_{\mathfrak{F}_n,N}$-algebra to the construction of the free-field realization for each copy of the $W_{\mathfrak{F}_n,1}$-algebra. Many examples suggest the existence of free-filed realizations of the above W-type algebras, but up to now we have no a unified proof for the general $W_{\mathfrak{F}_n,N}$-algebra. In the next section we illustrate our construction by taking a concrete algebra $\mathscr{A}_n$.

3. Free-field realizations of the $W_{\mathscr{A}_n,N}$-algebra

Let us denote

$$\mathscr{Z}_n=\left\{a=\sum_{k=1}^{n}a_k\Lambda^{k-1}\;\middle|\;a_k\in\mathbb{C},\,k=1,\ldots,n\right\},$$

where $\Lambda=(\Lambda_{ij})\in gl(n,\mathbb{C})$ with the elements

$$\Lambda_{ij}=\delta_{i,j+1}=\begin{cases}1, & i=j+1\\ 0, & \text{other cases}\end{cases}$$

and $\Lambda^0=\mathrm{I}_n$ is the n^{th}-order identity matrix. Observe that $\Lambda^n=0$, then $\mathscr{Z}_n$ is a maximal commutative subalgebra of $gl(n,\mathbb{C})$. In [21, 26], they have shown that the algebra $\mathscr{Z}_n$ has at least n-"basic" different ways to be realized as the Frobenius algebra $\mathscr{A}_k := \{\mathscr{Z}_n, \mathrm{I}_n, \mathrm{tr}_{\mathscr{A}_k}\}$ with the trace form defined by

$$\mathrm{tr}_{\mathscr{A}_k}(a) = a_k + a_n(1-\delta_{n,k}) \quad \text{for any} \quad a = \sum_{k=1}^{n} a_k \Lambda^{k-1} \in \mathscr{Z}_n. \tag{3.1}$$

Without loss of generality, in this section we will take the Frobenius algebra $\mathfrak{F}$ as $\mathscr{A}_n$ and construct a free-field realization of $W_{\mathscr{A}_n,N}$-algebra.

Suppose that the $\mathscr{A}_n$-valued differential operator

$$\mathscr{L} = \mathrm{I}_n \partial^N + U_{N-1}\partial^{N-1} + U_{N-2}\partial^{N-2} + \cdots U_0$$

could be represented as a product of

$$\mathscr{L} = \mathscr{L}_N \circ \mathscr{L}_{N-1} \circ \cdots \circ \mathscr{L}_1$$

of $\mathscr{A}_n$-valued differential operators $\mathscr{L}_j = \mathrm{I}_n\partial + V_j,\ j = 1,\ldots,N$. With the use of Theorem 2.1, we get

$$\begin{aligned}
\{\tilde{f},\tilde{g}\}^{(N)} &= \sum_{j=1}^{N} \{\tilde{f},\tilde{g}\}^{(1)}_{\mathscr{L}_j} \\
&= \sum_{j=1}^{N} \mathrm{tr}_{\mathfrak{F}} \int \mathrm{res}\left(\left(\mathscr{L}_j \circ \frac{\delta f}{\delta \mathscr{L}_j}\right)_+ \circ \mathscr{L}_j - \mathscr{L}_j \circ \left(\frac{\delta f}{\delta \mathscr{L}_j} \circ \mathscr{L}_j\right)_+ \right) \circ \frac{\delta g}{\delta \mathscr{L}_j}\, dx \\
&= \sum_{j=1}^{N} \mathrm{tr}_{\mathfrak{F}} \int \frac{\delta f}{\delta V_j} \circ \frac{\partial}{\partial x} \frac{\delta g}{\delta V_j}\, dx. \qquad \text{Here using} \quad \frac{\delta f}{\delta \mathscr{L}_j} = \partial^{-1} \frac{\delta f}{\delta V_j}.
\end{aligned} \tag{3.2}$$

More precisely,

$$\left\{ \mathrm{tr}_{\mathscr{A}_n} \int F V_i dx\, ,\, \mathrm{tr}_{\mathscr{A}_n} \int G V_j dx \right\}^{(N)} = \delta_{ij}\, \mathrm{tr}_{\mathscr{A}_n} \int F \frac{\partial}{\partial x} G\, dx,$$

where F and G are two $\mathscr{A}_n$-valued test functions.

Next, we want to study the reduced bracket under reduction to the submanifold $U_{N-1} = 0$.

Lemma 3.1. *The Poisson bracket* $\{\ ,\ \}^{(N)}$ *with the constraint* $U_{N-1} = 0$ *is reduced to*

$$\left\{ \mathrm{tr}_{\mathscr{A}_n} \int F V_i dx\, ,\, \mathrm{tr}_{\mathscr{A}_n} \int G V_j dx \right\}^{(N)}_D = \left(\delta_{ij} - \frac{1}{N}\right) \mathrm{tr}_{\mathscr{A}_n} \int F \frac{\partial}{\partial x} G\, dx, \tag{3.3}$$

where F and G are two $\mathscr{A}_n$-valued test functions. In particular,

$$\left\{ V_{i,q}(x)\, ,\, V_{j,r}(y) \right\}^{(N)}_D = \left(\delta_{ij} - \frac{1}{N}\right) \delta_{q+r,n+1}\, \delta'(x-y), \tag{3.4}$$

where $V_i = \sum_{q=1}^{n} V_{i,q} \Lambda^{q-1}$.

Proof. Taking an overcomplete set of vectors[b]

$$\vec{\mathbf{h}}_j = \left(h_j^1, \ldots, h_j^{N-1}\right), \quad j = 1, \ldots, N \tag{3.5}$$

in an $(N-1)$-dimensional Euclidean space with

$$\sum_{j=1}^{N} \vec{\mathbf{h}}_j = 0, \quad \sum_{j=1}^{N} h_j^a h_j^b = \delta_{ab}, \quad \sum_{a=1}^{N-1} h_i^a h_j^a = \delta_{ij} - \frac{1}{N}. \tag{3.6}$$

Observe that $U_{N-1} = \sum_{j=1}^{N} V_j$ and denoting $\mathscr{V}_a = \sum_{j=1}^{N} h_j^a V_j, \quad a = 1, \ldots, N-1$. With the help of (3.6), we have

$$V_j = \frac{1}{N} U_{N-1} + \sum_{a=1}^{N-1} h_j^a \mathscr{V}_a, \quad j = 1, \ldots, N$$

and

$$\frac{\delta f}{\delta V_j} = \frac{\delta f}{\delta U_{N-1}} + \sum_{a=1}^{N-1} h_j^a \frac{\delta f}{\delta \mathscr{V}_a}, \quad j = 1, \ldots, N. \tag{3.7}$$

So using (3.6) and (3.7), the Poisson bracket $\{\ ,\ \}^{(N)}$ in (3.2) can be rewritten as

$$\begin{aligned}
\{\tilde{f}, \tilde{g}\}^{(N)} &= \sum_{j=1}^{N} \mathrm{tr}_{\mathfrak{F}} \int \frac{\delta f}{\delta V_j} \circ \frac{\partial}{\partial x} \frac{\delta g}{\delta V_j}\, dx \\
&= N\, \mathrm{tr}_{\mathfrak{F}} \int \frac{\delta f}{\delta U_{N-1}} \circ \frac{\partial}{\partial x} \frac{\delta g}{\delta U_{N-1}}\, dx + \sum_{a=1}^{N-1} \mathrm{tr}_{\mathfrak{F}} \int \frac{\delta f}{\delta \mathscr{V}_a} \circ \frac{\partial}{\partial x} \frac{\delta g}{\delta \mathscr{V}_a}\, dx.
\end{aligned}$$

When we consider the reduction $U_{N-1} = 0$, from (2.7) we should take into account the following condition

$$\sum_{j=1}^{N} \left(\frac{\delta f}{\delta V_j}\right)_x = 0. \tag{3.8}$$

That is to say,

$$0 = \sum_{j=1}^{N} \left(\frac{\delta f}{\delta V_j}\right)_x = N \frac{\delta f}{\delta U_{N-1}} + \sum_{j=1}^{N} \sum_{a=1}^{N-1} h_j^a \frac{\delta f}{\delta \mathscr{V}_a} = N \frac{\delta f}{\delta U_{N-1}}.$$

Thus the reduced Poisson bracket $\{\ ,\ \}_D^{(N)}$ is given by

$$\{\tilde{f}, \tilde{g}\}_D^{(N)} = \sum_{a=1}^{N-1} \mathrm{tr}_{\mathfrak{F}} \int \frac{\delta f}{\delta \mathscr{V}_a} \circ \frac{\partial}{\partial x} \frac{\delta g}{\delta \mathscr{V}_a}\, dx$$

[b] e.g., such vectors have been explicitly written in [8].

and for two $\mathscr{A}_n$-valued test functions F and G,

$$\left\{\mathrm{tr}_{\mathscr{A}_n}\int FV_i dx\,,\,\mathrm{tr}_{\mathscr{A}_n}\int GV_j dx\right\}_D^{(N)} = \sum_{a=1}^{N-1} h_i^a h_j^a\,\mathrm{tr}_{\mathscr{A}_n}\int F\frac{\partial}{\partial x}G\,dx$$
$$= (\delta_{ij}-\frac{1}{N})\,\mathrm{tr}_{\mathscr{A}_n}\int F\frac{\partial}{\partial x}G\,dx.$$

The identity (3.4) follows from the formula (3.3) and the definition $\mathrm{tr}_{\mathfrak{F}}$ in (3.1). $\square$

Let $K=(K_{qr})$ be an $n\times n$ matrix with the elements $K_{qr}=\delta_{q+r,n+1}$. Obvious the matrix K is a real symmetric matrix, thus there exists an orthogonal matrix Q such that $K=Q\,\mathrm{diag}(\lambda_1,\ldots,\lambda_n)\,Q^t$, where λ_j are eigenvalues of K and Q^t is the transpose of Q. Assume

$$S=(S_{qr})=Q\,\mathrm{diag}\big(\sqrt{\lambda_1},\ldots,\sqrt{\lambda_n}\big)\in gl(n,\mathbb{C}), \tag{3.9}$$

then $K=SS^t$.

Taking $(N-1)n$ free fields $\varphi_{j,q}(x)$ with the currents $\mathbf{j}_{i,q}(x)=\varphi'_{i,q}(x)$ together with the Poisson bracket

$$\big\{\mathbf{j}_{i,q}(x)\,,\,\mathbf{j}_{j,r}(y)\big\}_D^{(N)}=\delta_{ij}\,\delta_{qr}\,\delta'(x-y), \tag{3.10}$$

where $i,j=1,\ldots,N-1$ and $q,r=1,\ldots,n$.

Theorem 3.1. *Setting*

$$\vec{\mathbf{J}}_k=(\mathbf{J}_{1,k}\,,\,\ldots\,,\,\mathbf{J}_{N-1,k})\,,\quad \mathbf{J}_{a,k}=\sum_{\alpha=1}^{n}S_{k,\alpha}\,\mathbf{j}_{a,\alpha}(x), \tag{3.11}$$

then the identification $\mathscr{L}=\mathscr{L}_N\circ\mathscr{L}_{N-1}\circ\cdots\circ\mathscr{L}_1$ with the element

$$\mathscr{L}_j=\mathrm{I}_n\partial+V_j=\mathrm{I}_n\partial+\sum_{k=1}^{n}(\vec{\mathbf{h}}_j\cdot\vec{\mathbf{J}}_k)\,\Lambda^{k-1},\quad j=1,\ldots,N \tag{3.12}$$

provides a free-field realization of the $W_{\mathscr{A}_n,N}$-algebra, where $\vec{\mathbf{h}}_j\cdot\vec{\mathbf{J}}_k:=\sum_{a=1}^{N-1}h_j^a\,\mathbf{J}_{a,k}$.

Proof. The constrained condition $U_{N-1}=0$ follows from

$$\sum_{j=1}^{N}V_j=\sum_{j=1}^{N}\sum_{k=1}^{n}(\vec{\mathbf{h}}_j\cdot\vec{\mathbf{J}}_k)\,\Lambda^{k-1}=\sum_{k=1}^{n}\left(\sum_{j=1}^{N}\vec{\mathbf{h}}_j\cdot\vec{\mathbf{J}}_k\right)\Lambda^{k-1}=0.$$

Denoting $V_j = \sum_{k=1}^{n} V_{j,k}(x)\Lambda^{k-1}$, then $V_{j,k}(x) = \vec{\mathbf{h}}_j \cdot \vec{\mathbf{J}}_k = \sum_{a=1}^{N-1} h_j^a \mathbf{J}_{a,k}$. Now, with the help of (3.6), (3.9) and (3.10), we have

$$\begin{aligned}
\left\{V_{i,q}(x)\,,V_{j,r}(y)\right\}_D^{(N)} &= \{\vec{\mathbf{h}}_i \cdot \vec{\mathbf{J}}_q\,, \vec{\mathbf{h}}_j \cdot \vec{\mathbf{J}}_r\}_D^{(m)} \\
&= \sum_{a,b=1}^{N-1} \left\{h_i^a \mathbf{J}_{a,q}(x)\,, h_j^a \mathbf{J}_{b,r}(x)\right\}_D^{(m)} \\
&= \sum_{a,b=1}^{N-1} h_i^a h_j^b \sum_{\alpha,\beta=1}^{n} \left\{S_{q,\alpha}\mathbf{j}_{a,\alpha}(x)\,, S_{r,\beta}\mathbf{j}_{b,\beta}(x)\right\}_D^{(m)} \\
&= \sum_{a,b=1}^{N-1} h_i^a h_j^b \sum_{\alpha,\beta=1}^{n} S_{q,\alpha}\, S_{r,\beta}\, \delta_{ab}\delta_{\alpha\beta}\delta'(x-y) \\
&= \sum_{a=1}^{N-1} h_i^a h_j^a\, \delta_{q+r,n-1}\, \delta'(x-y) \\
&= \vec{\mathbf{h}}_i \cdot \vec{\mathbf{h}}_j\ \delta_{q+r,n-1}\, \delta'(x-y) \\
&= \left(\delta_{ij} - \frac{1}{N}\right) \delta_{q+r,n-1}\, \delta'(x-y),
\end{aligned}$$

which is exactly the reduced Poisson bracket (3.4). We thereby obtain the free-field realization of the $\mathrm{W}_{\mathscr{A}_n,N}$-algebra. □

4. Conclusion

In summary, with the help of the KW-type theorem, we have constructed free-field realizations of the $\mathrm{W}_{\mathscr{A}_n,N}$ algebra associated with the $\mathscr{A}_n$-valued differential operator

$$\mathscr{L} = \mathrm{I}_n\partial^N + U_{N-1}\partial^{N-1} + U_{N-2}\partial^{N-2} + \cdots U_0.$$

By analogy with the above, a minor modification will give the free-field realizations of the $\mathrm{W}_{\mathscr{A}_k,N}$ algebra for $k = 1,\ldots,n-1$. But for general $\mathrm{W}_{\mathfrak{F},N}$ algebra, it is still open because of the uncertainty of the $\mathbb{K}$-linear form $\mathrm{tr}_{\mathfrak{F}}$.

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References

[1] I. Bakas, Higher Spin Fields and the Gelfand-Dickey Algebra, *Commun. Math. Phys.* **123** (1989) 627–639.

[2] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Infinite conformal symmetry in two dimensional quantum field theory, *Nuclear Phys. B* **241** (1984) 333–380.

[3] A. Bilal, Non-abelian Toda theory: a completely integrable model for strings on a black hole background, *Nucl. Phys. B* **422** (1994) 258–288.

[4] A. Bilal, Multi-component KdV hierarchy, V-algebra and non-abelian Toda theory, *Lett. Math. Phys.* **32** (1994) 103–120.
[5] A. Bilal, Non-Local Matrix Generalizations of W-Algebras, *Commu. Math. Phys.* **170** (1995) 117–150.
[6] P. Casati and G. Ortenzi, New integrable hierarchies from vertex operator representations of polynomial Lie algebras, *J. Geom. Phys.* **56** (2006) 418–449.
[7] Y. Cheng, Free-Field Realization of the $W_{\mathrm{KP}}^{(N)}$-Algebra, *Lett. Math. Phys.* **33** (1995) 159–169.
[8] Y. Cheng and Z. Li, Poisson Structures for Dispersionless Integrable Systems and Associated W-Algebras, *Lett. Math. Phys.* **42** (1997) 73–83.
[9] L.A. Dickey, Lectures on Classical W-Algebras, *Acta Applicandae Mathematicae* **47** (1997) 243–321.
[10] V.A. Fateev and S.L. Lukyanov, The models of two-dimensional conformal quantum field theory with Z_n symmetry, *Internat. J. Modern Phys. A* **3** (1988) 507–520.
[11] P.Di. Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory* (Springer-Verlag, New York, 1997).
[12] G.L. Gervais, Infinite family of polynomial functions of the Virasoro generators with vanishing Poisson brackets, *Phys. Lett. B* **160** (1985) 277–278.
[13] R. Hirota, X.B. Hu and X.Y. Tang, A vector potential KdV equation and vector Ito equation: soliton solutions, bilinear Bäcklund transformations and Lax pairs, *J. Math. Anal. Appl.* **288** (2003) 326–348.
[14] S. Hu and P. Liu, HOMFLY Polynomial from a Generalized Yang-Yang Function, *Commun. Math. Stat.* **3** (2015) 329–352.
[15] B.A. Kupershmidt and G. Wilson, Modifying Lax equations and the second Hamiltonian structure, *Invent. Math.* **62** (1981) 403–436.
[16] Johan van de Leur, Bäcklund transformations for new integrable hierarchies related to the polynomial Lie algebra $gl_\infty^{(n)}$, *J. Geom. Phys.* **57** (2007) 435–447.
[17] S.L. Lukyanov, Quantization of the Gelfand-Dickey bracket, *Funct. Anal. Appl.* **22** (1988) 255–262.
[18] F. Magri, A simple model of the integrable Hamiltonian equation, *J. Math. Phys.* **19** (1978) 1156–1162.
[19] P. Mathieu, Extended classical conformal algebras and the second hamiltonian structure of Lax equations, *Phys. Lett. B* **208** (1991) 101–106.
[20] P.J. Olver and V.V. Sokolov, Integrable evolution equations on associative algebras, *Commun. Math. Phys.* **2** (1998) 245–268.
[21] Ian.A.B. Strachan and D. Zuo, Integrability of the Frobenius algebra-valued Kadomtsev-Petviashvili hierarchy, *J. Math. Phys.* **56** (2015) 113509.
[22] Ian.A.B. Strachan and D. Zuo, Frobenius manifolds and Frobenius algebra-valued Integrable systems, *Lett. Math. Phys.* **107** (2017) 997–1026.
[23] A.B. Zamolodchikov, Infinite extra symmetries in two-dimensional conformal quantum field theory, *Teoret. Mat. Fiz.* **65** (1985) 347–359.
[24] H. Zhang and D. Zuo, Hamiltonian structures of the constrained F-valued KP hierarchy, *Rep. Math. Phys.* **76** (2015) 116–129.
[25] X. Zhu and D. Zhang, Lie algebras and Hamiltonian structures of multi-component Ablowitz-Kaup-Newell-Segur hierarchy, *J. Math. Phys.* **54** (2013) 53508.
[26] D. Zuo, The Frobenius-Virasoro algebra and Euler equations, *J. Geom. Phys.* **86** (2014) 203–210.