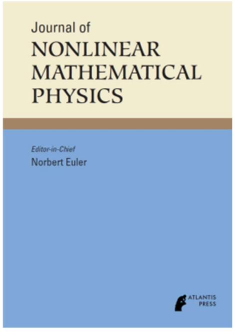

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LETTER TO THE EDITOR

On a surface isolated by Gambier

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We provide a Lax pair for the surfaces of Voss and Guichard, and we show that such particular surfaces considered by Gambier are characterized by a third Painlevé function.

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2000 Mathematics Subject Classification: 33E17, 34A05, 34Mxx.

1. Introduction. Surfaces of Voss and Guichard

Let us first recall two equivalent definitions of these surfaces: a geometric one and an analytic one. Our notation follows the review of Gambier [4].

Geometrically, the surfaces of Voss [11] and Guichard [6] are by definition those which admit a conjugate net made of geodesics. For instance, every minimal surface is such a surface.

Analytically, they can be characterized by their three fundamental quadratic forms d**F**.d**F**, $-$d**F**.d**N**, d**N**.d**N**, in which $\mathbf{F}(u,v)$ is the current point of the surface and $\mathbf{N}(u,v)$ a unit vector normal to the tangent plane. Choosing the coordinates (u,v) defined by the geodesic conjugate net, these are [4, p. 362]

$$\begin{cases} \mathrm{I} = X_u^2 \mathrm{d}u^2 + 2\cos(2\omega) X_u Y_v \mathrm{d}u\mathrm{d}v + Y_v^2 \mathrm{d}v^2, \\ \mathrm{II} = \sin(2\omega)(X_u \mathrm{d}u^2 + Y_v \mathrm{d}v^2), \\ \mathrm{III} = \mathrm{d}u^2 + 2\cos(2\omega)\mathrm{d}u\mathrm{d}v + \mathrm{d}v^2, \end{cases} \tag{1.1}$$

with the notation $X_u = \partial X(u,v)/\partial u, \dots$, they depend on three functions ω, X, Y of two variables, and 2ω is the angle between the two conjugate geodesics. It is remarkable that, among the three

Gauss-Codazzi equations [4, p. 362]

$$\begin{cases} \omega_{uv} - \dfrac{1}{2}\sin(2\omega) = 0, \\ X_v - \cos(2\omega) Y_v = 0, \\ Y_u - \cos(2\omega) X_u = 0, \end{cases} \tag{1.2}$$

the first one characterizes the surfaces with a constant total (Gauss) curvature.

2. Their Lax pair

Gambier succeeded in introducing a deformation parameter λ, thus upgrading the moving frame equations to a Lax pair, but he did not write this Lax pair explicitly, so let us do it here.

The moving frame equations (Gauss-Weingarten equations) only depend on the coefficients of the first and second fundamental forms, and the spectral parameter is introduced, as in the case of surfaces with a constant mean curvature, by noticing the invariance of the Gauss-Codazzi equations (1.2) under the scaling transformation $(u,v) \to (\lambda u, \lambda^{-1} v)$. The traceless Lax pair is

$$\partial_u \psi = L\psi,\ \partial_v \psi = M\psi, \tag{2.1}$$

$$L = \begin{pmatrix} \dfrac{2X_{uu}}{3X_u} + \dfrac{2X_u}{3X_v}\operatorname{cotg}(2\omega)\omega_v & 0 & Y_u \operatorname{tg}(2\omega) \\ -\dfrac{2Y_v}{\lambda^2 Y_u}\operatorname{cotg}(2\omega)\omega_u & -\dfrac{X_{uu}}{3X_u} - \dfrac{4X_u}{3X_v}\operatorname{cotg}(2\omega)\omega_v & 0 \\ -\dfrac{1}{\lambda^2 Y_u}\operatorname{cotg}(2\omega) & \dfrac{1}{Y_v}\operatorname{cotg}(2\omega) & -\dfrac{X_{uu}}{3X_u} + \dfrac{2X_u}{3X_v}\operatorname{cotg}(2\omega)\omega_v \end{pmatrix}, \tag{2.2}$$

$$M = \begin{pmatrix} -\dfrac{Y_{vv}}{3Y_v} - \dfrac{4Y_v}{3Y_u}\operatorname{cotg}(2\omega)\omega_u & -\dfrac{2\lambda^2 X_u}{X_v}\operatorname{cotg}(2\omega)\omega_v & 0 \\ 0 & \dfrac{2Y_{vv}}{3Y_v} + \dfrac{2Y_v}{3Y_u}\operatorname{cotg}(2\omega)\omega_u & X_v \operatorname{tg}(2\omega) \\ \dfrac{1}{X_u}\operatorname{cotg}(2\omega) & -\dfrac{\lambda^2}{X_v}\operatorname{cotg}(2\omega) & -\dfrac{Y_{vv}}{3Y_v} + \dfrac{2Y_v}{3Y_u}\operatorname{cotg}(2\omega)\omega_u \end{pmatrix}, \tag{2.3}$$

with the zero-curvature condition,

$$[\partial_u - L, \partial_v - M] = \begin{pmatrix} -FE_1 & EE_1 & -(EG-F^2)E_2 \\ -GE_1 & FE_1 & -(EG-F^2)E_3 \\ GE_2 - FE_3 & -FE_2 + EE_3 & 0 \end{pmatrix} = 0, \tag{2.4}$$

denoting E_j, $j = 1, 2, 3$ the lhs of (1.2), and E, F, G the coefficients of the first fundamental form,

$$E = X_u^2,\ \ F = X_u Y_v \cos(2\omega),\ \ G = Y_v^2. \tag{2.5}$$

3. Surfaces applicable on a surface of revolution

Gambier [3, p. 99] investigated surfaces whose first fundamental form I, Eq. (1.1), has coefficients X_u, Y_v, ω only depending on the single variable $x = u + v$. Denoting for shortness $X_u = \xi$, $Y_v = \eta$, he first obtains

$$
\begin{aligned}
& \mathrm{d}X = \xi \mathrm{d}u + (\xi + 2C_1)\mathrm{d}v,\ \mathrm{d}Y = (\eta + 2C_2)\mathrm{d}u + \eta \mathrm{d}v, \\
& \xi + 2C_1 = \eta \cos(2\omega),\ \eta + 2C_2 = \xi \cos(2\omega),
\end{aligned} \tag{3.1}
$$

with C_1, C_2 two integration constants. After a possible conformal transformation, this defines two reductions of the Gauss-Codazzi equations, either

$$
\frac{\mathrm{d}^2\omega}{\mathrm{d}x^2} = \frac{m^2}{2}\sin(2\omega),\ m = \text{arbitrary constant}, \tag{3.2}
$$

or

$$
\frac{\mathrm{d}^2\omega}{\mathrm{d}x^2} = \frac{e^x}{2}\sin(2\omega). \tag{3.3}
$$

The first reduction (3.2) integrates with elliptic functions and is handled in full detail by Gambier [3, pp. 100–105].

As to the second reduction (3.3), Gambier unexpectedly fails to integrate it. This ordinary differential equation (ODE) belongs to the class of second order first degree ODEs

$$
\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \sum_{j=0}^{3} A_j(x,u)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^j = 0, \tag{3.4}
$$

whose property is to be form-invariant under the group of point transformations

$$
(u,x) \to (U,X):\ u = \varphi(X,U),\ x = \psi(X,U),\ U = \Phi(x,u),\ X = \Psi(x,u). \tag{3.5}
$$

Roger Liouville [7] enumerated equivalence classes of (3.4) *modulo* the group (3.5) but, as later pointed out by Babich and Bordag [1], he forgot the important class, to which the ODE (3.3) belongs, when the invariants which he denotes ν_5 and w_1 both vanish.

When ν_5 and w_1 both vanish, the coefficients A_3, A_2, A_1 in the class (3.4) can be set to zero by a transformation (3.5), thus defining the five remarkable four-parameter nonautonomous ODEs

$$
\begin{aligned}
& \frac{\mathrm{d}^2U}{\mathrm{d}X^2} = \frac{(2\omega)^3}{\pi^2} \sum_{j=\infty,0,1,x} \theta_j^2 \wp'(2\omega U + \omega_j, g_2, g_3), \\
& \frac{\mathrm{d}^2U}{\mathrm{d}X^2} = -2\alpha \frac{\cosh U}{\sinh^3 U} - 2\beta \frac{\sinh U}{\cosh^3 U} - 2\gamma e^{2X}\sinh(2U) - \frac{1}{2}\delta e^{4X}\sinh(4U), \\
& \frac{\mathrm{d}^2U}{\mathrm{d}X^2} = \frac{1}{2}e^X(\alpha e^{2U} + \beta e^{-2U}) + \frac{1}{2}e^{2X}(\gamma e^{4U} + \delta e^{-4U}), \\
& \frac{\mathrm{d}^2U}{\mathrm{d}X^2} = -\alpha U + \frac{\beta}{2U^3} + \gamma\left(\frac{3}{4}U^5 + 2XU^3 + X^2U\right) + 2\delta(U^3 + XU), \\
& \frac{\mathrm{d}^2U}{\mathrm{d}X^2} = \delta(2U^3 + XU) + \gamma(6U^2 + X) + \beta U + \alpha,
\end{aligned}
$$

in which the summation in the first equation runs over the four half-periods ω_j of the Weierstrass elliptic function $\wp$.

The third one is precisely, up to rescaling, the ODE (3.3) isolated by Gambier, and the main result of Ref. [1] is the existence of a point transformation mapping these five four-parameter ODEs to the representation of the Painlevé equations chosen by Garnier [5], [2] (i.e. five equations with four parameters, the last one unifying P_{II} and P_I),

$$\begin{aligned}
P_{VI} &: u'' = \frac{1}{2}\left[\frac{1}{u}+\frac{1}{u-1}+\frac{1}{u-x}\right]u'^2 - \left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{u-x}\right]u' \\
&\quad + \frac{u(u-1)(u-x)}{x^2(x-1)^2}\left[\alpha+\beta\frac{x}{u^2}+\gamma\frac{x-1}{(u-1)^2}+\delta\frac{x(x-1)}{(u-x)^2}\right], \\
P_V &: u'' = \left[\frac{1}{2u}+\frac{1}{u-1}\right]u'^2 - \frac{u'}{x} + \frac{(u-1)^2}{x^2}\left[\alpha u+\frac{\beta}{u}\right] + \gamma\frac{u}{x} + \delta\frac{u(u+1)}{u-1}, \\
P_{III} &: u'' = \frac{u'^2}{u} - \frac{u'}{x} + \frac{\alpha u^2+\gamma u^3}{4x^2} + \frac{\beta}{4x} + \frac{\delta}{4u}, \\
P_{IV}' &: u'' = \frac{u'^2}{2u} + \gamma\left(\frac{3}{2}u^3+4xu^2+2x^2u\right) + 4\delta(u^2+xu) - 2\alpha u + \frac{\beta}{u}, \\
P_{II}' &: u'' = \delta(2u^3+xu) + \gamma(6u^2+x) + \beta u + \alpha.
\end{aligned} \tag{3.6}$$

The point transformations which realize this mapping are, respectively,

$$\begin{aligned}
& x = \frac{e_3-e_1}{e_2-e_1},\ u = \frac{\wp(2\omega U, g_2, g_3)-e_1}{e_2-e_1}, \\
& x = e^{2X},\ u = \coth^2 U, \\
& x = e^{2X},\ u = e^X e^{2U}, \\
& x = X,\ u = U^2, \\
& x = X,\ u = U.
\end{aligned}$$

Therefore the mapping between the ODE (3.3) for $\omega(x)$ and the third Painlevé equation (3.6) for $u(\xi)$ is either

$$e^{2i\omega} = 2\alpha e^{-x}u,\ \xi = -\frac{1}{4\alpha\beta}e^{2x},\ \alpha\beta \neq 0,\ \gamma = 0,\ \delta = 0, \tag{3.7}$$

or equivalently

$$e^{2i\omega} = \gamma e^{-x}u^2,\ \xi = \sqrt{-\frac{1}{\gamma\delta}}e^x,\ \alpha = 0,\ \beta = 0,\ \gamma\delta \neq 0. \tag{3.8}$$

As is well known, the third Painlevé equation has three kinds of solutions:

(i) two-parameter transcendental solutions, which is the generic case, and one cannot proceed beyond the description of Gambier [3, pp. 105–106];
(ii) one-parameter Riccati-type solutions, but for our case $\gamma\delta \neq 0$ this does not happen;
(iii) zero-parameter rational[a] solutions, the only ones being, with the choice (3.7),

$$u = \left(-\frac{\beta}{\alpha}\right)^{1/2}\xi^{1/2},\ \gamma = 0,\ \delta = 0, \tag{3.9}$$

[a]Algebraic solutions of (3.6) [8] are in fact rational solutions for another representative of P_{III} in its equivalence class under $(u,x) \to (g(x)u, f(x))$, with $f(x) = \sqrt{x}$, $g(x) = 1$. All algebraic solutions of P_n equations, $n = 2, 3, 4, 5$, are similarly rational.

or equivalently with the choice (3.8),

$$u = \left(-\frac{\delta}{\gamma}\right)^{1/4} \xi^{1/2},\ \alpha = 0,\ \beta = 0. \tag{3.10}$$

However, these rational solutions correspond to $\sin(2\omega) = 0$, forbidden because the second fundamental form would vanish. Consequently, all solutions of (3.3) are transcendental.

4. Future developments

The equation $(1.2)_1$ for constant total curvature surfaces (sine-Gordon equation) possesses many closed form solutions which obey neither (3.2) nor (3.3), for instance the factorized solution [10]

$$\operatorname{tg}\frac{\omega}{2} = \frac{J_1(u+v)}{J_2(u-v)}, \tag{4.1}$$

in which J_1 and J_2 are Jacobi elliptic functions, a degeneracy of which is

$$\operatorname{tg}\frac{\omega}{2} = \frac{\sin k(u+v)}{\sin k(u-v)}, \tag{4.2}$$

or the N-soliton solution [9], which depends on $2N$ arbitrary constants. The difficulty to build Voss-Guichard surfaces from such solutions is the integration of the linear system $(1.2)_{2,3}$ for $X(u,v)$ and $Y(u,v)$.

Another useful development would be to find a Darboux transformation for the system (1.2).

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