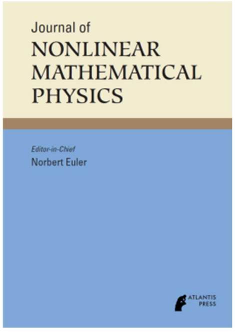

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On the modified discrete KP equation with self-consistent sources

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The modified discrete KP equation is the Bäcklund transformation for the Hirota's discrete KP equation or the Hirota-Miwa equation. We construct the modified discrete KP equation with self-consistent sources via source generation procedure and clarify the algebraic structure of the resulting coupled modified discrete KP system by presenting its discrete Gram-type determinant solutions. It is also shown that the commutativity between the source generation procedure and Bäcklund transformation is valid for the discrete KP equation. Finally, we demonstrate that the modified discrete KP equation with self-consistent sources yields the modified differential-difference KP equation with self-consistent sources through a continuum limit. The continuum limit of an explicit solution to the modified discrete KP equation with self-consistent sources also gives the explicit solution for the modified differential-difference KP equation with self-consistent sources.

Keywords: Modified discrete KP equation, Source generalization procedure, Discrete Gram-type determinant, Continuum limit

2000 Mathematics Subject Classification: 37K10, 37K40

1. Introduction

The study of discrete integrable systems has attracted a lot of interest in recent years. The most important and widely studied three-dimensional discrete integrable system is the Hirota-Miwa equation (or discrete KP equation in bilinear form) [1]- [3]:

$$\begin{aligned}&a_1(a_2-a_3)f(k_1+a_1)f(k_2+a_2,k_3+a_3)+a_2(a_3-a_1)f(k_2+a_2)f(k_1+a_1,k_3+a_3)\\&+a_3(a_1-a_2)f(k_3+a_3)f(k_1+a_1,k_2+a_2)=0 \qquad (1.1)\end{aligned}$$

where $f=f(k_1,k_2,k_3)$ and a_1,a_2,a_3 are the difference intervals for discrete independent variables k_1,k_2,k_3, respectively. We omit the dependence on unshifted variables in the above equation and in the following notations. The Hirota-Miwa equation (1.1) is considered as a master equation which by performing scaling continuum limit gives many well-known semi-discrete and continuous integrable systems.

Under the variable transformations $k_1 \to a_1k_1, k_2 \to a_2k_2, k_3 \to a_3k_3$, the discrete KP equation (1.1) can be written in a compact form by using bilinear operator:

$$[a_1(a_2-a_3)e^{\frac{1}{2}D_{k_1}-\frac{1}{2}D_{k_2}-\frac{1}{2}D_{k_3}}+a_2(a_3-a_1)e^{\frac{1}{2}D_{k_2}-\frac{1}{2}D_{k_1}-\frac{1}{2}D_{k_3}}+a_3(a_1-a_2)e^{\frac{1}{2}D_{k_3}-\frac{1}{2}D_{k_1}-\frac{1}{2}D_{k_2}}]f\cdot f=0, \tag{1.2}$$

where the bilinear operator is defined by

$$e^{\delta D_n}a(n)\cdot b(n)=a(n+\delta)b(n-\delta). \tag{1.3}$$

The discrete KP equation (1.1) or (1.2) has the following Bäcklund transformation [1,4]:

$$[a_1e^{\frac{1}{2}D_{k_1}-\frac{1}{2}D_{k_2}}-a_2e^{\frac{1}{2}D_{k_2}-\frac{1}{2}D_{k_1}}+(a_2-a_3)e^{\frac{1}{2}D_{k_1}+\frac{1}{2}D_{k_2}}]f\cdot g=0 \tag{1.4}$$
$$[a_1e^{\frac{1}{2}D_{k_1}-\frac{1}{2}D_{k_3}}-a_1e^{\frac{1}{2}D_{k_3}-\frac{1}{2}D_{k_1}}+(a_3-a_1)e^{\frac{1}{2}D_{k_1}+\frac{1}{2}D_{k_3}}]f\cdot g=0. \tag{1.5}$$

Through the variable transformations $k_1 \to \frac{1}{a_1}k_1, k_2 \to \frac{1}{a_2}k_2, k_3 \to \frac{1}{a_3}k_3$, the modified discrete KP equations (1.4-1.5) become

$$a_1f(k_1+a_1)g(k_2+a_2)-a_2f(k_2+a_2)g(k_1+a_1)+(a_2-a_1)f(k_1+a_1,k_2+a_2)g=0 \tag{1.6}$$
$$a_1f(k_1+a_1)g(k_3+a_3)-a_3f(k_3+a_3)g(k_1+a_1)+(a_3-a_1)f(k_1+a_1,k_3+a_3)g=0. \tag{1.7}$$

Soliton equations with self-consistent sources, describing important physical processes, have been studied by means of various mathematical approachs such as inverse scattering methods [5]-[9], Darboux transformation methods [10]- [13], Hirota's bilinear method and Wronskian technique [14]- [22]. However, most results have been achieved in continuous case. Comparatively less work has been done in discrete case, especially in the fully discrete case.

In [23], the authors propose a new algebraic method, called the source generalization procedure, to construct and solve the soliton equations with self consistent sources both in continuous and discrete cases. The purpose of this paper is to utilize the source generalization procedure to the modified discreet KP equation (1.6-1.7) to construct the modified discrete KP equation with self-consistent sources and clarify the bilinear structures of the modified discrete KP equations with self-consistent sources. Furthermore, we verity that the commutativity of the source generation procedure and Bäcklund transformation holds for the fully discrete case.

The outline of this paper is as follows. In Sec.2, we construct the modified discrete KP equations with self-consistent sources and give their discrete Gram-type determinant solutions. Sec.3 is devoted to show that the commutativity of the source generation procedure and Bäcklund transformation is valid for the discrete KP equation. We end this paper with a summary and discussion in section 4.

2. The modified discrete KP equation with self-consistent sources

In this section, we construct the modified discrete KP equation with self-consistent sources via source generation procedure. Furthermore, we present the discrete Gram-type determinant solution for the modified discrete KP equation with self-consistent sources and clarify its bilinear structure.

It is given in [24] that the bilinear difference equations (1.6-1.7) have the following discrete Gram-type determinant solution:

$$f(k_1,k_2,k_3) = |M|, \tag{2.1}$$

$$g(k_1,k_2,k_3) = \begin{vmatrix} M & \Phi(-1) \\ \bar{\Phi}(0)^T & 1 \end{vmatrix}. \tag{2.2}$$

where $M = \det(c_{ij}+m_{ij})_{1\le i,j\le N}$ in which $c_{ij}(1\le i,j\le N)$ is arbitrary constant and the matrix element $m_{ij}(1\le i,j\le N)$ is an function of k_1,k_2,k_3 satisfying the difference equation

$$\begin{aligned} &\Delta_{+k_\nu} m_{ij} = \phi_i(k_\nu+a_\nu;0)\bar{\phi}_j(0), \qquad (2.3)\\ &i,j=1,2,\cdots,N, \nu=1,2,3, \end{aligned}$$

where unshifted independent variables are suppressed and ϕi and $\bar{\phi}_i$ are arbitrary functions of k_1,k_2,k_3 and an integer s, satisfying the dispersion relations

$$\Delta_{-k_\nu}\phi_i(k_1,k_2,k_3,s) = \phi_i(k_1,k_2,k_3,s+1), \tag{2.4}$$
$$\Delta_{+k_\nu}\bar{\phi}_i(k_1,k_2,k_3,s) = \bar{\phi}_i(k_1,k_2,k_3,s+1), \tag{2.5}$$

where Δ_{-k_ν},Δ_{+k_ν} are defined by

$$\begin{aligned} \Delta_{-k_\nu}F(k_\nu) &= \frac{F(k_\nu)-F(k_\nu-a_\nu)}{a_\nu},\\ \Delta_{+k_\nu}F(k_\nu) &= \frac{F(k_\nu+a_\nu)-F(k_\nu)}{a_\nu}, \nu=0,1,2,3. \end{aligned}$$

In addition, $\Phi(s),\bar{\Phi}(s)$ (s is an integer) are Nth column vectors defined by

$$\Phi(s) = (\phi_1(s),\phi_2(s),\cdots,\phi_N(s))^T, \tag{2.6}$$
$$\bar{\Phi}(s) = (\bar{\phi}_1(s),\bar{\phi}_2(s),\cdots,\bar{\phi}_N(s))^T. \tag{2.7}$$

In order to construct the modified fully discrete KP equation with self-consistent sources, we change the discrete Gram-type determinant solutions (2.1-2.2) into the following form:

$$f(k_1,k_2,k_3) = \det(c_{ij}(k_1)+m_{ij})_{1\le i,j\le N} = |D|, \tag{2.8}$$

$$g(k_1,k_2,k_3) = \begin{vmatrix} D & \Phi(-1) \\ \bar{\Phi}(0)^T & 1 \end{vmatrix}, \tag{2.9}$$

where $N\times N$ matrix $D=(c_{ij}(k_1)+m_{ij})_{1\le i,j\le N}$ and $m_{ij},\Phi(s),\bar{\Phi}(s)$ are given in (2.3-2.7). In addition, $c_{ij}(k_1)$ satisfies

$$c_{ij}(k_1) = \begin{cases} c_i(k_1), & 1\le i\le K\le N \text{ and } j=1,2,\cdots,K, K\in Z^+,\\ d_{ij}, & \text{otherwise}, \end{cases} \tag{2.10}$$

with $c_i(k_1)$ being an arbitrary function of k_1 and K being a positive integer. Then we have the following difference formula by employing Eqs. (2.1-2.5):

$$f(k_1,k_2,k_3)=\begin{vmatrix} M_d & C(k_1)\\ \alpha^T & 1\end{vmatrix},\qquad g(k_1,k_2,k_3)=\begin{vmatrix} M_d & C(k_1) & \Phi(-1)\\ a^T & 1 & 0\\ \bar{\Phi}(0)^T & 0 & 1\end{vmatrix},\tag{2.11}$$

$$f(k_\mu+a_\mu)=\alpha_\mu\begin{vmatrix} M_d & C(k_1) & \Phi(0_\mu)\\ \alpha^T & 1 & 0\\ -\bar{\Phi}(0)^T & 0 & a_\mu^{-1}\end{vmatrix},\qquad f(k_1+a_1)=a_1\begin{vmatrix} M_d & C(k_1+a_1) & \Phi(0_1)\\ \alpha^T & 1 & 0\\ -\bar{\Phi}(0)^T & 0 & a_1^{-1}\end{vmatrix},\tag{2.12}$$

$$f(k_2+a_2,k_3+a_3)=\frac{(a_2a_3)^2}{(a_2-a_3)}\begin{vmatrix} M_d & C(k_1) & \Phi(0_2) & \Phi(0_3)\\ \alpha^T & 1 & 0 & 0\\ -\bar{\Phi}(1)^T & 0 & -a_2^{-2} & -a_3^{-2}\\ -\bar{\Phi}(0)^T & 0 & a_2^{-1} & a_3^{-1}\end{vmatrix},\tag{2.13}$$

$$f(k_1+a_1,k_\mu+a_\mu)=\frac{(a_1a_\mu)^2}{(a_1-a_\mu)}\begin{vmatrix} M_d & C(k_1+a_1) & \Phi(0_1) & \Phi(0_\mu)\\ \alpha^T & 1 & 0 & 0\\ -\bar{\Phi}(1)^T & 0 & -a_1^{-2} & -a_\mu^{-2}\\ -\bar{\Phi}(0)^T & 0 & a_1^{-1} & a_\mu^{-1}\end{vmatrix},\tag{2.14}$$

$$g(k_\mu+a_\mu)=-a_\mu^2\begin{vmatrix} M_d & C(k_1) & \Phi(-1) & \Phi(0_\mu)\\ \alpha^T & 1 & 0 & 0\\ \bar{\Phi}(1)^T & 0 & 0 & a_\mu^{-2}\\ \bar{\Phi}(0)^T & 0 & 1 & -a_\mu^{-1}\end{vmatrix},g(k_1+a_1)=-a_1^2\begin{vmatrix} M_d & C(k_1+a_1) & \Phi(-1) & \Phi(0_1)\\ \alpha^T & 1 & 0 & 0\\ \bar{\Phi}(1)^T & 0 & 0 & a_1^{-2}\\ \bar{\Phi}(0)^T & 0 & 1 & -a_1^{-1}\end{vmatrix},\tag{2.15}$$

where $\mu=2,3$; $C(k_1),\alpha,\Phi(s_\nu)$ are Nth column vectors given by

$$\begin{aligned}
&C(k_1)=(c_1(k_1),\cdots,c_K(k_1),\ 0,\cdots,0)^T,\\
&\alpha=(-1,\cdots,-1,\ 0,\cdots,0)^T,\qquad \text{number of}-1=K,\\
&\Phi(s_\nu)=(\phi_1(k_\nu+a_\nu;s),\phi_2(k_\nu+a_\nu;s)\cdots,\phi_N(k_\nu+a_\nu;s);)^T,\nu=1,2,3,
\end{aligned}$$

and M_d is given by

$$M_d=\begin{vmatrix}
m_{11} & \cdots & m_{1k} & m_{1,k+1}+d_{1,k+1} & \cdots & m_{1N}+d_{1N}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
m_{k1} & \cdots & m_{kk} & m_{k,k+1}+d_{k,k+1} & \cdots & m_{kN}+d_{kN}\\
m_{k+1,1}+d_{k+1,1} & \cdots & m_{k+1,k}+d_{k+1,k} & m_{k+1,k+1}+d_{k+1,k+1} & \cdots & m_{k+1,N}+d_{k+1,N}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
m_{N1}+d_{N1} & \cdots & m_{Nk}+d_{Nk} & m_{N,k+1}+d_{N,k+1} & \cdots & m_{NN}+d_{NN}
\end{vmatrix}.\tag{2.16}$$

According to the above results, f, g defined in (2.1-2.2) will not satisfy the modified fully discrete KP equation (1.6-1.7) again. In fact, they satisfy the following equations:

$$\begin{aligned}
&a_1 f(k_1+a_1)g(k_2+a_2) - a_2 f(k_2+a_2)g(k_1+a_1) + (a_2-a_1)f(k_1+a_1,k_2+a_2)g(k_1,k_2,k_3)\\
&\qquad = a_2 \sum_{i,j=1}^{K} P_{i,j}(k_1+a_1)Q_{i,j}(k_2+a_2), \qquad (2.17)\\
&a_1 f(k_1+a_1)g(k_3+a_3) - a_3 f(k_3+a_3)g(k_1+a_1) + (a_3-a_1)f(k_1+a_1,k_3+a_3)g(k_1,k_2,k_3,)\\
&\qquad = a_3 \sum_{i,j=1}^{K} P_{i,j}(k_1+a_1)Q_{i,j}(k_3+a_3), \qquad (2.18)
\end{aligned}$$

where $P_{i,j}, Q_{i,j}$ for $r,m = 1,\cdots,K$ are functions of variables k_1,k_2,k_3 given by

$$P_{i,j}(k_1,k_2,k_3) = \sqrt{c_i(k_1)-c_i(k_1-a_1)} \begin{vmatrix} D_{ij} & R_{ij} \\ \bar{\Phi}_j(0)^T & \bar{\phi}_j(0) \end{vmatrix}, \qquad (2.19)$$

$$Q_{i,j}(k_1,k_2,k_3) = -\sqrt{\Delta c_i(k_1)} \begin{vmatrix} D_{ij} & \Phi_i(-1) \\ M_{ij}^T & \phi_i(-1) \end{vmatrix}, \qquad (2.20)$$

in which D_{ij} is the $(N-1)$th order matrice obtained by eliminating the ith row and the jth column from the matrice $(c_{i,j}(k_2)+m_{ij}(k_1,k_2,k_3))_{1\le i,j\le N}$; $\Phi_j(s), \bar{\Phi}_j(s)$ are $(N-1)$th column vectors obtained by eliminating the jth element from $\Phi(s)$ and $\bar{\Phi}(s)$, respectively; $\Delta c_i(k_1)$ is defined by

$$\Delta c_i(k_1) = c_i(k_1+a_1) - c_i(k_1) \qquad (2.21)$$

and R_{ij}, M_{ij} are given by

$$\begin{aligned}
R_{ij} &= (c_1(k_1)+m_{1j},\cdots,c_{i-1}(k_1)+m_{i-1,j},c_{i+1}(k_1)+m_{i+1,j},\cdots,\\
&\qquad c_K(k_1)+m_{K,j},d_{K+1,j}+m_{K+1,j},\cdots,d_{N,j}+m_{N,j})^T,\\
M_{ij} &= (c_i(k_1)+m_{i1},\cdots,c_i(k_1)+m_{i,j-1},c_i(k_1)+m_{i,j+1},\cdots,\\
&\qquad c_i(k_1)+m_{i,K},d_{i,K+1}+m_{i,K+1},\cdots,d_{i,N}+m_{i,N})^T.
\end{aligned}$$

We can also show that functions (2.1-2.2) and new fields (2.19-2.20) for $r,m = 1,\cdots,K$ satisfy the following bilinear equations:

$$\begin{aligned}
&a_2 f(k_2+a_2)P_{ij}(k_3+a_3) - a_3 f(k_3+a_3)P_{ij}(k_2+a_2) +\\
&\qquad (a_3-a_2)f(k_2+a_2,k_3+a_3)P_{ij}(k_1,k_2,k_3) = 0, \qquad (2.22)\\
&a_2 Q_{ij}(k_2+a_2)g(k_3+a_3) - a_3 Q_{ij}(k_3+a_3)g(k_2+a_2) +\\
&\qquad (a_3-a_2)Q_{ij}(k_2+a_2,k_3+a_3)g(k_1,k_2,k_3) = 0. \qquad (2.23)
\end{aligned}$$

So equations (2.17-2.18,2.22-2.23) constitute the modified discrete KP equation with self-consistent sources. In the following, we will verify that functions (2.1-2.2) and (2.19-2.20) for $r,m = 1,\cdots,K$ solve bilinear equations (2.17-2.18,2.22-2.23).

We can compute the following difference formula for $P_{i,j}, Q_{i,j}$, $r,m = 1,\cdots,K$ by employing equations (2.3-2.5):

$$P_{ij}(k_1+a_1)=a_1^2\sqrt{\Delta C_i(k_1)}\begin{vmatrix} E_{ij} & \Phi_i(0_1) & N_{ij} \\ -\bar{\Phi}_j(1)^T & -a_1^{-2} & -\bar{\phi}_j(1)^T \\ -\bar{\Phi}_j(0)^T & a_1^{-1} & -\bar{\phi}_j(0)^T \end{vmatrix}, \tag{2.24}$$

$$P_{i,j}(k_\nu+a_\nu)=-a_\nu^2\sqrt{c_i(k_1)-c_i(k_1-a_1)}\begin{vmatrix} D_{ij} & R_{ij} & \Phi_i(0_\nu) \\ -\bar{\Phi}_j(1)^T & -\bar{\phi}_j(1) & -a_\nu^{-2} \\ -\bar{\Phi}_j(0)^T & -\bar{\phi}_j(0) & a_\nu^{-1} \end{vmatrix}, \tag{2.25}$$

$$Q_{i,j}(k_\nu+a_\nu)=a_\nu\sqrt{\Delta c_i(k_1)}\begin{vmatrix} D_{ij} & \Psi_i(-1) & \Psi_i(0_\nu) \\ -\Phi_j(0)^T & -1 & a_\nu^{-1} \\ M_{ij}^T & \varphi_i(-1) & \varphi_i(0_\nu) \end{vmatrix}, \tag{2.26}$$

$$Q_{i,j}(k_2+a_2,k_3+a_3)=\frac{a_2^2a_3^2}{a_2-a_3}\sqrt{\Delta c_i(k_1)}\begin{vmatrix} D_{ij} & \Phi_i(-1) & \Phi_i(0_2) & \Phi_i(0_3) \\ -\bar{\Phi}_j(1)^T & 0 & -a_2^{-2} & -a_3^{-2} \\ M_{ij}^T & \phi_i(-1) & \phi_i(0_2) & \phi_i(0_3) \\ -\bar{\Phi}_j(0)^T & -1 & a_2^{-1} & a_3^{-1} \end{vmatrix}, \tag{2.27}$$

where $\nu=2,3$, E_{ij} is the $(N-1)$th order matrice obtained by eliminating the ith row and the jth column from the matrix $(c_{i,j}(k_1+a_1)+m_{ij}(k_1,k_2,k_3))_{1\le i,j\le N}$, respectively; $\Phi_i(s_\nu)$ is $(N-1)$th column vector obtained by eliminating the jth element from $\Phi(s_\nu)$. In addition, N_{ij} is given by

$$\begin{aligned} N_{ij}=(&c_1(k_1+a_1)+m_{ij},\cdots,c_{i-1}(k_1+a_1)+m_{i-1,j},c_{i+1}(k_1+a_1)+m_{i+1,j},\cdots,\\ &c_K(k_1+a_1)+m_{K,j},d_{K+1,j}+m_{K+1,j},\cdots,d_{N,j}+m_{N,j})^T. \end{aligned}$$

Substituting equations (2.11-2.15) and (2.24-2.27) into modified discrete KP equation (1.6), we obtain the following determinant identity:

$$\begin{aligned}
&-\begin{vmatrix} M_d & C(k_1+a_1) & \Phi(0_1) \\ a^T & 1 & 0 \\ -\bar{\Phi}(0)^T & 0 & a_1^{-1} \end{vmatrix}\begin{vmatrix} M_d & C(k_1) & \Phi(-1) & \Phi(0_2) \\ a^T & 1 & 0 & 0 \\ -\bar{\Phi}(1)^T & 0 & 0 & -a_2^{-2} \\ -\bar{\Phi}(0)^T & 0 & -1 & a_2^{-1} \end{vmatrix}\\
&+\begin{vmatrix} M_d & C(k_1) & \Phi(0_2) \\ a^T & 1 & 0 \\ -\bar{\Phi}(0)^T & 0 & a_2^{-1} \end{vmatrix}\begin{vmatrix} M_d & C(k_1+a_1) & \Phi(-1) & \Phi(0_1) \\ a^T & 1 & 0 & 0 \\ -\bar{\Phi}(1)^T & 0 & 0 & -a_1^{-2} \\ -\bar{\Phi}(0)^T & 0 & -1 & a_1^{-1} \end{vmatrix}\\
&+\begin{vmatrix} M_d & C(k_1+a_1) & \Phi(0_1) & \Phi(0_2) \\ a^T & 1 & 0 & 0 \\ -\bar{\Phi}(1)^T & 0 & -a_1^{-2} & -a_2^{-2} \\ -\bar{\Phi}(0)^T & 0 & a_1^{-1} & a_2^{-1} \end{vmatrix}\begin{vmatrix} M_d & C(k_1) & \Phi(-1) \\ a^T & 1 & 0 \\ -\bar{\Phi}(0)^T & 0 & -1 \end{vmatrix}\\
&-\begin{vmatrix} M_d & \Phi(-1) & \Phi(0_2) \\ a^T & 1 & 0 \\ -\bar{\Phi}(0)^T & -1 & a_2^{-1} \end{vmatrix}\begin{vmatrix} M_d & C(k_1+a_1) & \Phi(0_1) & C(k_1) \\ a^T & 1 & 0 & 1 \\ -\bar{\Phi}(1)^T & 0 & -a_1^{-2} & 0 \\ -\bar{\Phi}(0)^T & 0 & a_1^{-1} & 0 \end{vmatrix}=0.
\end{aligned} \tag{2.28}$$

In order to show that determinant identity (2.28) holds, let us introduce the following $2(N+3)\times 2(N+3)$ determinant which vanishes,

$$\begin{vmatrix} M_d & 0 & 0 & 0 & 0 & C(k_1) & \Phi(-1) & \Phi(0_2) \\ \alpha^T & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\bar{\Phi}(1)^T & 0 & 0 & 0 & 1 & 0 & 0 & -a_2^{-2} \\ -\bar{\Phi}(0)^T & 0 & 0 & 0 & 0 & 0 & -1 & a_2^{-1} \\ 0 & M_d & C(k_1+a_1) & \Phi(0_1) & 0 & C(k_1) & \Phi(-1) & \Phi(0_2) \\ 0 & a^T & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\bar{\Phi}(1)^T & 0 & -a_1^{-2} & 1 & 0 & 0 & -a_2^{-2} \\ 0 & -\bar{\Phi}(0)^T & 0 & a_1^{-1} & 0 & 0 & -1 & a_2^{-1} \end{vmatrix} = 0. \tag{2.29}$$

Applying the Laplace expansion in $(N+3)\times(N+3)$ minors to the left-hand side of (2.29), we obtain the determinant identity (2.28). Therefore, functions $f, g, P_{i,j}, Q_{i,j}$ given in equations (2.1-2.2,2.19-2.20) are solutions of equation (1.6). In the same way, substitution of equations (2.11-2.15,2.24-2.27) into equation (1.7) gives the determinant identity:

$$\begin{aligned} &-\begin{vmatrix} M_d & C(k_1+a_1) & \Phi(0_1) \\ \alpha^T & 1 & 0 \\ -\bar{\Phi}(0)^T & 0 & a_1^{-1} \end{vmatrix} \begin{vmatrix} M_d & C(k_1) & \Phi(-1) & \Phi(0_3) \\ \alpha^T & 1 & 0 & 0 \\ -\bar{\Phi}(1)^T & 0 & 0 & -a_3^{-2} \\ -\bar{\Phi}(0)^T & 0 & -1 & a_3^{-1} \end{vmatrix} \\ &+\begin{vmatrix} M_d & C(k_1) & \Phi(0_3) \\ \alpha^T & 1 & 0 \\ -\bar{\Phi}(0)^T & 0 & a_3^{-1} \end{vmatrix} \begin{vmatrix} M_d & C(k_1+a_1) & \Phi(-1) & \Phi(0_1) \\ \alpha^T & 1 & 0 & 0 \\ -\bar{\Phi}(1)^T & 0 & 0 & -a_3^{-2} \\ -\bar{\Phi}(0)^T & 0 & -1 & a_3^{-1} \end{vmatrix} \\ &+\begin{vmatrix} M_d & C(k_1) & \Phi(-1) \\ \alpha^T & 1 & 0 \\ -\bar{\Phi}(0)^T & 0 & -1 \end{vmatrix} \begin{vmatrix} M_d & C(k_1+a_1) & \Phi(0_1) & \Phi(0_3) \\ \alpha^T & 1 & 0 & 0 \\ -\bar{\Phi}(1)^T & 0 & -a_1^{-2} & -a_3^{-2} \\ -\bar{\Phi}(0)^T & 0 & a_1^{-1} & a_3^{-1} \end{vmatrix} \\ &-\begin{vmatrix} M_d & \Phi(-1) & \Phi(0_3) \\ \alpha^T & 1 & 0 \\ -\bar{\Phi}(0)^T & -1 & a_3^{-1} \end{vmatrix} \begin{vmatrix} M_d & C(k_1+a_1) & \Phi(0_1) & C(k_1) \\ \alpha^T & 1 & 0 & 1 \\ -\bar{\Phi}(1)^T & 0 & -a_1^{-2} & 0 \\ -\bar{\Phi}(0)^T & 0 & a_1^{-1} & 0 \end{vmatrix} = 0, \end{aligned} \tag{2.30}$$

which can also be proved similarly.

Now we prove f, g given in (2.1-2.2) and $P_{i,j}, Q_{i,j}$ for $r, m = 1, \cdots, K$ given in (2.19-2.20) are solutions to equations (2.22-2.23). Substituting equations (2.11-2.15,2.19,2.24-2.27) into equations (2.22-2.23), we get the following two identities for the determinant, respectively:

$$\begin{aligned} &-\begin{vmatrix} M_d & C(k_1) & \Phi(0_2) \\ \alpha^T & 1 & 0 \\ -\Psi(0)^T & 0 & a_2^{-1} \end{vmatrix} \begin{vmatrix} D_{ij} & R_{ij} & \Phi_i(0_3) \\ -\bar{\Phi}_j(1)^T & -\bar{\phi}_j(1) & -a_3^{-2} \\ -\bar{\Phi}_j(0)^T & -\bar{\phi}_j(0) & a_3^{-1} \end{vmatrix} \\ &+\begin{vmatrix} M_d & C(k_1) & \Phi(0_3) \\ \alpha^T & 1 & 0 \\ -\bar{\Phi}(0)^T & 0 & a_3^{-1} \end{vmatrix} \begin{vmatrix} D_{ij} & R_{ij} & \Phi_i(0_2) \\ -\bar{\Phi}_j(1)^T & -\bar{\phi}_j(1) & -a_2^{-2} \\ -\bar{\Phi}_j(0)^T & -\bar{\phi}_j(0) & a_2^{-1} \end{vmatrix} \end{aligned}$$

$$+\begin{vmatrix} M_d & C(k_1) & \Phi(0_2) & \Phi(0_3) \\ \alpha^T & 1 & 0 & 0 \\ -\bar{\Phi}(1)^T & 0 & -a_2^{-2} & -a_3^{-2} \\ -\bar{\Phi}(0)^T & 0 & a_2^{-1} & a_3^{-1} \end{vmatrix} \begin{vmatrix} D_{ij} & R_{ij} \\ -\bar{\Phi}_j(0)^T & -\bar{\phi}_j(0) \end{vmatrix} = 0, \tag{2.31}$$

and

$$\begin{vmatrix} D_{ij} & \Phi_i(-1) & \Phi_i(0_2) \\ -\bar{\Phi}_j(0)^T & -1 & a_2^{-1} \\ M_{ij}^T & \phi_i(-1) & \phi_i(0_2) \end{vmatrix} \begin{vmatrix} M_d & C(k_1) & \Phi(-1) & \Phi(0_3) \\ \alpha^T & 1 & 0 & 0 \\ -\bar{\Phi}(1)^T & 0 & 0 & -a_3^{-2} \\ -\bar{\Phi}(0)^T & 0 & -1 & a_3^{-1} \end{vmatrix}$$
$$-\begin{vmatrix} D_{ij} & \Phi_i(-1) & \Phi_i(0_3) \\ -\bar{\Phi}_j(0)^T & -1 & a_3^{-1} \\ M_{ij}^T & \phi_i(-1) & \phi_i(0_3) \end{vmatrix} \begin{vmatrix} M_d & C(k_1) & \Phi(-1) & \Phi(0_2) \\ \alpha^T & 1 & 0 & 0 \\ -\bar{\Phi}(1)^T & 0 & 0 & -a_2^{-2} \\ -\bar{\Phi}(0)^T & 0 & -1 & a_2^{-1} \end{vmatrix}$$
$$-\begin{vmatrix} D_{ij} & \Phi_i(-1) & \Phi_i(0_2) & \Phi_i(0_3) \\ -\bar{\Phi}_j(1)^T & 0 & -a_2^{-2} & -a_3^{-2} \\ M_{ij}^T & \phi_i(-1) & \phi_i(0_2) & \phi_i(0_3) \\ -\bar{\Phi}_j(0)^T & -1 & a_2^{-1} & a_3^{-1} \end{vmatrix} \begin{vmatrix} M_d & C(k_1) & \Phi(-1) \\ \alpha^T & 1 & 0 \\ -\bar{\Phi}(0)^T & 0 & -1 \end{vmatrix} = 0. \tag{2.32}$$

Now we prove the determinant identity (2.31). Let us introduce the following $2(N+2)\times 2(N+2)$ determinant which is equal to zero:

$$\begin{vmatrix} D_i^R & \Phi_i(0_2) & 0 & \Phi_i(0_3) & 0 & 0 \\ M_i & \phi_i(0_2) & 0 & \phi_i(0_3) & 0 & 0 \\ -\bar{\Phi}(1)^T & -a_2^{-2} & 1 & -a_3^{-2} & 0 & 0 \\ -\bar{\Phi}(0)^T & a_2^{-1} & 0 & a_3^{-1} & 0 & 0 \\ 0 & \Phi_i(0_2) & 0 & \Phi_i(0_3) & D_i^R & 0 \\ 0 & \phi_i(0_2) & 0 & \phi_i(0_3) & M_i & 1 \\ 0 & -a_2^{-2} & 1 & -a_3^{-2} & -\bar{\Phi}(1)^T & 0 \\ 0 & a_2^{-1} & 0 & a_3^{-1} & -\bar{\Phi}(0)^T & 0 \end{vmatrix} = 0, \tag{2.33}$$

where D_i^R denotes the $(N-1)\times N$ matrix obtained by eliminating the ith row from the matrix $(c_{i,j}(k_1)+m_{ij}(k_1,k_2,k_3))_{1\le i,j\le N}$, and M_i is given by

$$M_i = (c_i(k_1)+m_{i1},\cdots,c_i(k_1)+m_{i,K},d_{i,K+1}+m_{i,K+1},\cdots,d_{i,N}+m_{i,N})^T.$$

Applying the Laplace expansion in $(N+2)\times(N+2)$ minors to the left-hand side of equation (2.33), we obtain the determinant identity (2.31).

Similarly, the Laplace expansion of the following $2(N+2)\times 2(N+2)$ determinant which is equal to zero in $(N+2)\times(N+2)$ minors:

$$\begin{vmatrix} D_j^C & \Phi(-1) & 0 & 0 & 0 & 0 & \Phi(0_2) & \Phi(0_3) \\ -\bar{\Phi}_j(1)^T & 0 & 1 & 0 & 0 & 0 & -a_2^{-2} & -a_3^{-2} \\ -\bar{\Phi}(0)^T & -1 & 0 & 0 & 0 & 0 & a_2^{-1} & a_3^{-1} \\ 0 & 0 & 0 & \Phi(-1) & D_j^C & R_j & \Phi(0_2 & \Phi(0_3) \\ 0 & 0 & 1 & 0 & -\bar{\Phi}_j(1)^T & -\bar{\phi}_j(1) & -a_2^{-2} & -a_3^{-2} \\ 0 & 0 & 0 & -1 & -\bar{\Phi}_j(0)^T & -\bar{\phi}_j(0) & a_2^{-1} & a_3^{-1} \end{vmatrix} = 0,$$

where D_j^C denotes the $N\times(N-1)$ matrix obtained by eliminating the jth column from the matrix $(c_{i,j}(k_1)+m_{ij}(k_1,k_2,k_3))_{1\le i,j\le N}$, and R_j is given by

$$R_j=(c_1(k_1)+m_{1j},\cdots,c_K(k_1)+m_{K,jK},d_{K+1,j}+m_{K+1,j},\cdots,d_{N,j}+m_{N,j})^T,$$

yields the determinant identity (2.32).

So functions (2.1-2.2) and (2.19-2.20) for $r,m=1,\cdots,K$ are solutions to the modified discrete KP equation with self-consistent sources (2.17-2.18, 2.22-2.23).

If we choose the arbitrary function c_{ij} in (2.10) as a function of k_2 instead of k_1, we can produce another form of the modified discrete KP equation with self-consistent sources via source generation procedure in a similar way:

$$\begin{aligned}&a_1f(k_1+a_1)g(k_2+a_2)-a_2f(k_2+a_2)g(k_1+a_1)+(a_2-a_1)f(k_1+a_1,k_2+a_2)g(k_1,k_2,k_3)\\&\qquad=a_1\sum_{i,j=1}^{K}P_{i,j}(k_2+a_2)Q_{i,j}(k_1+a_1),\end{aligned}\tag{2.34}$$

$$\begin{aligned}&a_1f(k_1+a_1)g(k_3+a_3)-a_3f(k_3+a_3)g(k_1+a_1)+(a_3-a_1)f(k_1+a_1,k_3+a_3)g(k_1,k_2,k_3,)\\&\qquad=0,\end{aligned}\tag{2.35}$$

$$\begin{aligned}&a_1f(k_1+a_1)P_{ij}(k_3+a_3)-a_3f(k_3+a_3)P_{ij}(k_1+a_1)+(a_3-a_1)f(k_1+a_1,k_3+a_3)P_{ij}(k_1,k_2,k_3)\\&\qquad=0,\end{aligned}\tag{2.36}$$

$$\begin{aligned}&a_1Q_{ij}(k_1+a_1)g(k_3+a_3)-a_3Q_{ij}(k_3+a_3)g(k_1+a_1)+(a_3-a_1)Q_{ij}(k_1+a_1,k_3+a_3)g(k_1,k_2,k_3)\\&\qquad=0.\end{aligned}\tag{2.37}$$

Equations (2.34-2.37) can be written in a compact way using the bilinear operator (1.3):

$$\begin{aligned}&[a_1e^{\frac12a_1D_{k_1}-\frac12a_2D_{k_2}}-a_2e^{\frac12a_2D_{k_2}-\frac12a_1D_{k_1}}+(a_2-a_1)e^{\frac12a_1D_{k_1}+\frac12a_2D_{k_2}}]f\cdot g\\&\qquad\qquad=a_1\sum_{i,j=1}^{K}e^{\frac12a_2D_{k_2}-\frac12a_1D_{k_1}}P_{ij}\cdot Q_{ij},\end{aligned}\tag{2.38}$$

$$[a_1e^{\frac12a_1D_{k_1}-\frac12a_3D_{k_3}}-a_3e^{\frac12a_3D_{k_3}-\frac12a_1D_{k_1}}+(a_3-a_1)e^{\frac12a_1D_{k_1}+\frac12a_3D_{k_3}}]f\cdot g=0,\tag{2.39}$$

$$[a_1e^{\frac12a_1D_{k_1}-\frac12a_3D_{k_3}}-a_3e^{\frac12a_3D_{k_3}-\frac12a_1D_{k_1}}+(a_3-a_1)e^{\frac12a_1D_{k_1}+\frac12a_3D_{k_3}}]f\cdot P_{ij}=0,\tag{2.40}$$

$$[a_1e^{\frac12a_1D_{k_1}-\frac12a_3D_{k_3}}-a_3e^{\frac12a_3D_{k_3}-\frac12a_1D_{k_1}}+(a_3-a_1)e^{\frac12a_1D_{k_1}+\frac12a_3D_{k_3}}]Q_{ij}\cdot g=0.\tag{2.41}$$

We can derive the following discrete Gram-type determinant solution for the modified discrete KP equation with self-consistent sources (2.34-2.37) or (2.38-2.41):

$$f(k_1,k_2,k_3)=\det(\bar{c}_{ij}(k_2)+m_{ij})_{1\le i,j\le N}=|\bar{D}|,\tag{2.42}$$

$$g(k_1,k_2,k_3)=\begin{vmatrix}\bar{D}&\Phi(-1)\\\bar{\Phi}(0)^T&1\end{vmatrix},\tag{2.43}$$

where $m_{ij}(1\le i,j\le N),\Phi(s),\bar{\Phi}(s)$ are given in (2.3-2.7). In addition, $\bar{c}_{ij}(k_2)$ satisfies

$$\bar{c}_{ij}(k_2)=\begin{cases}\bar{c}_i(k_2),&1\le i\le K\le N\text{ and }j=1,2,\cdots,K,K\in Z^+,\\\bar{d}_{ij},&\text{otherwise,}\end{cases}$$

with $\bar{c}_i(k_2)(1 \le i \le K)$ being an arbitrary function of k_2, $\bar{d}_{ij}(1 \le i, j \le N)$ being an arbitrary constant and K being a positive integer, and

$$P_{i,j}(k_1,k_2,k_3) = \sqrt{c_i(k_2) - c_i(k_2 - a_2)} \begin{vmatrix} \bar{D}_{ij} & \bar{R}_{ij} \\ \bar{\Phi}_j(0)^T & \bar{\phi}_j(0) \end{vmatrix}, \tag{2.44}$$

$$Q_{i,j}(k_1,k_2,k_3) = a_1 \sqrt{\Delta c_i(k_2)} \begin{vmatrix} \bar{D}_{ij} & \Phi_i(-1) \\ \bar{M}_{ij}^T & \phi_i(-1) \end{vmatrix}, \tag{2.45}$$

where $\bar{D}_{ij}$ is the $(N-1)$th order matrice obtained by eliminating the ith row and the jth column from the $N \times N$ matrice $\bar{D}$, and

$$\begin{aligned} \bar{R}_{ij} = (&\bar{c}_1(k_2) + m_{1j}, \cdots, \bar{c}_{i-1}(k_2) + m_{i-1,j}, \bar{c}_{i+1}(k_2) + m_{i+1,j}, \cdots, \\ &\bar{c}_K(k_2) + m_{K,j}, \bar{d}_{K+1,j} + m_{K+1,j}, \cdots, \bar{d}_{N,j} + m_{N,j})^T, \\ \bar{M}_{ij} = (&\bar{c}_i(k_2) + m_{i1}, \cdots, \bar{c}_i(k_2) + m_{i,j-1}, \bar{c}_i(k_2) + m_{i,j+1}, \cdots, \\ &\bar{c}_i(k_2) + m_{i,K}, \bar{d}_{i,K+1} + m_{i,K+1}, \cdots, \bar{d}_{i,N} + m_{i,N})^T. \end{aligned}$$

3. Commutativity of the source generation procedure and Bäcklund transformation

In this section, we show that the commutativity of the source generation procedure and Bäcklund transformation holds for the discrete KP equation. For this purpose, we derive another form of the modified discrete KP with self-consistent sources which is the Bäcklund transformation for the discrete KPequation with self-consistent sources.

We has shown that the Grammian determinants $f(k_1,k_2,k_3)$,$g(k_1,k_2,k_3)$,$P_{i,j}(k_1,k_2,k_3)$, $Q_{i,j}(k_1,k_2,k_3)$ given in (2.42-2.45) satisfy the modified discrete KP equation with self-consistent sources (2.38-2.41). Now we take

$$f'(k_1,k_2,k_3) = g(k_1,k_2,k_3), \tag{3.1}$$
$$g'_{i,j}(k_1,k_2,k_3) = Q_{ij}(k_1,k_2,k_3), \tag{3.2}$$
$$h_{i,j}(k_1,k_2,k_3) = P_{i,j}(k_1,k_2,k_3), \tag{3.3}$$

and introduce two new fields

$$g_{i,j}(k_1,k_2,k_3) = -a_1 a_3 \sqrt{\Delta c_i(k_2)} \begin{vmatrix} \bar{D}_j^C & \Phi(k_3) \end{vmatrix}, \tag{3.4}$$

$$h'_{i,j}(k_1,k_2,k_3) = -a_3 \sqrt{c_i(k_2) - c_i(k_2 - a_2)} \begin{vmatrix} \bar{D}_i^R & \Phi_i(-1) \\ \bar{\Phi}(1)^T & 0 \\ \bar{\Phi}(0)^T & 1 \end{vmatrix}, \tag{3.5}$$

where $\bar{D}_j^C$ denotes the $N \times (N-1)$ matrix obtained by eliminating the jth column from the $N \times N$ matrice $\bar{D}$ and $\bar{D}_i^R$ denotes the $(N-1) \times N$ matrix obtained by eliminating the ith row from the $N \times N$ matrice $\bar{D}$.

It is not difficult to show that Grammian determinants $f, g_{i,j}, h_{i,j}$ and $f', g'_{i,j}, h'_{i,j}$ are two solutions to the discrete KP equation with self-consistent sources derived in [23]:

$$(a_1(a_2-a_3)e^{1/2(-D_{k_1}+D_{k_2}+D_{k_3})}+a_2(a_3-a_1)e^{1/2(D_{k_1}-D_{k_2}+D_{k_3})}+a_3(a_1-a_2)e^{1/2(-D_{k_1}-D_{k_2}+D_{k_3})})f\cdot f \\ =\sum_{i,j=1}^{K} e^{1/2(D_{k_1}-D_{k_2}+D_{k_3})}g_{i,j}\cdot h_{i,j}, \tag{3.6}$$

$$(a_3e^{1/2(D_{k_3}-D_{k_1})}-a_1e^{1/2(D_{k_1}-D_{k_3})}+(a_1-a_3)e^{1/2(D_{k_1}+D_{k_3})})f\cdot h_{i,j}=0,\quad j=1,\cdots,K, \tag{3.7}$$

$$(a_3e^{1/2(D_{k_3}-D_{k_1})}-a_1e^{1/2(D_{k_1}-D_{k_3})}+(a_1-a_3)e^{1/2(D_{k_1}+D_{k_3})})g_{i,j}\cdot f=0,\quad j=1,\cdots,K. \tag{3.8}$$

Furthermore, we can verify that the Grammian determinants $f, f', g_{i,j}, g'_{i,j}, h_{i,j}, h'_{i,j}$ satisfy the following six bilinear equations:

$$[a_1e^{\frac{1}{2}D_{k_1}-\frac{1}{2}D_{k_2}}-a_2e^{\frac{1}{2}D_{k_2}-\frac{1}{2}D_{k_1}}+(a_2-a_1)e^{\frac{1}{2}D_{k_1}+\frac{1}{2}D_{k_2}}]f\cdot f'=\sum_{i,j=1}^{K}e^{\frac{1}{2}D_{k_2}-\frac{1}{2}D_{k_1}}h_{ij}\cdot g'_{ij}, \tag{3.9}$$

$$[a_1e^{\frac{1}{2}D_{k_1}-\frac{1}{2}D_{k_3}}-a_3e^{\frac{1}{2}D_{k_3}-\frac{1}{2}D_{k_1}}+(a_3-a_1)e^{\frac{1}{2}D_{k_1}+\frac{1}{2}D_{k_3}}]f\cdot f'=0, \tag{3.10}$$

$$[a_1e^{\frac{1}{2}D_{k_1}-\frac{1}{2}D_{k_3}2}-a_3e^{\frac{1}{2}D_{k_3}-\frac{1}{2}D_{k_1}}+(a_3-a_1)e^{\frac{1}{2}D_{k_1}+\frac{1}{2}D_{k_3}}]g_{i,j}\cdot g'_{i,j}=0,\quad j=1,\cdots,K, \tag{3.11}$$

$$[a_1e^{\frac{1}{2}D_{k_1}-\frac{1}{2}D_{k_3}}-a_3e^{\frac{1}{2}D_{k_3}-\frac{1}{2}D_{k_1}}+(a_3-a_1)e^{\frac{1}{2}D_{k_1}+\frac{1}{2}D_{k_3}}]h_{i,j}\cdot h'_{i,j}=0,\quad j=1,\cdots,K,, \tag{3.12}$$

$$e^{\frac{1}{2}D_{k_3}}g_{i,j}\cdot f'=(e^{-\frac{1}{2}D_{k_3}}-e^{\frac{1}{2}D_{k_3}})f\cdot g'_{i,j},\quad j=1,\cdots,K, \tag{3.13}$$

$$e^{\frac{1}{2}D_{k_3}}f\cdot h'_{i,j}=(e^{-\frac{1}{2}D_{k_3}}-e^{\frac{1}{2}D_{k_3}})h_{i,j}\cdot f',\quad j=1,\cdots,K. \tag{3.14}$$

Equations (3.9-3.14) constitute another form of the modified discrete KP with self-consistent sources. It is proved in [23] that the discrete KP equation with self-consistent sources (3.6-3.8) possesses the following bilinear Bäcklund transformation:

$$[\gamma e^{\frac{1}{2}D_{k_1}-\frac{1}{2}D_{k_2}}+z_2e^{\frac{1}{2}D_{k_2}-\frac{1}{2}D_{k_1}}+\lambda_1z_3e^{\frac{1}{2}D_{k_1}+\frac{1}{2}D_{k_2}}]f\cdot f'=\lambda_2\sum_{i,j=1}^{K}e^{\frac{1}{2}D_{k_2}-\frac{1}{2}D_{k_1}}h_{ij}\cdot g'_{ij}, \tag{3.15}$$

$$[\beta_1e^{\frac{1}{2}D_{k_1}-\frac{1}{2}D_{k_3}}-e^{\frac{1}{2}D_{k_3}-\frac{1}{2}D_{k_1}}+\lambda_1e^{\frac{1}{2}D_{k_1}+\frac{1}{2}D_{k_3}}]f\cdot f'=0, \tag{3.16}$$

$$[\beta_1e^{\frac{1}{2}D_{k_1}-\frac{1}{2}D_{k_3}}-e^{\frac{1}{2}D_{k_3}-\frac{1}{2}D_{k_1}}+\lambda_1e^{\frac{1}{2}D_{k_1}+\frac{1}{2}D_{k_3}}]g_{i,j}\cdot g'_{i,j}=0,\quad j=1,\cdots,K, \tag{3.17}$$

$$[\beta_1e^{\frac{1}{2}D_{k_1}-\frac{1}{2}D_{k_3}}-e^{\frac{1}{2}D_{k_3}-\frac{1}{2}D_{k_1}}+\lambda_1e^{\frac{1}{2}D_{k_1}+\frac{1}{2}D_{k_3}}]h_{i,j}\cdot h'_{i,j}=0,\quad j=1,\cdots,K,, \tag{3.18}$$

$$e^{\frac{1}{2}D_{k_3}}g_{i,j}\cdot f'=(\beta_2e^{-\frac{1}{2}D_{k_3}}+\lambda_2e^{\frac{1}{2}D_{k_3}})f\cdot g'_{i,j},\quad j=1,\cdots,K, \tag{3.19}$$

$$e^{\frac{1}{2}D_{k_3}}f\cdot h'_{i,j}=(\beta_2e^{-\frac{1}{2}D_{k_3}}+\lambda_2e^{\frac{1}{2}D_{k_3}})h_{i,j}\cdot f',\quad j=1,\cdots,K, \tag{3.20}$$

where γ is an arbitrary constant and $\beta_1, \beta_2, \lambda_1, \lambda_2$ are constants satisfying the relation $\lambda_1\beta_2a_1 = \lambda_2\beta_1(a_1-a_3)$.

If we take $\gamma=-a_1, \lambda_1=\frac{a_3-a_1}{a_3}, \lambda_2=-1, \beta_1=\frac{a_1}{a_3}, \beta_2=1$, then equations (3.15-3.20) become (3.9-3.14). Hence the modified discrete KP with self-consistent sources (3.9-3.14) is a Bäcklund transformation for the discrete KP equation with self-consistent sources (3.6-3.8).

4. Summary and discussion

In this paper, we have produced two forms of the the modified discrete KP equation with self-consistent sources (2.17-2.23) and (2.38-2.41) via the source generation procedure, and given their discrete Gram-type determinant solutions. We have also constructed another form of the modified discrete KP equation with self-consistent sources (3.9-3.14) which is the Bäcklund transformation for the discrete KP equation with self-consistent sources derived in [23].

We investigate a continuum limit of the modified discrete KP with self-consistent sources (2.34-2.37) with one pair of sources $P_{11}(k_1,k_2,k_3),Q_{11}(k_1,k_2,k_3)$, for simplicity. If we take

$$a_1=\delta,a_2=\varepsilon,a_3=1,D_{k_1}=D_y,D_{k_2}=D_y-\frac{1}{2}\varepsilon D_t,D_{k_3}=D_n, \tag{4.1}$$

$$f(k_1,k_2,k_3)=F(n,y,t),g(k_1,k_2,k_3)=G(n,y,t), \tag{4.2}$$

$$P_{11}(k_1,k_2,k_3)=\varepsilon S_{11}(n,y,t),Q_{11}(k_1,k_2,k_3)=\frac{1}{2}\varepsilon H_{11}(n,y,t), \tag{4.3}$$

in (2.38-2.41) with one pair of sources P,Q and compare $\varepsilon^2\delta$ order of the two sides of the equation (2.38) and δ order of the two sides of the equations (2.39-2.41), then we obtain the modified differential-difference KP equation with self-consistent sources:

$$(D_y^2+D_t)F\cdot G=H_{11}S_{11}, \tag{4.4}$$

$$D_y e^{\frac{1}{2}D_n}F\cdot G=(e^{\frac{1}{2}D_n}-e^{-\frac{1}{2}D_n})F\cdot G, \tag{4.5}$$

$$D_y e^{\frac{1}{2}D_n}F\cdot S_{11}=(e^{\frac{1}{2}D_n}-e^{-\frac{1}{2}D_n})F\cdot S_{11}, \tag{4.6}$$

$$D_y e^{\frac{1}{2}D_n}H_{11}\cdot G=(e^{\frac{1}{2}D_n}-e^{-\frac{1}{2}D_n})H_{11}\cdot G, \tag{4.7}$$

which can be derived by applying source generation procedure to the modified differential-difference KP equation [25]:

$$(D_y^2+D_t)F\cdot G=0,$$
$$D_y e^{\frac{1}{2}D_n}F\cdot G=(e^{\frac{1}{2}D_n}-e^{-\frac{1}{2}D_n})F\cdot G.$$

Now we show that the exact solution to the modified discrete KP with self-consistent sources (2.34-2.37) yields the exact solution to the modified differential-difference KP equation with self-consistent sources (4.4-4.7) through the continuum limit.

We obtain the N-soliton solution for the modified discrete KP with self-consistent sources (2.34-2.37) by choosing

$$m_{i,j}(k_1,k_2,k_3)=\frac{1}{p_i+\bar{p}_j}\left(\frac{1+\bar{p}_ja_1}{1-p_ia_1}\right)^{\frac{k_1}{a_1}}\left(\frac{1+\bar{p}_ja_2}{1-p_ia_2}\right)^{\frac{k_2}{a_2}}\left(\frac{1+\bar{p}_ja_3}{1-p_ia_3}\right)^{\frac{k_3}{a_3}},1\le i,j\le N, \tag{4.8}$$

$$\phi_i(k_1,k_2,k_3;s)=p_i^s(1-p_ia_1)^{-\frac{k_1}{a_1}}(1-p_ia_2)^{-\frac{k_2}{a_2}}(1-p_ia_3)^{-\frac{k_3}{a_3}},1\le i\le N, \tag{4.9}$$

$$\bar{\phi}_j(k_1,k_2,k_3;s)=\bar{p}_j^s(1+\bar{p}_ja_1)^{-\frac{k_1}{a_1}}(1+\bar{p}_ja_2)^{-\frac{k_2}{a_2}}(1+\bar{p}_ja_3)^{-\frac{k_3}{a_3}},1\le j\le N, \tag{4.10}$$

in equations (2.42-2.45). In equations (4.8-4.10), $p_i,\bar{p}_i(1\le i\le N)$ are arbitrary constants. For example, if we take $N=1,K=1$ and

$$\phi_1(k_1,k_2,k_3;s)=p_1^s(1-p_1a_1)^{-\frac{k_1}{a_1}}(1-p_1a_2)^{-\frac{k_2}{a_2}}(1-p_1a_3)^{-\frac{k_3}{a_3}}, \tag{4.11}$$

$$\bar{\phi}_j(k_1,k_2,k_3;s)=\bar{p}_1^s(1-\bar{p}_1a_1)^{-\frac{k_1}{a_1}}(1-\bar{p}_1a_2)^{-\frac{k_2}{a_2}}(1-\bar{p}_1a_3)^{-\frac{k_3}{a_3}}, \tag{4.12}$$

$$c_1(k_2)=\frac{(1+2a_2\beta_1(k_2))^{\frac{1}{a_2}}}{p_1+\bar{p}_1}, \tag{4.13}$$

where $\beta_1(k_2)$ is an arbitrary function of k_2, we obtain the one-soliton solution of the modified discrete KP with self-consistent sources (2.34-2.37):

$$f = \frac{(1+2a_2\beta_1(k_2))^{\frac{1}{a_2}}}{p_1+\bar{p}_1} + \frac{1}{p_1+\bar{p}_1}(\frac{1+\bar{p}_1a_1}{1-p_1a_1})^{\frac{k_1}{a_1}}(\frac{1+\bar{p}_1a_2}{1-p_1a_2})^{\frac{k_2}{a_2}}(\frac{1+\bar{p}_1a_3}{1-p_1a_3})^{\frac{k_3}{a_3}}, \tag{4.14}$$

$$g = \frac{(1+2a_2\beta_1(k_2))^{\frac{1}{a_2}}}{p_1+\bar{p}_1} - \frac{\bar{p}_1}{p_1(p_1+\bar{p}_1)}(\frac{1+\bar{p}_1a_1}{1-p_1a_1})^{\frac{k_1}{a_1}}(\frac{1+\bar{p}_1a_2}{1-p_1a_2})^{\frac{k_2}{a_2}}(\frac{1+\bar{p}_1a_3}{1-p_1a_3})^{\frac{k_3}{a_3}}, \tag{4.15}$$

$$P_{11} = \sqrt{c_1(k_2)-c_1(k_2-a_2)}(1+\bar{p}_1a_1)^{\frac{k_1}{a_1}}(1+\bar{p}_1a_2)^{\frac{k_2}{a_2}}(1+\bar{p}_1a_3)^{\frac{k_3}{a_3}}, \tag{4.16}$$

$$Q_{11} = \frac{1}{p_1}\sqrt{c_1(k_2+a_2)-c_1(k_2)}(1-p_1a_1)^{-\frac{k_1}{a_1}}(1-p_1a_2)^{-\frac{k_2}{a_2}}(1-p_1a_3)^{-\frac{k_3}{a_3}}. \tag{4.17}$$

We derive the following variable transformation from equation (4.1):

$$t = -\frac{1}{2}\varepsilon k_2, \quad y = k_1+k_2, \quad n = k_3. \tag{4.18}$$

Applying equations (4.18) and setting $\beta_1(k_2) = b_1(t), c_1(k_2) = C_1(t)$, we obtain

$$c_1(k_2) = \frac{(1+2a_2\beta_1(k_2))^{\frac{1}{a_2}}}{p_1+\bar{p}_1} = \frac{e^{\frac{1}{\varepsilon}\ln(1+2\varepsilon\beta_1(k_2)}}{p_1+\bar{p}_1} = \frac{e^{2\beta_1(k_2)+O(\varepsilon)}}{p_1+\bar{p}_1}, \tag{4.19}$$

which gives $C_1(t) = \frac{e^{2b_1(t)}}{p_1+\bar{p}_1}$ in the small limit of ε. Furthermore, we can derive the following equations using (4.1-4.3) and (4.18):

$$\begin{aligned} f &= \frac{(1+2a_2\beta_1(k_2))^{\frac{1}{a_2}}}{p_1+\bar{p}_1} + \frac{1}{p_1+\bar{p}_1}(\frac{1+\bar{p}_1a_1}{1-p_1a_1})^{\frac{k_1}{a_1}}(\frac{1+\bar{p}_1a_2}{1-p_1a_2})^{\frac{k_2}{a_2}}(\frac{1+\bar{p}_1a_3}{1-p_1a_3})^{\frac{k_3}{a_3}} \\ &= \frac{e^{2\beta_1(k_2)+O(\varepsilon)}}{p_1+\bar{p}_1} + \frac{1}{p_1+\bar{p}_1}e^{\frac{k_1}{\delta}\ln(1+\bar{p}_1\delta)}e^{\frac{k_2}{\varepsilon}\ln(1+\bar{p}_1\varepsilon)}e^{k_3\ln(1+\bar{p}_1)}e^{-\frac{k_1}{\delta}\ln(1-p_1\delta)}e^{-\frac{k_2}{\varepsilon}\ln(1-p_2\varepsilon)}e^{-k_3\ln(1-p_1)} \\ &= \frac{1}{p_1+\bar{p}_1}(e^{2b_1(t)+O(\varepsilon)} + e^{(p_1+\bar{p}_1)y+(\bar{p}_1^2-p_1^2)t+(\ln(1+\bar{p}_1)-\ln(1-p_1))n+O(\delta)+O(\varepsilon^2)}), \end{aligned} \tag{4.20}$$

(4.21)

$$\begin{aligned} g &= \frac{(1+2a_2\beta_1(k_2))^{\frac{1}{a_2}}}{p_1+\bar{p}_1} - \frac{\bar{p}_1}{p_1(p_1+\bar{p}_1)}(\frac{1+\bar{p}_1a_1}{1-p_1a_1})^{\frac{k_1}{a_1}}(\frac{1+\bar{p}_1a_2}{1-p_1a_2})^{\frac{k_2}{a_2}}(\frac{1+\bar{p}_1a_3}{1-p_1a_3})^{\frac{k_3}{a_3}} \\ &= \frac{1}{p_1+\bar{p}_1}(e^{2b_1(t))+O(\varepsilon)} - \frac{\bar{p}_1}{p_1}e^{(p_1+\bar{p}_1)y+(\bar{p}_1^2-p_1^2)t+(\ln(1+\bar{p}_1)-\ln(1-p_1))n+O(\delta)+O(\varepsilon^2)}), \end{aligned} \tag{4.22}$$

(4.23)

$$\begin{aligned} P_{11} &= \sqrt{c_1(k_2)-c_1(k_2-a_2)}(1+\bar{p}_1a_1)^{\frac{k_1}{a_1}}(1+\bar{p}_1a_2)^{\frac{k_2}{a_2}}(1+\bar{p}_1a_3)^{\frac{k_3}{a_3}} \\ &= \sqrt{\varepsilon\frac{dc_1(k_2)}{dk_2}+O(\varepsilon^2)}e^{\frac{k_1}{\delta}\ln(1+\bar{p}_1\delta)}e^{\frac{k_2}{\varepsilon}\ln(1+\bar{p}_1\varepsilon)}e^{k_3\ln(1+\bar{p}_1)} \\ &= \sqrt{-\frac{1}{2}\varepsilon^2\frac{dC_1(t)}{dt}+O(\varepsilon^4)}e^{\bar{p}_1y+\bar{p}_1^2t+\ln(1+\bar{p}_1)n+O(\delta)+O(\varepsilon^2)} \\ &= \varepsilon\sqrt{-\frac{1}{2}\frac{dC_1(t)}{dt}+O(\varepsilon^2)}e^{\bar{p}_1y+\bar{p}_1^2t+\ln(1+\bar{p}_1)n+O(\delta)+O(\varepsilon^2)}, \end{aligned} \tag{4.24}$$

$$\begin{aligned}
Q_{11} &= \frac{1}{p_1}\sqrt{c_1(k_2+a_2)-c_1(k_2)}(1-p_1a_1)^{-\frac{k_1}{a_1}}(1-p_1a_2)^{-\frac{k_2}{a_2}}(1-p_1a_3)^{-\frac{k_3}{a_3}} \\
&= \sqrt{\varepsilon\frac{dc_1(k_2)}{dk_2}+O(\varepsilon^2)}e^{-\frac{k_1}{\delta}\ln(1-p_1\delta)}e^{-\frac{k_2}{\varepsilon}\ln(1-p_1\varepsilon)}e^{-k_3\ln(1-p_1)} \\
&= \sqrt{-\frac{1}{2}\varepsilon^2\frac{dC_1(t)}{dt}+O(\varepsilon^4)}e^{p_1y-p_1^2t-\ln(1-p_1)n+O(\delta)+O(\varepsilon^2)} \\
&= \varepsilon\sqrt{-\frac{1}{2}\frac{dC_1(t)}{dt}+O(\varepsilon^2)}e^{p_1y-p_1^2t-\ln(1-p_1)n+O(\delta)+O(\varepsilon^2)}.
\end{aligned} \tag{4.25}$$

From equations (4.3) and (4.24-4.25), we have

$$H_{11} = \sqrt{-\frac{1}{2}\frac{dC_1(t)}{dt}+O(\varepsilon^2)}e^{\bar{p}_1y+\bar{p}_1^2t+\ln(1+\bar{p}_1)n+O(\delta)+O(\varepsilon^2)}, \tag{4.26}$$

$$S_{11} = \sqrt{-\frac{1}{2}\frac{dC_1(t)}{dt}+O(\varepsilon^2)}e^{p_1y-p_1^2t-\ln(1-p_1)n+O(\delta)+O(\varepsilon^2)}. \tag{4.27}$$

Through the small limit of δ and ε, the functions f,g,H_{11},S_{11} given in (4.20-4.22,4.26-4.27) become

$$f = \frac{e^{2b_1(t)}}{p_1+\bar{p}_1}(1+e^{(p_1+\bar{p}_1)y+(\bar{p}_1^2-p_1^2)t+(\ln(1+\bar{p}_1)-\ln(1-p_1))n-2b_1(t)}), \tag{4.28}$$

$$g = \frac{e^{2b_1(t)}}{p_1+\bar{p}_1}(1-\frac{\bar{p}_1}{p_1}e^{(p_1+\bar{p}_1)y+(\bar{p}_1^2-p_1^2)t+(\ln(1+\bar{p}_1)-\ln(1-p_1))n-2b_1(t)}), \tag{4.29}$$

$$H_{11} = \sqrt{-\frac{1}{2}\frac{dC_1(t)}{dt}}e^{\bar{p}_1y+\bar{p}_1^2t+\ln(1+\bar{p}_1)n} = \sqrt{-\frac{\dot{b}_1(t)}{p_1+\bar{p}_1}}e^{\bar{p}_1y+\bar{p}_1^2t+\ln(1+\bar{p}_1)n+b_1(t)}, \tag{4.30}$$

and

$$S_{11} = \sqrt{-\frac{1}{2}\frac{dC_1(t)}{dt}}e^{p_1y-p_1^2t-\ln(1-p_1)n} = \sqrt{-\frac{\dot{b}_1(t)}{p_1+\bar{p}_1}}e^{p_1y-p_1^2t-\ln(1-p_1)n+b_1(t)}, \tag{4.31}$$

respectively. We can verify that the functions f,g,H_{11},S_{11} given in (4.28-4.31) solve the modified differential-difference KP equation with self-consistent sources (4.4-4.7).

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