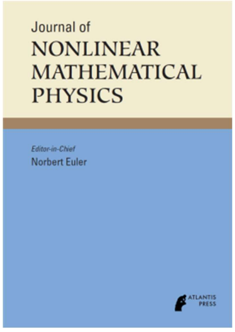

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On generalized Lax equations of the Lax triple of the BKP and CKP hierarchies

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Based on the Lax triple (B_m, B_n, L) of the BKP and CKP hierarchies, we derive the nonlinear evolution equations from the generalized Lax equation. The solutions of some evolution equations are presented, such as soliton and rational solutions.

Keywords: Lax Equation; BKP Hierarchy; CKP Hierarchy; Lax Triple.

2000 Mathematics Subject Classification: 37K10, 35Q53

1. Introduction

The Kadomtsev-Petviashvili (KP) hierarchy [10, 11, 18, 28] is one of the most important classical integrable systems. It has many applications including the theory of infinite-dimensional Lie algebras [16, 17], orthogonal polynomials and random matrix model [14, 27]. The KP equation, which is a natural generalization of the KdV equation [25], can be used to model water waves of long wavelength with weakly non-linear restoring forces and frequency dispersion. It can also be used to model nonlinear waves in ferromagnetic media, as well as two-dimensional matter-wave pulses in Bose-Einstein condensates. The BKP [10] and CKP hierarchies [9] are two kinds of sub-hierarchies of KP hierarchy. In other words, the two hierarchies are two important reductions of the KP hierarchy.

Nambu mechanics [22,23] is a generalization of classical Hamiltonian mechanics. Since Nambu [22] proposed 3-bracket for the generalized Hamiltonian dynamics, the 3-algebras have been studied

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intensively. With the development of string theory and M-branches, it appears naturally an algebra with ternary operation called Bagger-Lambert algebra [1]. More motivation comes from various 3-Lie algebras, in particular the infinite-dimensional cases such as (q-deformed) Virasoro-Witt 3-algebra [7, 8, 12] and (super) w_∞ 3-algebra [3, 6, 13]. Recently there has been an increasing interest in the relationship between the infinite-dimensional 3-algebra and integrable systems [4, 5, 30]. It is well known that the first Hamiltonian structure of KP hierarchy is related to the $W_{1+\infty}$ algebra [29]. The $W_{1+\infty}$ 3-algebra related to the KP hierarchy has been constructed in [4]. The BKP and CKP hierarchies are also associated with two infinite dimensional algebras $o(\infty)$ and $sp(\infty)$ [16]. The general symmetry reduction of the BKP hierarchy has been studied by means of tau functions [19]. In addition, many other Lax types have been investigated, such as the triple bracket equations of Lax type [15] and the generalized Lax pairs [2]. There are a lot of methods to describe the algebraic properties of the KP hierarchy. One was described by the associated Lax pair (B_n, L). By means of the operator Nambu 3-bracket, the generalized Lax equation of the KP hierarchy with the Lax triple (B_m, B_n, L) was studied in [26], where the KP equation and other integrable (nonintegrable) equations were derived. It is well known that the BKP and CKP hierarchies are two important reductions of the KP hierarchy. The constraints of the BKP and CKP hierarchies imply the vanishing of the even time variables. Thus the BKP and CKP hierarchies can be expressed by the Lax pair (B_{2n+1}, L). The aim of this paper is to derive the nonlinear evolution equations from the generalized Lax equation in term of the Lax triple (B_m, B_n, L) of the BKP and CKP hierarchies.

This paper is arranged as follows. In Section 2, the usual Lax equation is introduced, and the BKP and CKP hierarchies are given. In Section 3, we investigate the generalized Lax equation with the Lax triple (B_m, B_n, L) and derive the generalized BKP and CKP equations. In Sections 4, we give the solutions of the generalized nonlinear evolution equations. Finally, a short conclusion and further discussion are presented.

2. The usual Lax equation

The KP hierarchy is a paradigm of the hierarchies of integrable systems. It is defined as an infinite system of equations given in Lax form [11]

$$\frac{\partial L}{\partial t_n} = [B_n, L] = B_n L - L B_n, \quad n = 1, 2, \cdots . \tag{2.1}$$

where $B_n = (L^n)_+$ is the differential part of L^n. L is the pseudo-differential operator

$$L = \partial + \sum_{i=0}^{+\infty} v_i(t)\partial^{-i-1}, \tag{2.2}$$

$t = (t_1, t_2, \cdots)$ are the time variables and $\partial = \partial/\partial_x, x = t_1$.

The first few members of B_n are

$$\begin{aligned}
B_1 &= \partial, \\
B_2 &= \partial^2 + 2v_0, \\
B_3 &= \partial^3 + 3v_0\partial + 3v_1 + 3v_{0,x}, \\
B_4 &= \partial^4 + 4v_0\partial^2 + (4v_1 + 6v_{0,x})\partial + 4v_2 + 6v_0^2 + 6v_{1,x} + 4v_{0,xx}, \\
B_5 &= \partial^5 + 5v_0\partial^3 + 5(v_1 + 2v_{0,x})\partial^2 + 5(v_2 + 2v_0^2 + 2v_{1,x} + 2v_{0,xx})\partial \\
&\quad +5(v_3 + 4v_0v_1 + 4v_0v_{0,x} + 2v_{2,x} + 2v_{1,xx} + v_{0,xxx}), \\
B_6 &= \partial^6 + 6v_0\partial^4 + 3(2v_1 + 5v_{0,x})\partial^3 + (6v_2 + 15v_0^2 + 15v_{1,x} + 20v_{0,xx})\partial^2 \\
&\quad +(6v_3 + 30v_0v_1 + 15v_{2,x} + 45v_0v_{0,x} + 20v_{1,xx} + 15v_{0,xxx})\partial \\
&\quad +6v_4 + 6v_{0,xxxx} + 20v_{2,xx} + 15v_{1,xxx} + 15v_{3,x} + 30v_0v_2 \\
&\quad +45v_0v_{1,x} + 30v_1v_{0,x} + 35v_0v_{0,xx} + 20v_0^3 + 25v_{0,x}^2 + 15v_1^2, \\
&\vdots,
\end{aligned} \tag{2.3}$$

where the subscript x denotes the derivative with respect to the variable x.

The BKP and CKP hierarchies are also defined as an infinite dimensional of nonlinear evolution equations given in Lax form (2.1). However the operator L needs to satisfy the constraints $L^* = -\partial L\partial^{-1}$ and $L^* = -L$, respectively.

The BKP hierarchy is obtained from the KP hierarchy by imposing the condition

$$L^* = -\partial L\partial^{-1}, \tag{2.4}$$

which implies the vanishing of the even time variables (i.e., $t_2 = t_4 = \cdots = 0$) and of the constant terms of B_n $(n = 3, 5, \cdots)$ [10, 18, 28].

Equating the coefficients of the operator $\partial^{-i}(i = 2, 3, \cdots)$ in (2.4), we can derive

$$\begin{aligned}
v_1 &= -v_{0,x}, \\
v_3 &= -2v_{2,x} + v_{0,xxx}, \\
v_5 &= -3v_{4,x} + 5v_{2,xxx} - 3v_{0,xxxxx}, \\
v_7 &= -4v_{6,x} + 14v_{4,xxx} - 28v_{2,xxxxx} + 17v_{0,xxxxxxx}, \\
&\vdots.
\end{aligned} \tag{2.5}$$

The CKP hierarchy is obtained from the KP hierarchy by imposing the condition

$$L^* = -L, \tag{2.6}$$

which also implies the vanishing of the even time variables (i.e., $t_2 = t_4 = \cdots = 0$) [18].

Equating the coefficients of the operator $\partial^{-i}(i=2,3,\cdots)$ in (2.6), we can derive

$$
\begin{aligned}
v_1 &= -\frac{1}{2}v_{0,x},\\
v_3 &= -\frac{3}{2}v_{2,x}+\frac{1}{4}v_{0,xxx},\\
v_5 &= -\frac{5}{2}v_{4,x}+\frac{5}{2}v_{2,xxx}-\frac{1}{2}v_{0,xxxxx},\\
v_7 &= -\frac{7}{2}v_{6,x}+\frac{35}{4}v_{4,xxx}-\frac{21}{2}v_{2,xxxxx}+\frac{17}{8}v_{0,xxxxxxx},\\
&\vdots.
\end{aligned}\qquad (2.7)
$$

Equations (2.5) and (2.7) show that the odd dynamical variables of $\{v_j\}$ can be expressed by the even ones of $\{v_j\}$, and the even dynamical variables of $\{v_j\}$ are independent.

Let us list the usual BKP and CKP equations. Taking $B_n=B_3$ in (2.1) and equating the coefficient of $\partial^{-i-1}(i=0,2,4,\cdots)$ with the left and right-hands side of (2.1), we obtain

$$
\begin{aligned}
\frac{\partial v_0}{\partial y} &= 3v_{2,x}+3v_{1,xx}+v_{0,xxx}+6v_0v_{0,x},\\
\frac{\partial v_2}{\partial y} &= 3v_{4,x}+3v_{3,xx}+v_{2,xxx}+3v_0v_{2,x}+9v_2v_{0,x}-3v_0v_{1,xx}+6v_1v_{1,x}-3v_1v_{0,xx},\\
\frac{\partial v_4}{\partial y} &= 3v_{6,x}+3v_{5,xx}+v_{4,xxx}+3v_0v_{4,x}+15v_4v_{0,x}+12v_3v_{1,x}-18v_3v_{0,xx}-18v_2v_{1,xx}\\
&\quad +12v_2v_{0,xxx}+12v_1v_{1,xxx}-3v_0v_{1,xxxx}-3v_1v_{0,xxxx},\\
&\vdots,
\end{aligned}\qquad (2.8)
$$

where $y=t_3$.

From (2.5) and (2.8), we derive the constraints of the BKP hierarchy as follows

$$
\begin{aligned}
v_1 &= -v_{0,x},\\
v_2 &= \frac{1}{3}\partial^{-1}v_{0,y}-v_0^2+\frac{2}{3}v_{0,xx},\\
v_3 &= -\frac{2}{3}v_{0,y}+4v_0v_{0,x}-\frac{1}{3}v_{0,xxx},\\
v_4 &= \frac{1}{9}\partial^{-2}v_{0,yy}-\partial^{-1}(v_0v_{0,y})+\frac{7}{9}v_{0,xy}-\partial^{-1}(v_{0,x}\partial^{-1}v_{0,y})+\frac{5}{3}v_0^3-\frac{15}{2}v_{0,x}^2\\
&\quad -5\partial^{-1}(v_0v_{0,xxx})+\frac{1}{9}v_{0,xxxx},\\
&\vdots.
\end{aligned}\qquad (2.9)
$$

From (2.7) and (2.8), we obtain the constraints of the CKP hierarchy as follows

$$
\begin{aligned}
v_1 &= -\frac{1}{2}v_{0,x},\\
v_2 &= \frac{1}{3}\partial^{-1}v_{0,y} - v_0^2 + \frac{1}{6}v_{0,xx},\\
v_3 &= -\frac{1}{2}v_{0,y} + 3v_0v_{0,x},\\
v_4 &= \frac{1}{9}\partial^{-2}v_{0,yy} - \partial^{-1}(v_0v_{0,y}) + \frac{4}{9}v_{0,xy} - \partial^{-1}(v_{0,x}\partial^{-1}v_{0,y}) + \frac{5}{3}v_0^3 - \frac{17}{4}v_{0,x}^2\\
&\quad -3\partial^{-1}(v_0v_{0,xxx}) - \frac{1}{18}v_{0,xxxx},\\
&\vdots
\end{aligned}
\tag{2.10}
$$

Taking $B_n = B_5$ in (2.1) and equating the coefficient of ∂^{-1}, we have

$$
\begin{aligned}
\frac{\partial v_0}{\partial t} &= 20v_0v_{2,x} + 20v_1v_{1,x} + 20v_2v_{0,x} + 20v_0v_{1,xx} + 10v_1v_{0,xx} + 10v_0v_{0,xxx} + 10v_{2,xxx}\\
&\quad +5v_{1,xxxx} + v_{0,xxxxx} + 10v_{3,xx} + 5v_{4,x} + 30v_{0,x}v_0^2 + 20v_{0,x}v_{0,xx} + 30v_{1,x}v_{0,x},
\end{aligned}
\tag{2.11}
$$

where $t = t_5$.

Substituting (2.9) into (2.11), we obtain the usual BKP equation

$$
\begin{aligned}
\frac{\partial v_0}{\partial t} &= \frac{5}{3}v_0v_{0,y} - 5v_{0,x}v_0^2 - \frac{5}{3}v_0v_{0,xxx} - \frac{5}{3}v_{0,x}v_{0,xx} + \frac{5}{3}v_{0,x}\partial^{-1}v_{0,y}\\
&\quad +\frac{5}{9}v_{0,xxy} - \frac{1}{9}v_{0,xxxxx} + \frac{5}{9}\partial^{-1}v_{0,yy}.
\end{aligned}
\tag{2.12}
$$

Substituting (2.10) into (2.11), we obtain the usual CKP equation

$$
\begin{aligned}
\frac{\partial v_0}{\partial t} &= \frac{5}{3}v_0v_{0,y} - 5v_{0,x}v_0^2 - \frac{5}{3}v_0v_{0,xxx} - \frac{25}{6}v_{0,x}v_{0,xx} + \frac{5}{3}v_{0,x}\partial^{-1}v_{0,y}\\
&\quad +\frac{5}{9}v_{0,xxy} - \frac{1}{9}v_{0,xxxxx} + \frac{5}{9}\partial^{-1}v_{0,yy}.
\end{aligned}
\tag{2.13}
$$

3. Generalized Lax equation with the Lax triple

The operator Nambu 3-bracket is defined by [22]

$$
[\tilde{A},\tilde{B},\tilde{C}] = [\tilde{A},\tilde{B}]\tilde{C} + [\tilde{B},\tilde{C}]\tilde{A} + [\tilde{C},\tilde{A}]\tilde{B},
\tag{3.1}
$$

where $[\tilde{A},\tilde{B}] = \tilde{A}\tilde{B} - \tilde{B}\tilde{A}$.

By means of (3.1), the generalized zero curvature formulations with the Lax triple (B_m,B_n,L) is as follows

$$
\frac{\partial L}{\partial t_{mn}} = [B_m,B_n,L]_-, \quad (m,n = 0,1,2\cdots),
\tag{3.2}
$$

where $B_0 = 1$, the operator Nambu 3-bracket $[,,]_-$ denotes the formal integration operator part of the derived pseudo-differential operator.

Taking $B_m = B_0$ in (3.2), it is easy to verify that (3.2) leads to the Lax equation (2.1),

$$\frac{\partial L}{\partial t_{0n}} = [B_0, B_n, L]_- = [B_n, L]. \tag{3.3}$$

Thus it is natural to derive the BKP and CKP hierarchies from (3.3).

3.1. *Generalized Lax equation*

• Taking the operator pair (B_1, B_2) in (3.2), we obtain

$$\frac{\partial v_0}{\partial t} = v_{2,x} + v_{1,xx} + 2v_0 v_{0,x}, \tag{3.4}$$

where $t = t_{12}$.

• Taking the operator pair (B_1, B_3) in (3.2), we obtain

$$\frac{\partial v_0}{\partial t} = 6v_1 v_{0,x} + 6v_0 v_{1,x} + 3v_0 v_{0,xx} + 3v_{0,x}^2 + 2v_{3,x} + 3v_{2,xx} + v_{1,xxx}, \tag{3.5}$$

where $t = t_{13}$.

• Taking the operator pair (B_1, B_4) in (3.2), we obtain

$$\begin{aligned}\frac{\partial v_0}{\partial t} = {} & 12v_2 v_{0,x} + 12v_1 v_{1,x} + 6v_1 v_{0,xx} + 12v_0 v_{2,x} + 18v_{0,x} v_0^2 + 12v_0 v_{1,xx} + 6v_0 v_{0,xxx} \\ & +18v_{1,x} v_{0,x} + 8v_{0,x} v_{0,xx} + 3v_{4,x} + v_{1,xxxx} + 6v_{3,xx} + 4v_{2,xxx},\end{aligned} \tag{3.6}$$

where $t = t_{14}$.

• Taking the operator pair (B_2, B_3) in (3.2), we obtain

$$\begin{aligned}\frac{\partial v_0}{\partial t} = {} & v_{4,x} + 2v_{3,xx} + v_{2,xxx} + 3v_0 v_{1,xx} + 3v_1 v_{0,xx} + 3v_{0,x} v_{0,xx} \\ & +3v_2 v_{0,x} + v_0 v_{0,xxx} + 6v_{0,x} v_{1,x} + 6v_1 v_{1,x} + 3v_0 v_{2,x},\end{aligned} \tag{3.7}$$

where $t = t_{23}$.

• Taking the operator pair (B_1, B_5) in (3.2), we obtain

$$\begin{aligned}\frac{\partial v_0}{\partial t_{15}} = {} & 4v_{5,x} + 10v_{4,xx} + 10v_{3,xxx} + 5v_{2,xxxx} + v_{1,xxxxx} + 20v_{3,x} v_0 + 20v_3 v_{0,x} \\ & +20v_1 v_{0,xxx} + 20v_0 v_{1,xxx} + 10(v_{0,xx})^2 + 20v_0^2 v_{0,xx} + 20v_1 v_{1,xx} + 20v_{1,x}^2 \\ & +80v_1 v_0 v_{0,x} + 5v_0 v_{0,xxxx} + 30v_{1,x} v_{0,xx} + 30v_{0,x} v_{1,xx} + 15v_{0,x} v_{0,xxx} + 40v_0 v_{0,x}^2 \\ & +40v_{2,x} v_{0,x} + 30v_0 v_{2,xx} + 10v_2 v_{0,xx} + 20v_2 v_{1,x} + 20v_1 v_{2,x} + 40v_0^2 v_{1,x},\end{aligned} \tag{3.8}$$

where $t = t_{15}$.

• Taking the operator pair (B_2, B_4) in (3.2), we obtain

$$\begin{aligned}\frac{\partial v_0}{\partial t} = {} & 2v_{5,x} + 5v_{4,xx} + 4v_{3,xxx} + v_{2,xxxx} + 8v_{3,x} v_0 + 8v_3 v_{0,x} + 12v_0 v_{0,x}^2 \\ & +4v_1 v_{0,xxx} + 12v_0 v_{1,xxx} + 2(v_{0,xx})^2 + 6v_0^2 v_{0,xx} + 12v_1 v_{1,xx} + 12v_{1,x}^2 \\ & +24v_1 v_0 v_{0,x} + 4v_0 v_{0,xxxx} + 10v_{1,x} v_{0,xx} + 18v_{0,x} v_{1,xx} + 6v_{0,x} v_{0,xxx} \\ & +18v_{2,x} v_{0,x} + 12v_0 v_{2,xx} + 6v_2 v_{0,xx} + 12v_2 v_{1,x} + 12v_1 v_{2,x} + 12v_0^2 v_{1,x},\end{aligned} \tag{3.9}$$

where $t = t_{24}$.

3.2. *Generalized BKP equations*

- For the pair (B_1, B_2), substituting (2.9) into (3.4), we obtain

$$\frac{\partial v_0}{\partial t} = \frac{1}{3}(v_{0,y} - v_{0,xxx}). \tag{3.10}$$

- For the pair (B_1, B_3), substituting (2.9) into (3.5), we obtain

$$\frac{\partial v_0}{\partial t} = -v_{0,x}^2 - v_0 v_{0,xx} - \frac{1}{3}v_{0,xy} + \frac{1}{3}v_{0,xxxx}. \tag{3.11}$$

- For the pair (B_1, B_4), substituting (2.9) into (3.6), we obtain

$$\frac{\partial v_0}{\partial t} = v_{0,x}\partial^{-1}v_{0,y} - 3v_{0,x}v_0^2 + 7v_{0,x}v_{0,xx} + v_0 v_{0,y} + 3v_0 v_{0,xxx} + \frac{1}{3}\partial^{-1}v_{0,yy} - \frac{1}{3}v_{0,xxy}. \tag{3.12}$$

- For the pair (B_2, B_3), substituting (2.9) into (3.7), we obtain

$$\frac{\partial v_0}{\partial t} = -4v_{0,x}v_0^2 + 5v_{0,x}v_{0,xx} + v_0 v_{0,xxx} + \frac{1}{9}\partial^{-1}v_{0,yy} - \frac{2}{9}v_{0,xxy} + \frac{1}{9}v_{0,xxxxx}. \tag{3.13}$$

- For the pair (B_1, B_5), substituting (2.9) into (3.8), we obtain

$$\begin{aligned}\frac{\partial v_0}{\partial t} = & -\frac{2}{9}v_{0,yy} - \frac{8}{3}v_{0,x}v_{0,y} - \frac{4}{3}v_0 v_{0,xy} + \frac{1}{9}v_{0,xxxy} - \frac{4}{3}v_{0,xx}\partial^{-1}v_{0,y} - \frac{20}{3}v_{0,xx}^2 \\ & -\frac{25}{3}v_{0,x}v_{0,xxx} - \frac{5}{3}v_0 v_{0,xxxx} + \frac{1}{9}v_{0,xxxxxx}.\end{aligned} \tag{3.14}$$

- For the pair (B_2, B_4), substituting (2.9) into (3.9), we obtain

$$\begin{aligned}\frac{\partial v_0}{\partial t} = & -\frac{1}{9}v_{0,yy} - \frac{4}{3}v_{0,x}v_{0,y} - \frac{1}{3}v_0 v_{0,xy} + \frac{2}{9}v_{0,xxxy} - v_{0,xx}\partial^{-1}v_{0,y} + 3v_0^2 v_{0,xx} + 6v_0 v_{0,x}^2 \\ & -3v_{0,xx}^2 - \frac{20}{3}v_{0,x}v_{0,xxx} - \frac{11}{3}v_0 v_{0,xxxx} - \frac{1}{9}v_{0,xxxxxx}.\end{aligned} \tag{3.15}$$

3.3. *Generalized CKP equations*

- For the pair (B_1, B_2), substituting (2.10) into (3.4), we obtain

$$\frac{\partial v_0}{\partial t} = \frac{1}{3}(v_{0,y} - v_{0,xxx}). \tag{3.16}$$

- For the pair (B_1, B_3), substituting (2.10) into (3.5), we obtain

$$\frac{\partial v_0}{\partial t} = 0. \tag{3.17}$$

- For the pair (B_1, B_4), substituting (2.10) into (3.6), we obtain

$$\frac{\partial v_0}{\partial t} = v_{0,x}\partial^{-1}v_{0,y} - 3v_{0,x}v_0^2 + \frac{11}{2}v_{0,x}v_{0,xx} + v_0 v_{0,y} + 3v_0 v_{0,xxx} + \frac{1}{3}\partial^{-1}v_{0,yy} - \frac{1}{3}v_{0,xxy}. \tag{3.18}$$

- For the pair (B_2, B_3), substituting (2.10) into (3.7), we obtain

$$\frac{\partial v_0}{\partial t} = -4v_{0,x}v_0^2 + 4v_{0,x}v_{0,xx} + v_0 v_{0,xxx} + \frac{1}{9}\partial^{-1}v_{0,yy} - \frac{2}{9}v_{0,xxy} + \frac{1}{9}v_{0,xxxxx}. \tag{3.19}$$

• For the pair (B_1, B_5), substituting (2.10) into (3.8), we obtain

$$\frac{\partial v_0}{\partial t} = 0. \tag{3.20}$$

• For the pair (B_2, B_4), substituting (2.10) into (3.9), we obtain

$$\frac{\partial v_0}{\partial t} = 0. \tag{3.21}$$

4. The solutions of the generalized evolution equations

For (B_1, B_2), we note (3.10) and (3.16) are the same linear partial differential equations. However, the other equations on the generalized BKP equations do not pass the Painlev*é* test. For the case of (B_1, B_4) and (B_2, B_3), by means of the Painleve analysis approach [20], we find both of the evolution equations (3.18) and (3.19) on the generalized CKP equations do not pass the Painlev*é* test. For the case of (B_1, B_3), (B_1, B_5) and (B_2, B_4), we also observe that the evolution equations (3.17), (3.20) and (3.21) disappear. Now let us give the solutions of the generalized evolution equations.

• For the pair (B_1, B_2), the generalized BKP (3.10) and CKP equations (3.16) are the same linear partial differential equation. Its traveling wave solution is

$$v_0 = ce^{ax+by+kt}, \tag{4.1}$$

where a, b, c are the arbitrary constants and

$$k = \frac{1}{3}(b - a^3). \tag{4.2}$$

• For the pair (B_1, B_3), the generalized CKP equation (3.17) disappears. The generalized BKP equation (3.11) does not pass the Painlev*é* test. However, we may construct its solution by using the similarity transformations method via Lie group theory [21, 24].

We consider the one-parameter (ε) Lie group of infinitesimal transformations.

$$\begin{aligned} x^* &= x + \varepsilon\xi^{(1)}(x, y, t, v_0) + O(\varepsilon^2), \\ y^* &= y + \varepsilon\xi^{(2)}(x, y, t, v_0) + O(\varepsilon^2), \\ t^* &= t + \varepsilon\tau(x, y, t, v_0) + O(\varepsilon^2), \\ v_0^* &= v_0 + \varepsilon\eta(x, y, t, v_0) + O(\varepsilon^2), \end{aligned} \tag{4.3}$$

where ξ^1, $\xi^{(2)}$, τ and η are the infinitesimals for the variables x, y, t and v_0 respectively.

The vector field of the form

$$V = \xi^{(1)}\frac{\partial}{\partial x} + \xi^{(2)}\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial v_0}, \tag{4.4}$$

is the infinitesimal generator of the one-parameter Lie transformation group.

With the aid of the symbolic computation system *Maple*, we arrive at the infinitesimals

$$\begin{aligned} &\xi^{(1)} = \frac{1}{4}C_1 x + f'(y), \quad \xi^{(2)} = \frac{3}{4}C_1 y + C_3, \quad \tau = C_1 t + C_2, \\ &\eta = \frac{1}{3}f''(y) - \frac{1}{2}C_1 v_0, \end{aligned} \tag{4.5}$$

where C_1, C_2 and C_3 are the arbitrary constants, $f(y)$ is an arbitrary function with respect to y. The prime denotes differentiation with respect to y.

Thus we may derive the linearly independent infinitesimal generators as follows

$$V_1 = \frac{1}{4}x\frac{\partial}{\partial x} + \frac{3}{4}y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} - \frac{1}{2}v_0\frac{\partial}{\partial v_0}, \ \ V_2 = \frac{\partial}{\partial t},$$
$$V_3 = \frac{\partial}{\partial y}, \ \ V_4 = f'(y)\frac{\partial}{\partial x} + \frac{1}{3}f''(y)\frac{\partial}{\partial v_0}. \tag{4.6}$$

In the case of $V = V_3 + V_4$, according to the corresponding characteristic system

$$\frac{\mathrm{d}x}{\xi^{(1)}} = \frac{\mathrm{d}y}{\xi^{(2)}} = \frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}v_0}{\eta}, \tag{4.7}$$

we obtain the similarity form of the solution of (3.11) as

$$v_0(x,y,t) = \frac{1}{3}f'(y) + F(X,t), \tag{4.8}$$

with the similarity variable

$$X = x - f(y). \tag{4.9}$$

Substituting (4.8) into (3.11), we obtain a new partial differential equation

$$F_t - \frac{1}{3}F_{XXXX} + FF_{XX} + F_X^2 = 0. \tag{4.10}$$

By using the symmetry reduction process again, we may conclude that (4.10) has the infinitesimal generator

$$V = \frac{1}{4}X\frac{\partial}{\partial X} + t\frac{\partial}{\partial t}. \tag{4.11}$$

Then we obtain the similarity solution of Eq.(4.10) as

$$F(X,t) = \frac{K(Y)}{X^2}, \tag{4.12}$$

with the similarity variable

$$Y = \frac{t}{X^4}. \tag{4.13}$$

The reduction equation is given by

$$\begin{aligned} &-256K_{YYYY}Y^4 + 48KK_{YY}Y^2 + 48K_Y^2Y^2 - 2432K_{YYY}Y^3 + 156KK_YY \\ &-5616K_{YY}Y^2 + 30K^2 - 2904K_YY - 120K + 3K_Y = 0. \end{aligned} \tag{4.14}$$

(4.14) can be rewritten as

$$-256K_{YYYY}Y^4 - 2432K_{YYY}Y^3 - 5616K_{YY}Y^2 - 2904K_YY - 120K = 0, \tag{4.15}$$
$$48KK_{YY}Y^2 + 48K_Y^2Y^2 + 156KK_YY + 30K^2 + 3K_Y = 0. \tag{4.16}$$

Solving (4.15), we have

$$K(Y) = \frac{c_1}{Y^{\frac{5}{4}}} + \frac{c_2}{Y} + \frac{c_3}{Y^{\frac{1}{2}}} + \frac{c_4}{Y^{\frac{3}{4}}}. \tag{4.17}$$

Substitution Eq.(4.17) into (4.16), we obtain

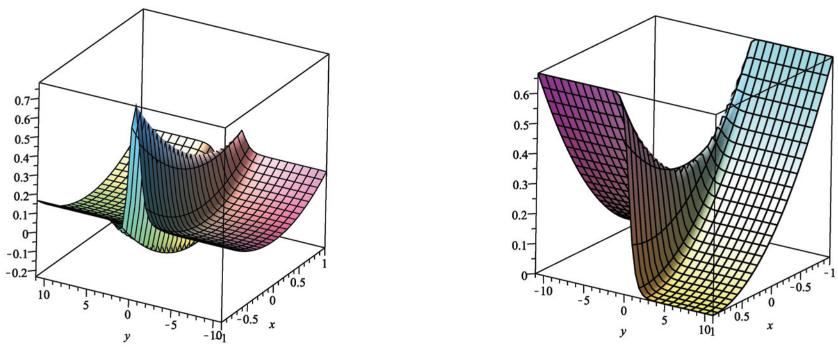
$$c_2 = \frac{1}{6}, \quad c_1 = c_3 = c_4 = 0. \tag{4.18}$$

Thus we have a rational solution of (3.11)

$$v_0 = \frac{1}{3} f'(y) + \frac{(x - f(y))^2}{6t}, \tag{4.19}$$

where $f(y)$ is an arbitrary function with respect to y.

Fig.1 shows the profiles of the exact rational solution v_0 described by Eq.(4.19) at $t = 1$ for (a) $f(y) = \mathrm{sech}^2 y$ and (b) $f(y) = \tanh y$.



(a) $f(y) = \mathrm{sech}^2 y$ (b) $f(y) = \tanh y$

Fig.1 Profiles of exact solution v_0 described by Eq.(4.19) at $t = 1$.

• For the pair (B_1, B_4), we set

$$v_0 = a\,\mathrm{sech}^2(C_1(\omega t + x + py) + C_2), \tag{4.20}$$

where a, C_1, ω, p, C_2 are undetermined parameters. Substituting (4.20) into (3.12) and taking the coefficients to zero, then we obtain

$$a = -26C_1^2, \quad \omega = \frac{5005}{27} C_1^4, \quad p = -\frac{65}{3} C_1^2. \tag{4.21}$$

Thus we find the single soliton solution of the generalized BKP equation (3.12)

$$v_0 = -26C_1^2 \mathrm{sech}^2\left(\frac{5005}{27} C_1^5 t + C_1 x - \frac{65}{3} C_1^3 y + C_2\right), \tag{4.22}$$

where C_1, C_2 are the arbitrary constants.

Similarly, the single soliton solution of the generalized CKP equation (3.18) is

$$v_0 = -23C_1^2 \mathrm{sech}^2\left(\frac{185725}{1323} C_1^5 t + C_1 x - \frac{391}{21} C_1^3 y + C_2\right), \tag{4.23}$$

where C_1, C_2 are the arbitrary constants.

• For the pair (B_2, B_3), we find the single soliton solution of the generalized BKP equation (3.13) as follows,

$$v_0 = 3C_1^2 - (\frac{21}{4} - \frac{1}{4}\sqrt{601})C_1^2 \mathrm{sech}^2(\frac{73}{4}C_1^5 t + C_1 x + \frac{47}{2}C_1^3 y + C_2), \tag{4.24}$$

where C_1, C_2 are the arbitrary constants.

Similarly, the single soliton solution of the generalized CKP equation (3.19) is

$$v_0 = -\frac{5C_1^3 - 2C_2}{15C_1} + C_1^2 \mathrm{sech}^2(\frac{C_2^2}{25C_1}t + C_1 x + C_2 y + C_3), \tag{4.25}$$

where C_1, C_2, C_3 are the arbitrary constants, and $C_1 \neq 0$.

The BKP (CKP) hierarchy is obtained by imposing a symmetry constraint in KP hierarchy. The KP hierarchy can be reduced to the KdV hierarchy by dimensional reductions. Now we tend to provide the reduction equation of the generalized BKP (CKP) hierarchy. With the reduction $y = 0$, (3.13) and (3.19) leads to the fifth order KdV equations

$$\frac{\partial v_0}{\partial t} = -4v_{0,x}v_0^2 + 5v_{0,x}v_{0,xx} + v_0 v_{0,xxx} + \frac{1}{9}v_{0,xxxxx}, \tag{4.26}$$

and

$$\frac{\partial v_0}{\partial t} = -4v_{0,x}v_0^2 + 4v_{0,x}v_{0,xx} + v_0 v_{0,xxx} + \frac{1}{9}v_{0,xxxxx}, \tag{4.27}$$

respectively. As is well known, the fifth order KdV equations describe the motions of long waves in shallow water. Thus the generalized BKP (CKP) evolution equations could be widely applied in many fields.

5. Summary

We investigated the generalized Lax equation of the BKP (CKP) hierarchy with the Lax triple (L, B_m, B_n) in the frame of the Nambu Mechanics. In terms of the Lax triple (L, B_m, B_n), some evolution equations were derived from the generalized Lax equation. We noted that they do not pass the Painlev*é* test except for the linear equation. However, we still concluded some solutions for these evolution equations. Moreover it should be noted that the evolution equations seem to disappear in the case of (L, B_m, B_n) with $m+n$ even for the generalized CKP hierarchy. The deeper relationship of the generalized BKP (CKP) hierarchy with the Lax triple (L, B_m, B_n) is still deserved to further study.

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