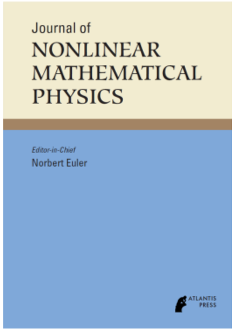

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1402-9251 **ISSN (Print): 1776-0852**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

Exact solutions and Riccati-type first integrals

J. Mendoza, C. Muriel

To cite this article: J. Mendoza, C. Muriel (2017) Exact solutions and Riccati-type first integrals, Journal of Nonlinear Mathematical Physics 24: Supplement 1, 75–89, DOI: https://doi.org/10.1080/14029251.2017.1418055

To link to this article: https://doi.org/10.1080/14029251.2017.1418055

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 24, Supplement 1 (2017) 75–89

Exact solutions and Riccati-type first integrals

J. Mendoza
Departamento de Matemáticas. Universidad del Atlántico
Km 7 vía Puerto Colombia, 081001, Barranquilla, Colombia
josemendozae@mail.uniatlantico.edu.co

C. Muriel
Departamento de Matemáticas. Universidad de Cádiz
Campus universitario de Puerto Real, 11510, Spain
concepcion.muriel@uca.es

Received 1 October 2017

Accepted 25 October 2017

The $\lambda-$symmetry approach is applied to a family of second-order ODEs whose algebra of Lie point symmetries is insufficient to integrate them. The general solution and two functionally independent first integrals of a subclass of the studied equations can be expressed in terms of a fundamental set of solutions of a related second-order linear ordinary differential equation. Remarkably, no equation in the family passes the Lie's test of linearisation, although all of them can be linearised by generalised Sundman transformations. Several examples, including equations lacking Lie point symmetries, illustrate the presented procedure.

Keywords: $\lambda-$symmetry; Sundman transformation; Riccati-type first integral.

2000 Mathematics Subject Classification: 34A05,34A34

1. Introduction

Lie symmetry method [7, 20, 23] is one of the most powerful tool to determine exact solutions of ordinary differential equations (ODEs). However, many equations lack Lie point symmetries or admit a symmetry algebra which is insufficient to integrate completely the equations under study [6, 18, 19]. This is the case, for instance, of the second-order nonlinear differential equation [9, 11]

$$u_{xx} = 3uu_x + 3au^2 + 4a^2u + b, \tag{1.1}$$

where a and b are arbitrary parameters, and the subscripts denote derivation with respect to the independent variable x. The vector field $\mathbf{v} = \partial_x$ is the unique Lie point symmetry admitted by equation (1.1). By using the differential invariants

$$y = u, \qquad w = 1/u_x \tag{1.2}$$

of $\mathbf{v}^{(1)} = \partial_x$, the order of (1.1) can be reduced by one and the corresponding reduced equation becomes

$$w_y + (3ay^2 + 4a^2y + b)w^3 + 3yw^2 = 0. \tag{1.3}$$

Equation (1.3) is an Abel's equation of the first kind whose general solution $I(y,w) = C_1$, where $C_1 \in \mathbb{R}$, can be expressed in terms of Bessel functions. The recovery of the solutions of the original second-order ODE (1.1) requires to solve the auxiliary equation

$$I\left(u, \frac{1}{u_x}\right) = C_1, \tag{1.4}$$

which cannot be evaluated due to the presence of special functions. In fact, equation (1.3) was studied by Liouville in [11], who used the change of variables (1.2) to convert equation (1.3) into equation (1.1). By using an *ad hoc* procedure, Liouville determined a reduced equation of (1.1):

$$u_x - \frac{3}{2}u^2 - 2au = \frac{b + e^{-ax}}{2a}. \tag{1.5}$$

The equation (1.5) can be mapped into a Riccati ODE solvable in terms of special functions, which provides an indirect method to find only a one-parameter family of exact solutions of equation (1.1).

In this paper we study the integrability of a wide class of second-order ODEs which includes equations lacking Lie point symmetries or whose Lie symmetry algebra is insufficient to integrate them (as in the case of equation (1.1)). With this aim, we exploit the $\lambda-$symmetries [14] of the equations under study and use them to calculate first integrals by a systematic and general procedure.

In many cases the obtained first integrals provide, as in (1.5), Riccati-type equations. We prove that, in these cases, the general solution of the second-order ODE can be explicitly expressed in terms of a fundamental set of solutions of a related second-order linear ODE. Remarkably, none of the equations under study are linearisable by point transformations. Nevertheless, the equations in the family can be linearised by generalised Sundman transformations [2–4, 12], which can be constructed from the coefficients of the associated first integral [16, 17]. The relationships between generalised Sundman transformations and Sundman symmetries [3], or between $\lambda-$symmetries and telescopic vector fields [21], exponential vector fields [20], semi-classical nonlocal symmetries [1,5], can be used to derive these different types of symmetries of the equations under study (see [15] and references therein).

2. Introductory example

2.1. *The λ-symmetry method*

Let us recall [13,14] that a $\lambda-$symmetry (in canonical form) of a second-order ODE $u_{xx} = \phi(x,u,u_x)$ is a pair $(\mathbf{v},\lambda)$, where $\mathbf{v} = \partial_u$ and the function $\lambda = \lambda(x,u,u_x)$ is a particular solution of the corresponding determining equation:

$$\lambda_x + u_x\lambda_u + \phi\lambda_{u_x} + \lambda^2 = \phi_u + \phi_{u_x}\lambda. \tag{2.1}$$

For equation (1.1), $\phi(x,u,u_x) = 3uu_x + 3au^2 + 4a^2u + b$, and the determining equation (2.1) becomes

$$\lambda_x + u_x\lambda_u + (3uu_x + 3au^2 + 4a^2u + b)\lambda_{u_x} + \lambda^2 = 3u_x + 6au + 4a^2 + 3u\lambda. \tag{2.2}$$

We assume an ansatz for λ in the form

$$\lambda = \alpha(x,u)u_x + \beta(x,u) \tag{2.3}$$

and equation (2.2) splits into the following system of partial differential equations for $\alpha = \alpha(x,u)$ and $\beta = \beta(x,u)$:

$$\alpha^2 + \alpha_u = 0, \tag{2.4}$$

$$2\alpha\beta + \alpha_x + \beta_u - 3 = 0, \tag{2.5}$$

$$4\alpha a^2 u + 3\alpha a u^2 + \alpha b + \beta^2 - 3\beta u - 4a^2 - 6au + \beta_x = 0. \tag{2.6}$$

Solving equations (2.4)-(2.6), we find that the only admissible solution is $\alpha = 0$ and $\beta = 2a + 3u$. Therefore

$$\lambda = 3u + 2a \tag{2.7}$$

is a particular solution of equation (2.2) and the pair $(\mathbf{v}, \lambda)$, where $\mathbf{v} = \partial_u$ and $\lambda = 2a + 3u$, is a $\lambda-$symmetry of equation (1.1). By using the $\lambda-$prolongation formula [14], we obtain

$$\mathbf{v}^{[\lambda,(2)]} = \partial_u + (2a + 3u)\partial_{u_x} + (4a^2 + 12au + 9u^2 + 3u_x)\partial_{u_{xx}}. \tag{2.8}$$

As for standard prolongations [20], a complete system of invariants of $\mathbf{v}^{[\lambda,(2)]}$ can be computed by derivation of lower-order invariants, x and

$$w = u_x - \frac{3}{2}u^2 - 2au. \tag{2.9}$$

By derivation of w with respect to x we get $w_x = -2au_x - 3uu_x + u_{xx}$. Equation (1.1) can be written in terms of $\{x, w, w_x\}$ as the reduced equation

$$w_x = -2aw + b. \tag{2.10}$$

This first-order differential equation is linear and its general solution becomes

$$\begin{aligned} w &= \frac{b}{2a} + C_1 e^{-2ax}, \quad &\text{if} \quad a \neq 0,\\ w &= bx + C_1, &\text{if} \quad a = 0, \end{aligned} \tag{2.11}$$

where $C_1 \in \mathbb{R}$. By substituting (2.9) into (2.11), we obtain the auxiliary first-order differential equations

$$\begin{aligned} &u_x - \frac{3}{2}u^2 - 2au - \frac{b}{2a} - C_1 e^{-2ax} = 0, \quad &\text{if} \quad a \neq 0,\\ &u_x - \frac{3}{2}u^2 - bx - C_1 = 0, &\text{if} \quad a = 0. \end{aligned} \tag{2.12}$$

We observe that by setting $C_1 = 0$ in the first equation in (2.12) we obtain the particular reduction (1.5), obtained in [11] by an *ad hoc* procedure. The $\lambda-$symmetry method provides a systematic method to calculate, not only such particular reduction, but a one-parameter family of reduced equations (2.12) from which the general solution of equation (1.1) can be completely reconstructed.

Equations (2.12) are Riccati-type equations that can be transformed into the respective linear second-order equation

$$\begin{aligned} &4a\phi''(x) - 8a^2\phi'(x) + (3b + 6aC_1e^{-2ax})\phi(x) = 0, \quad &\text{if} \quad a \neq 0, \\ &2\phi''(x) + 3(bx + C_1)\phi(x) = 0, &\text{if} \quad a = 0, \end{aligned} \tag{2.13}$$

through the standard transformation

$$u(x) = -\frac{2}{3}\frac{\phi'(x)}{\phi(x)}. \tag{2.14}$$

Let $\phi_1 = \phi_1(x;C_1)$ and $\phi_2 = \phi_2(x;C_1)$ denote two linearly independent solutions of the corresponding linear equation in (2.13). Then, it can be checked (see, for instance, Proposition 4.1 in [22]), that the general solution of the respective Riccati equation (2.12) can be expressed in the implicit form as follows

$$\frac{3u\phi_1(x;C_1) + 2\phi_1'(x;C_1)}{3u\phi_2(x;C_1) + 2\phi_2'(x;C_1)} = C_2, \tag{2.15}$$

where $C_2 \in \mathbb{R}$.

From (2.15), we finally obtain that the general solution of (1.1) can be expressed in terms of a fundamental set of solutions of the linear equation in (2.13) in the form

$$u(x) = -\frac{2}{3}\frac{C_2\phi_2'(x;C_1) - \phi_1'(x;C_1)}{C_2\phi_2(x;C_1) - \phi_1(x;C_1)}, \quad (C_1, C_2 \in \mathbb{R}). \tag{2.16}$$

As far as we know, this expression for the general solution of (1.1) is new in the literature. It should be noted that the application of the $\lambda-$symmetry method leads in a systematic way to the connection of equation (1.1) with the linear equation (2.13), although equation (1.1) is not linearisable by point transformations: the symmetry algebra of any linearisable second-order ODE is eight-dimensional and equation (1.1) only admits one Lie point symmetry. In the next subsection we investigate the linearisation of (1.1) by nonlocal transformations.

2.2. *Linearisation by generalised Sundman transformations*

A nonlocal transformation of the form

$$X = F(x,u), \qquad dU = G(x,u)dx \tag{2.17}$$

is called a generalised Sundman transformation [2–4, 12]. Several properties and characterizations of the class $\mathscr{A}$ of the second-order ODEs that can be transformed into the linear equation $U_{XX} = 0$ by means of (2.17) were derived in [2, 16, 17]. Such equations must be of the form

$$u_{xx} + a_2(x,u)u_x^2 + a_1(x,u)u_x + a_0(x,u) = 0 \tag{2.18}$$

and their coefficients must satisfy the conditions

$$S_1 \equiv a_{1u} - 2a_{2x} = 0 \text{ and } S_2 \equiv (a_0a_2 + a_{0u})_u + (a_{2x} - a_{1u})_x + (a_{2x} - a_{1u})a_1 = 0 \tag{2.19}$$

or if $S_1 \neq 0$, then

$$S_3 \equiv \left(\frac{S_2}{S_1}\right)_u - a_{2x} + a_{1u} = 0 \text{ and } S_4 \equiv \left(\frac{S_2}{S_1}\right)_x + \left(\frac{S_2}{S_1}\right)^2 + a_1\left(\frac{S_2}{S_1}\right) + a_0a_2 + a_{0u} = 0. \quad (2.20)$$

The equations in class $\mathscr{A}$ can also be characterized by the type of $\lambda-$symmetries that they have: they are the unique equations of the form (2.18) that admit the pair $(\partial_u, -a_2(x,u)u_x + \beta(x,u))$ as $\lambda-$symmetry, for some function $\beta = \beta(x,u)$. A very important property, which also characterizes the equations in $\mathscr{A}$, is that the first integral associated to that $\lambda-$symmetry is linear in the first-order derivative:

$$I_1(x,u,u_x) = A(x,u)u_x + B(x,u). \quad (2.21)$$

The family of $\mathscr{A}$ formed by the equations whose coefficients satisfy (2.19) (resp. (2.20)) is called subclass $\mathscr{A}_1$ (resp. $\mathscr{A}_2$). Any equation in $\mathscr{A}_1$ can be linearised by a point transformation [16, Theorem 7] and admit two independent first integrals of the form (2.21) [17, Theorem 5]. We do not consider in this paper equations in $\mathscr{A}_1$ because they can be integrated by classical procedures and focus our attention in the equations in $\mathscr{A}_2$. This class is very interesting, because, in contrast to $\mathscr{A}_1$, none of the equations in $\mathscr{A}_2$ passes the Lie's test of linearisation [10]. Whereas the symmetry algebra of equations in $\mathscr{A}_1$ is eight-dimensional, all of the examples of equations in $\mathscr{A}_2$ we have found in the literature so far do not admit Lie point symmetries or admit just one symmetry. Nevertheless, all of them admit the $\lambda-$symmetry (∂_u, λ) for

$$\lambda = -a_2(x,u)u_x + \frac{S_2(x,u)}{S_1(x,u)}. \quad (2.22)$$

We observe that equation (1.1) is of the form (2.18), where $a_2(x,u) = 0$, $a_1(x,u) = -3u$ and $a_0(x,u) = -3au^2 - 4a^2u - b$. It is easy to check that $S_1 = -3 \neq 0$, $S_2 = -6a - 9u$, and $S_3 = S_4 = 0$. Consequently, equation (1.1) belongs to subclass $\mathscr{A}_2$. It can be checked that the $\lambda-$symmetry (∂_u, λ) defined by (2.22) corresponds to the function found in (2.7) and that the associated first integral of the form (2.21) becomes

$$\begin{aligned} I_1(x,u,u_x) &= e^{2ax}\left(u_x - \frac{3}{2}u^2 - 2au - \frac{b}{2a}\right), \quad && \text{if} \quad a \neq 0, \\ I_1(x,u,u_x) &= u_x - \frac{3}{2}u^2 - bx, && \text{if} \quad a = 0. \end{aligned} \quad (2.23)$$

Moreover, $I_1(x,u,u_x) = C_1$ leads to the corresponding auxiliary equation obtained in (2.12).

The following relation between the coefficients $A(x,u)$ and $B(x,u)$ of the first integral (2.21) and a generalised Sundman transformation (2.17) was firstly discovered in [16]

$$BF_u - AF_x = 0, \qquad G = F_u/A. \quad (2.24)$$

This relation can be used to construct a generalised Sundman transformation (2.17) for equation (1.1) by using the coefficients $A(x,u)$ and $B(x,u)$ of the respective first integral given in (2.23). It can be checked that a generalised Sundman transformation (2.17) for equation (1.1) with $a \neq 0$, is defined by the functions

$$F(x,u) = -x - \frac{2a\,\mathrm{arctanh}\left(\dfrac{a(3u+2a)}{\sqrt{4a^4-3ba}}\right)}{\sqrt{4a^4-3ba}}, \quad G(x,u) = 2\,\frac{e^{-2ax}a}{4a^2u+3au^2+b}. \tag{2.25}$$

The expressions of functions F and G that define a generalised Sundman transformation for equation (1.1) with $a=0$ can be expressed in terms of two independent solutions φ_1 and φ_2 of the Airy equation $\varphi''(x)+3/2bx\varphi(x)=0$, as follows

$$F(x,u) = -\frac{u\varphi_1(x)-2\varphi_1'(x)}{u\varphi_2(x)-2\varphi_2'(x)}, \; G(x,u) = -\frac{W}{(u\varphi_2(x)-2\varphi_2'(x))^2}, \tag{2.26}$$

where $W = W(\varphi_1,\varphi_2)(x) = \varphi_2(x)\varphi_1'(x)-\varphi_2'(x)\varphi_1(x)$ denotes the Wronskian of φ_1 and φ_2, which, by the Jacobi-Liouville formula, is a constant.

To end this example we point out that different types of nonlocal symmetries for equation (1.1) can be constructed, once the $\lambda-$symmetry (∂_u,λ) defined by the function λ given in (2.7) has been determined:

- **Sundman symmetries:** Equation (1.1) is in the class of second-order ODE for which Euler and Euler determined in [3] nontrivial Sundman symmetries. In fact, as a consequence of Corollary 3.1 in [3], the Sundman transformations (2.25)-(2.26) can be used to construct infinitely many Sundman symmetries of the form

$$F(\tilde{x},\tilde{u}) = P(F(x,u)), \qquad G(\tilde{x},\tilde{u})d\tilde{x} = G(x,u)\frac{dP(F(x,u))}{dF}dx, \tag{2.27}$$

  with no further conditions on the differentiable function P.
- **Exponential vector fields:** Exponential vector fields were firstly considered in the book of P. Olver [20, p. 81] in order to show that not every integration method comes from the classical method of Lie and they were the origin of nonlocal and hidden symmetries theories. The relationship between these vector fields and $\lambda-$symmetries has been studied in [14, Theorem 5.1]. From (2.7) we deduce that

$$\mathbf{v}^* = e^{2ax}e^{3\int u dx}\partial_u$$

  is an exponential vector field of equation (1.1).
- **Telescopic vector fields:** A telescopic vector field [21] can be considered as a $\lambda-$prolongation, where the two first infinitesimals can depend on the first-order derivative of the dependent variable [15]. Conversely, the first-order $\lambda-$prolongation of a $\lambda-$symmetry defines a telescopic vector field of the equation. In consequence,

$$\mathbf{v}^{[\lambda,(1)]} = \partial_u + (2a+3u)\partial_{u_x}$$

  is a telescopic vector field of equation (1.1).
- **Semi-classical nonlocal symmetries:** Relationships between $\lambda-$symmetries and nonlocal symmetries associated to $\lambda-$coverings (called semi-classical nonlocal symmetries in [1])

were studied in [15]. As a direct consequence of Theorem 7 in [15], the vector field

$$\tilde{\mathbf{v}} = e^{v}(\partial_u + \psi(x,u,u_x))\partial_v$$

is a nonlocal symmetry of equation (1.1) associated to the covering

$$\begin{cases} u_{xx} = 3uu_x + 3au^2 + 4a^2u + b, \\ v_x = 2a + 3u, \end{cases}$$

where $\psi = \psi(x,u,u_x)$ is any particular solution of the equation (see Eq. (5.1) in [15])

$$\psi_x + u_x\psi_u + (3uu_x + 3au^2 + 4a^2u + b)\psi_{u_x} + (2a+3u)\psi = 3.$$

We omit here the calculation of the infinitesimal ψ, because it was proven in [15] that the order reduction associated to the semi-classical nonlocal symmetry is equivalent to the reduction (2.10) obtained by using the $\lambda-$symmetry.

In the next section we extend the results obtained for the particular equation (1.1) to a wide family of second-order ODEs equations, which contains several particular equations not completely solved so far in the literature.

3. The general procedure

In this section we investigate the class of equations of the form

$$u_{xx} + a_1(x,u)u_x + a_0(x,u) = 0 \tag{3.1}$$

which can be solved by techniques similar to those used in the previous section for equation (1.1).

We restrict our attention to the equations of the form (3.1) which belong to $\mathscr{A}_2$, because the equations in class $\mathscr{A}_1$ can be solved by classical methods. By (2.19) and (2.20), equation (3.1) belongs to $\mathscr{A}_2$ if the coefficients $a_0 = a_0(x,u)$ and $a_1 = a_1(x,u)$ satisfy $S_1 = a_{1u} \neq 0$ and

$$S_3 \equiv \left(\frac{S_2}{S_1}\right)_u + a_{1u} = 0, \qquad S_4 \equiv \left(\frac{S_2}{S_1}\right)_x + \left(\frac{S_2}{S_1}\right)^2 + a_1\left(\frac{S_2}{S_1}\right) + a_{0u} = 0 \tag{3.2}$$

where $S_2 = -a_1a_{1u} + a_{0uu} - a_{1xu}$. The condition $S_3 = 0$ implies that

$$\left(\frac{a_{0uu} - a_{1xu}}{a_{1u}}\right)_u = 0. \tag{3.3}$$

In terms of the function

$$h(x) = \frac{a_{0uu} - a_{1xu}}{a_{1u}} \tag{3.4}$$

the condition $S_4 = 0$ can be written as follows:

$$h'(x) - a_1h(x) + h^2(x) + a_{0u} - a_{1x} = 0. \tag{3.5}$$

Any equation of the form (3.1) whose coefficients satisfy conditions (3.3) and (3.5) admits the $\lambda-$symmetry (∂_u, λ) for

$$\lambda = \frac{S_2}{S_1} = h(x) - a_1,$$

where h is defined in (3.4). According to [17, Theorem 2], the coefficients $A = A(x,u)$ and $B = B(x,u)$ of a first integral (2.21) associated to this $\lambda-$symmetry can be computed by quadratures as follows:

$$A = \exp(a(x)), \tag{3.6}$$

where the function $a = a(x)$ is any primitive of the function $h = h(x)$ given in (3.4), i.e.,

$$a'(x) = h(x), \tag{3.7}$$

and $B = B(x,u)$ is any particular solution to the system

$$B_x = a_0 \exp(a(x)), \quad B_u = (a_1 - h(x)) \exp(a(x)). \tag{3.8}$$

Once the functions A and B have been determined, the auxiliary equations

$$A(x)u_x + B(x,u) = C_1, \quad C_1 \in \mathbb{R} \tag{3.9}$$

provide the general solution of equation (3.1) in class $\mathscr{A}_2$.

The functions A and B can be used to find a generalised Sundman transformation to transform equation (3.1) into $U_{XX} = 0$, as it was done for equation (1.1) in Section 2.2. Besides, the $\lambda-$symmetry $(\partial_u, h(x) - a_1(x,u))$ can be used to construct other types of symmetries, such as exponential vector fields, telescopic vector fields or nonlocal symmetries associated to $\lambda-$coverings, for the equations (3.1) in subclass $\mathscr{A}_2$. The construction of these symmetries will be omitted for the examples studied in Section 4 of this paper; the interested reader can directly obtain all them by following the same steps described at the end of Section 2.2 for equation (1.1).

As in the case of the introductory example (1.1), the general solution of a wide family of equations (3.1) in class $\mathscr{A}_2$ can be expressed in terms of a fundamental set of solutions of a related second-order linear equation, although such equations are not linearisable by point transformations because they belong to $\mathscr{A}_2$. Such family corresponds to the case when the coefficient $a_1(x,u)$ is linear in the dependent variable u. In this case, a solution of (3.8) must be of the form $B(x,u) = b_2(x)u^2 + b_1(x)u + b_0(x)$, with $b_2 \neq 0$. Therefore the auxiliary equation (3.9) is a Riccati-type equation:

$$A(x)u_x + b_2(x)u^2 + b_1(x)u + b_0(x) = C_1,$$

which can be transformed into the linear second-oder ODE

$$A(x)^2\phi''(x) = \left(\frac{b_2'(x)}{b_2(x)}A(x)^2 - b_1(x)A(x) - A(x)^2h'(x)\right)\phi'(x) + (b_0(x) - C_1)b_2(x)\phi(x) \tag{3.10}$$

through the standard transformation

$$u(x) = \frac{A(x)}{b_2(x)}\frac{\phi'(x)}{\phi(x)}.$$

Therefore, the general solution of the original second-order ODE can be expressed in terms of two linearly independent solutions $\phi_1(x;C_1), \phi_2(x;C_1)$ of the linear second-oder ODE (3.10), as it was done in (2.16) for equation (1.1):

$$u(x) = \frac{A(x)}{b_2(x)}\left(\frac{\phi_1(x;C_1) - C_2\,\phi_1'(x;C_1)}{\phi_2(x;C_1) - C_2\,\phi_2'(x;C_1)}\right). \tag{3.11}$$

As a by-product of the procedure, a remaining first integral $I_2 = I_2(x,u,u_x)$ of the original equation can be expressed in terms of the fundamental set of solutions of the linear second-oder ODE (3.10) as follows. For $i=1,2$, the integration constant C_1 is replaced by $I_1 = A(x)u_x + B(x,u)$ into $\phi_i(x;C_1)$ and $\phi_i'(x;C_1)$ and denote the resultant expressions by $\phi_i(x;I_1)$ and $\phi_i'(x;I_1)$, respectively. Then it can be checked, by using for instance [22, Proposition 4.1], that the function

$$I_2(x,u,u_x) = \frac{u(x)b_2(x)\phi_1(x;I_1) + A(x)\phi_1'(x;I_1)}{u(x)b_2(x)\phi_2(x;I_1) + A(x)\phi_2'(x;I_1)} \tag{3.12}$$

is a first integral of the original equation. Moreover, the two first integrals $I_1 = A(x)u_x + B(x,u)$ and I_2 are functionally independent.

In the next section we illustrate the described procedure to integrate four equations which lack Lie point symmetries.

4. Some examples

4.1. *Example I*

Let us consider the equation

$$u_{xx} + kuu_x + \frac{1}{2}kx^2u^2 - \left(x^4 + 2x\right)u + \frac{1}{x} = 0, \qquad k \neq 0. \tag{4.1}$$

It can be checked that equation (4.1) does not admit Lie point symmetries. The equation belongs to class $\mathscr{A}_2$ because its coefficients satisfy $S_1 = k \neq 0, S_2 = kx^2 - k^2u$ and $S_3 = S_4 = 0$. By (3.4) we find that $h(x) = x^2$. By setting $a(x) = 1/3x^3$ in (3.7), system (3.8) becomes

$$B_x = \frac{\exp(1/3x^3)}{2x}\left(ku^2x^3 - 2ux^5 - 4ux^2 + 2\right), \qquad B_u = \exp(1/3x^3)\left(ku - x^2\right).$$

A solution of this system is

$$B = \exp(1/3x^3)\left(\frac{1}{3}ku^2 - ux^2\right) + B_1(x) \tag{4.2}$$

where $B_1 = B_1(x)$ satisfies

$$B_1'(x) = \frac{\exp(1/3x^3)}{x}. \tag{4.3}$$

Previous calculations show:

(1) Equation (4.1) lacks Lie point symmetries but admits the $\lambda-$symmetry $(\partial_u, -x^2 + ku)$.
(2) A first integral of equation (4.1) associated to the $\lambda-$symmetry $(\partial_u, -x^2 + ku)$ becomes

$$I_1(x,u,u_x) = \exp(1/3x^3)\left(u_x + \frac{1}{3}ku^2 - ux^2\right) + B_1(x),$$

where B_1 verifies (4.3).

The corresponding auxiliary equation (3.9) is a Riccati-type equation, which can be transformed into the linear second-order ODE

$$\phi''(x) = x^2\phi'(x) + \frac{k}{2}\exp(-1/3x^3)k(C_1 - B_1(x))\phi(x) \tag{4.4}$$

through the standard transformation

$$u(x) = \frac{2}{k}\frac{\phi'(x)}{\phi(x)}.$$

By using [22, Proposition 4.1] the general solution of equation (4.1) can be expressed in terms of two linearly independent solutions $\phi_1(x;C_1)$ and $\phi_2(x;C_1)$ of (4.4) as follows:

$$u(x) = \frac{2}{k}\frac{C_2\,\phi_2'(x;C_1) - \phi_1'(x;C_1)}{C_2\,\phi_2(x;C_1) - \phi_1(x;C_1)}, \tag{4.5}$$

where $C_1, C_2 \in \mathbb{R}$.

4.2. *Example II*

The family of equations

$$u_{xx} + (2g(x)u + f(x))u_x + g'(x)u^2 + f'(x)u = 0 \tag{4.6}$$

does not admit Lie point symmetries for arbitrary smooth functions f and g. The coefficients of (4.6) satisfy $S_1 = 2g(x), S_2 = -4g(x)u - 2f(x)$. We assume that $S_1 = g(x) \neq 0$, and then $S_3 = S_4 = 0$.

For equation (4.6) the corresponding function (3.4) is $h(x) = 0$. By setting $a(x) = 0$, we obtain $A(x) = 1$ and the corresponding system (3.8) becomes

$$B_x = f'(x)u + g'(x)u^2, \qquad B_u = 2g(x)u + f(x). \tag{4.7}$$

This system can be easily integrated and yields the particular solution $B(x,u) = g(x)u^2 + f(x)u$. Therefore,

(1) The pair (∂_u, λ), for $\lambda = -2g(x)u - f(x)$, is a $\lambda-$symmetry of equation (4.6).
(2) The function $I_1(x,u,u_x) = u_x + g(x)u^2 + f(x)u$ is a first integral of (4.6) associated to the former $\lambda-$symmetry.

The auxiliary equation $I_1 = C_1$, where $C_1 \in \mathbb{R}$, is the Riccati equation $u_x + g(x)u^2 + f(x)u = C_1$. The associated linear second-order ODE by means of the transformation

$$u(x) = \frac{1}{g(x)}\frac{\phi'(x)}{\phi(x)}$$

becomes

$$\phi''(x) + \left(f(x) - \frac{g'(x)}{g(x)}\right)\phi'(x) - C_1 g(x)\phi(x) = 0. \tag{4.8}$$

In terms of a fundamental set of solutions $\phi_1(x;C_1)$ and $\phi_2(x;C_2)$ of (4.8), the general solution of (4.6) can be expressed in the following form:

$$u(x) = \frac{1}{g(x)}\left(\frac{C_2\,\phi_2'(x;C_1) - \phi_1'(x;C_1)}{C_2\,\phi_2(x;C_1) - \phi_1(x;C_1)}\right), \tag{4.9}$$

where $C_1, C_2 \in \mathbb{R}$.

The solutions of the linear equation (4.8) can also be used to obtain the following expression of a first integral I_2 of equation (4.6):

$$I_2(x,u,u_x) = \frac{-u\,g(x)\phi_1(x;I_1)+\phi_1'(x;I_1)}{-u\,g(x)\phi_2(x;I_1)+\phi_2'(x;I_1)}.$$

The two first integrals $I_1 = u_x + g(x)u^2 + f(x)u$ and I_2 are functionally independent.

It is interesting to notice that our procedure reveals a method to obtain by integration a one-parameter family of solutions of equation (4.6). This family of solutions corresponds to $C_1 = 0$, in which case two linearly independent solutions of (4.8) become

$$\phi_1(x;0) = 1, \phi_2(x;0) = \int \exp\Big(\int \Big(\frac{g'(x)}{g(x)} - f(x)\Big)\,dx\Big)dx.$$

Therefore

$$u(x) = \frac{1}{g(x)} \frac{C_2 \exp\Big(\int \Big(\frac{g'(x)}{g(x)} - f(x)\Big)\,dx\Big)}{C_2 \int \exp\Big(\int \Big(\frac{g'(x)}{g(x)} - f(x)\Big)\,dx\Big)dx - 1}, \tag{4.10}$$

where $C_2 \in \mathbb{R}$, is a one-parameter family of exact solutions of equation (4.6) obtained by direct integration.

4.2.1. *A particular case: Eq. 6.36 in Kamke's book*

The family of equations

$$u_{xx} + (2u + f(x))u_x + f'(x)u = 0, \tag{4.11}$$

where f is an arbitrary smooth function, can be reduced by integration to a Riccati equation after using the change of the dependent variable $y = u + f(x)/2$ (see [9, Eq. 6.36] and [8]). However, as far as we know, no further information on its general solution or first integrals has been obtained from this *ad hoc* procedure.

Since this equation is a particular case of equation (4.6), for $g(x) = 1$, the systematic approach developed in Subsection 4.2 can be followed to obtain its general solution and two functionally independent first integrals by considering the corresponding linear second-order ODE (4.8):

$$\phi''(x) + f(x)\phi'(x) - C_1\phi(x) = 0. \tag{4.12}$$

In terms of a fundamental set of solutions $\phi_1(x;C_1)$ and $\phi_2(x;C_2)$ of (4.12), the general solution of (4.11) can be expressed in the following form:

$$u(x) = \frac{C_2\,\phi_2'(x;C_1) - \phi_1'(x;C_1)}{C_2\,\phi_2(x;C_1) - \phi_1(x;C_1)}, \tag{4.13}$$

where $C_1, C_2 \in \mathbb{R}$. Besides, the functions

$$I_1 = u_x + u^2 + f(x)u, \qquad I_2 = \frac{-u\phi_1(x;I_1)+\phi_1'(x;I_1)}{-u\phi_2(x;I_1)+\phi_2'(x;I_1)}$$

are two functionally independent first integrals of equation (4.11).

The one-parameter family of exact solutions of equation (4.11) that can be obtained by direct integration, corresponding to $C_1 = 0$, becomes

$$u(x) = \frac{C_2 \exp\left(-\int f(x)dx\right)}{C_2 \int \exp\left(-\int f(x)dx\right)dx - 1}. \tag{4.14}$$

To end this example, we recall that, except for a few particular functions $f = f(x)$, equation (4.6) does not admit Lie point symmetries. This is the case, for instance, of the equation

$$u_{xx} + \left(2u + \frac{1-x}{x^2}\right)u_x + \frac{x-2}{x^3}u = 0 \tag{4.15}$$

corresponding to the function $f(x) = \dfrac{1-x}{x^2}$.

4.3. *Example III*

In this example we study the equation 6.37 in Kamke's book [9], which depends on two arbitrary smooth functions, f and g :

$$u_{xx} + (2u + f(x))u_x + f(x)u^2 - g(x) = 0. \tag{4.16}$$

The coefficients $a_1(x,u) = 2u + f(x)$ and $a_0(x,u) = f'(x)u^2 - g(x)$ satisfy $S_1 = 2, S_2 = -4u$ and $S_3 = S_4 = 0$. Therefore, any equation of the form (4.16) admits the pair (∂_u, λ), where $\lambda = -2u$, as $\lambda-$symmetry. It is important to notice that, for arbitrary functions f and g, equation (4.16) does not admit Lie point symmetries.

Next we calculate the first integral (2.21) of equation (4.16) associated to the $\lambda-$symmetry $(\partial_u, -2u)$. The corresponding function (3.4) becomes $h(x) = f(x)$. Then $a = a(x)$ is any primitive of function $f = f(x)$ and (3.6) yields

$$A(x) = \exp\left(\int f(x)dx\right).$$

System (3.8) becomes

$$B_x = \big(f(x)u^2 - g(x)\big)A(x), \qquad B_u = 2uA(x), \tag{4.17}$$

which provides the particular solution

$$B(x,u) = A(x)f(x)u^2 - \int g(x)A(x)dx.$$

The associated linear second-order ODE to the Riccati equation

$$A(x)u_x + A(x)f(x)u^2 - \int g(x)A(x)dx = C_1, \qquad C_1 \in \mathbb{R}$$

by setting $u(x) = \phi'(x)/\phi(x)$, becomes

$$\phi''(x) - \frac{1}{A(x)}\left(\int g(x)A(x)dx - C_1\right)\phi(x) = 0. \tag{4.18}$$

In terms of a fundamental set of solutions $\phi_1(x;C_1),\phi_2(x;C_1)$ of (4.18), the general solution of equation (4.16) becomes

$$u(x)=\frac{C_2\,\phi_2'(x;C_1)-\phi_1'(x;C_1)}{C_2\,\phi_2(x;C_1)-\phi_1(x;C_1)}. \tag{4.19}$$

For illustration purposes, we consider a particular equation of the family (4.16) which does not admit Lie point symmetries. For $f(x)=1/x$ and $g(x)=x^3$ equation (4.16) becomes

$$u_{xx}+\left(2u+\frac{1}{x}\right)u_x+\frac{u^2}{x}-x^3=0 \tag{4.20}$$

It can be checked that this equation does not admit Lie symmetries. The application of the procedure provides its general solution (4.19) in terms of a fundamental set of solutions of the linear second-order ODE

$$\phi''(x)-\frac{1}{x}\left(\frac{x^5}{5}-C_1\right)\phi(x)=0. \tag{4.21}$$

The procedure also provides two functionally independent first integrals of the equation (4.20) in terms of a fundamental system of solutions of equation (4.21):

$$I_1=xu_x+u^2-\frac{x^5}{5},\qquad I_2=\frac{u\,\phi_1(x;I_1)+x\phi_1'(x;I_1)}{u\,\phi_2(x;I_1)+x\phi_2'(x;I_1)}.$$

4.4. *Example IV*

Let us consider the one-parameter family of equations

$$u_{xx}+\left(\frac{u-1}{x}\right)u_x+4k^2=0,\quad k>0, \tag{4.22}$$

which lacks Lie point symmetries. For equation (4.22) we have $S_1=\dfrac{1}{x},S_2=\dfrac{2-u}{x^2}$ and $S_2=S_4=0$. Therefore equation (4.22) admits the $\lambda-$symmetry

$$\left(\partial_u,\frac{2-u}{x}\right)$$

and the associated first integral is of the form $I_1(x,u,u_x)=A(x)u_x+B(x,u)$. By using the primitive $a(x)=\ln(x)$ of the function $h(x)=1/x$ corresponding to (3.4), the coefficient $A=A(x)$ becomes $A=1/x$ and the coefficient $B=B(x,u)$ is any particular solution to system (3.8):

$$B_x=4k^2x,\qquad B_u=u-2. \tag{4.23}$$

Bu using the solution $B=u^2/2-2u+2k^2x^2$, the linear second-order ODE associated to the Riccati equation $A(x)u_x+B(x,u)=C_1,C_1\in\mathbb{R}$, becomes

$$\phi''(x)-\frac{1}{x}\phi'(x)+\left(k^2-\frac{C_1}{2x}\right)\phi(x)=0. \tag{4.24}$$

Two linearly independent solutions to equation (4.24) can be expressed in terms of the Bessel functions of the first and second kind, J_ν and Y_ν, as follows

$$\phi_1(x;C_1)=xJ_\nu(kx),\qquad \phi_2(x;C_1)=xY_\nu(kx), \tag{4.25}$$

where $\nu = 1/2\sqrt{4+2C_1}$. In consequence, the general solution of equation (4.22) becomes:

$$u(x) = 2\left(1 + kx\,\frac{C_2 Y_\nu'(kx) - J_\nu'(kx)}{C_2 Y_\nu(kx) - J_\nu(kx)}\right). \tag{4.26}$$

5. Concluding remarks and extensions

A class of second-order ODEs that can be linearised by generalised Sundman transformations, but that do not satisfy the Lie criteria of linearisation, has been analysed in the paper. It has been proven that the general solutions of the subclass of these equations that admit a first integral of the Riccati type can be explicitly expressed in terms of a fundamental set of solutions of a related second-order linear equation. This fundamental set of solutions has been also used to derive the expression of a second first integral, which is functionally independent to the first integral of the Riccati type. The procedure has been applied to solve several ODEs lacking Lie point symmetries in terms of fundamental sets of solutions of second-order linear equations.

The results presented in this paper need to be extended in different directions: in a forthcoming paper we plan to preform a similar study of the equations of the form (2.18) with $a_2 \neq 0$ that admit a first integral of the Riccati type. It is also interesting to extent the study to first integrals of Abel type, although this is a quite more complicated problem, because only a few integration strategies for some Abel ODEs are available.

Finally, it is expected that the application of the procedure and its future generalisations to relevant equations in mathematical physics (for instance, Linéard-type equations), will provide exact solutions of new classes of integrable equations that are usually studied by numerical methods.

Acknowledgements

J. Mendoza thanks the support of *Programa Iberoamericano de Formación de Doctores en Ciencias Básicas* during his stay at the University of Cádiz in 2017. C. Muriel acknowledges the support from the University of Cádiz (*Plan Propio de Investigación*) and from Junta de Andalucía to the research group FQM 377.

References

[1] D. Catalano-Ferraioli, Nonlocal aspects of λ-symmetries and ODEs reduction, *J. Phys. A: Math. Theor.* **40**(21) (2007) 5479–5489.

[2] L. G. S. Duarte, I. C. Moreira and F. C. Santos, Linearization under non-point transformations, *J. Phys. A: Math. Gen.* **27** (1994) L739–L743.

[3] N. Euler and M. Euler, Sundman symmetries of nonlinear second-order and third-order ordinary differential equations, *J. Nonlinear Math. Phys.* **11** (2004) 399–421.

[4] N. Euler, T. Wolf, P. Leach and M. Euler, Linearisable third-order ordinary differential equations and generalised Sundman transformations: the case $X''' = 0$, *Acta Appl. Math.* **76** (2003) 89–115.

[5] M. L. Gandarias, Nonlocal symmetries and reductions for some ordinary differential equations, *Theoretical and Mathematical Physics,* **159**(3) (2009) 779–786.

[6] A. González-López, Symmetry and integrability by quadratures of ordinary differential equations, *Phys. Lett. A* **133**(4-5) (1988) 190–194.

[7] N. H. Ibragimov, *A Practical Course in Differential Equations and Mathematical Modelling: Classical and New Methods, Nonlinear Mathematical Models, Symmetry and Invariance Principles* (World Scientific, Beijing, 2010).

[8] E. Ince, *Ordinary Differential Equations* (Dover Publications, New York, 1956).

[9] E. Kamke, *Differentialgleichungen Lösungsmethoden und Lösungen* (Chelsea, New York, 1948).

[10] S. Lie, Klassifikation und integration von gewöhnlichen differentialgleichungen zwischen x, y die eine gruppe von transformationen gestatten. III, *Arch. Mat. Naturvidenskab* **8** (1883) 371–458.

[11] R. Liouville, Sur une équation différentielle du premier ordre, *Acta Mathematica* **27**(1) (1903) 55–78.

[12] S. Moyo and S. Meleshko, Application of the generalised Sundman transformation to the linearisation of two second-order ordinary differential equations, *J. Nonlinear Math. Phys.* **18**(1) (2011) 213–236.

[13] C. Muriel and J. L. Romero, First integrals, integrating factors and λ-symmetries of second-order differential equations, *J. Phys. A: Math. Theor.* **42**(36) (2009) 365207 (17pp).

[14] C. Muriel and J. L. Romero, New methods of reduction for ordinary differential equations, *IMA J. Appl. Math.* **66**(2) (2001) 111–125.

[15] C. Muriel and J. L. Romero, Nonlocal symmetries, telescopic vector fields and λ-symmetries of ordinary differential equations, *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications* **8** (2012) 106 (21pp).

[16] C. Muriel and J. L. Romero, Nonlocal transformations and linearization of second-order ordinary differential equations, *J. Phys. A: Math. Theor.* **43**(43) (2010) 434025 (13pp).

[17] C. Muriel and J. L. Romero, Second-order ordinary differential equations and first integrals of the form $A(t,x)\dot{x} + B(t,x)$, *J. Nonlinear Math. Phys.* **16**(1) (2009) 209–222.

[18] C. Muriel and J. L. Romero, C^∞-Symmetries and reduction of equations without Lie point symmetries, *J. Lie Theory* **13**(1) (2003) 167–188.

[19] M. C. Nucci, Lie symmetries of a Painlevé-type equation without Lie symmetries, *J. Nonlinear Math. Phys.* **15 (2)** (2008) 201–211.

[20] P. Olver, *Applications of Lie groups to differential equations*, second edn. (Springer-Verlag, New York, 1993).

[21] E. Pucci and G. Saccomandi, On the reduction methods for ordinary differential equations, *J. Phys. A: Math. Gen.* **35**(29) (2002) 6145–6155.

[22] A. Ruiz and C. Muriel, First integrals and parametric solutions of third-order odes admitting $\mathfrak{sl}(2,\mathbb{R})$, *J. Phys. A: Math. Theor.* **50** (2017) 205201 (21pp).

[23] H. Stephani, *Differential equations: their solution using symmetries* (Cambridge University Press, Cambridge, 1989).