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Symmetry Constraint of the Differential-difference KP Hierarchy and a Second Discretization of the ZS-AKNS System

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In this paper we construct a squared-eigenfunction symmetry of the scalar differential-difference KP hierarchy. Through a constraint of the symmetry, Lax triad of the differential-difference KP hierarchy is reduced to a known discrete spectral problem and a semidiscrete AKNS hierarchy. The discrete spectral problem corresponds to a bidirectional discretization of the derivatives $\phi_{1,x}$ and $\phi_{2,x}$ in the ZS-AKNS spectral problem and therefore it is a discretization of the later. The discrete spectral problem is also known as a Darboux transformation of the ZS-AKNS spectral problem. Isospectral and nonisospectral flows derived from the spectral problem compose a Lie algebra. Infinitely many symmetries of the nonisospectral hierarchy are obtained. By considering infinite dimensional subalgebras of the algebra and continuum limit of recursion operator, three semi-discrete AKNS hierarchies are constructed.

Keywords: differential-difference KP hierarchy; squared-eigenfunction; symmetry; symmetry constraint; ZS-AKNS spectral problem; semi-discrete AKNS hierarchies.

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1. Introduction

A beautiful and remarkable connection between (1+1)-dimensional and (2+1)-dimensional continuous integrable systems is that the fundamental Zakharov-Shabat-Ablowitz-Kaup-Newell-Segur (ZS-AKNS) spectral problem [1, 30]

$$\Phi_x = \begin{pmatrix} \eta & q \\ r & -\eta \end{pmatrix} \Phi, \quad \Phi = (\phi_1, \phi_2)^T \tag{1.1}$$

and its isospectral evolution equation hierarchy can be viewed as a symmetry constraint of the Kadomtsev-Petviashvili (KP) system [7, 8, 17, 18]. In discrete case such a connection is still unrevealed. One discrete counterpart of the KP equation is the differential-difference KP ($\mathrm{D}^2\Delta$KP) equation [9] with 2 continuous and 1 discrete independent variables, i.e.

$$\Delta\left(\frac{\partial u}{\partial t_2} + 2\frac{\partial u}{\partial x} - 2u\frac{\partial u}{\partial x}\right) = (2+\Delta)\frac{\partial^2 u}{\partial x^2}, \tag{1.2}$$

which is related to the spectral problem [28]

$$\mathfrak{L}\varphi_n = \xi\varphi_n, \quad \mathfrak{L} = \Delta + u_{0,n} + u_{1,n}\Delta^{-1} + u_{2,n}\Delta^{-2} + \cdots, \tag{1.3}$$

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where $\Delta = E-1$, $Ef_n = f_{n+1}$, $\mathfrak{L}$ is a pseudo-difference operator and in (1.2) $u = u_{0,n}$. In this paper we will prove that by a squared-eigenfunction symmetry constraint, the Lax triad of the $\mathrm{D}^2\Delta$KP hierarchy is converted to a discrete spectral problem

$$\Theta_{n+1} = \begin{pmatrix} \lambda^2 + Q_nR_n & Q_n \\ R_n & 1 \end{pmatrix} \Theta_n, \quad \Theta_n = (\theta_{1,n}, \theta_{2,n})^T \tag{1.4}$$

and its isospectral differential-difference (semidiscrete) equation hierarchy. This discrete spectral problem was proposed by Ragnisco and Tu in 1989 [27] and then studied in [23,34]. The isospectral hierarchy generated from (1.4) can recover the AKNS hierarchy in continuum limit (cf. [23]).

In this paper we will also revisit the spectral problem (1.4) and its related hierarchies. At the first glance (1.4) has nothing with the ZS-AKNS spectral problem (1.1). However, in this paper we will show that (1.4) is gauge equivalent to the form (see Appendix Appendix B)

$$\begin{pmatrix} \phi_{1,n+1} \\ -\phi_{2,n-1} \end{pmatrix} = \begin{pmatrix} \lambda & Q_n \\ R_n & -\lambda \end{pmatrix} \begin{pmatrix} \phi_{1,n} \\ \phi_{2,n} \end{pmatrix}, \tag{1.5}$$

which easily recovers the ZS-AKNS spectral problem (1.1) in continuum limit by defining

$$\Phi(n+j) = \Phi(x+j\varepsilon), \ \ (Q_n, R_n) = \varepsilon(q,r), \quad \lambda = e^{\varepsilon\eta}, \tag{1.6}$$

and taking $\varepsilon \to 0$. Obviously, (1.5) is different from the well known standard discretisation of (1.1), i.e. [2,3]

$$\Phi_{n+1} = \begin{pmatrix} \lambda & Q_n \\ R_n & 1/\lambda \end{pmatrix} \Phi_n, \quad \Phi_n = (\phi_{1,n}, \phi_{2,n})^T, \tag{1.7}$$

which is known as the Ablowitz-Ladik (AL) spectral problem and also recovers (1.1) by the same continuum limit (1.6). With regard to the discretisation of derivatives $\phi_{i,x}$, in (1.7) $\phi_{i,x}$ are discretised as $\phi_{i,x} \sim \frac{\phi_{1,n+1}-\phi_{1,n}}{\varepsilon}$ for $i = 1,2$, while in (1.5) $\phi_{i,x}$ are discretised as

$$\phi_{1,x} \sim \frac{\phi_{1,n+1}-\phi_{1,n}}{\varepsilon}, \ \ \phi_{2,x} \sim \frac{\phi_{2,n}-\phi_{1,n-1}}{\varepsilon}, \tag{1.8}$$

which is a bidirectional discretisation for $\phi_{1,x}$ and $\phi_{2,x}$.

In addition to the bidirectional discretisation, (1.4) is related to the ZS-AKNS spectral problem (1.1) as its Darboux transformation [4] with $\lambda^2 = 2(\eta-\gamma)$ where γ is a soliton parameter. It is well known that a Darboux transformation $\widetilde{\Phi} = D(u, \widetilde{u}, \lambda)\Phi$ of a continuous spectral problem $\Phi_x = M(u,\eta)\Phi$, where $\widetilde{\Phi}$ and $\widetilde{u}$ stand for new eigenfunction and potential function corresponding to λ, can act as a discrete spectral problem (by considering $\widetilde{\Phi} = \Phi_{n+1}$ and $\widetilde{u} = u_{n+1}$)

$$\Phi_{n+1} = D(u_n, u_{n+1}, \lambda)\Phi_n \tag{1.9}$$

to generate semidiscrete integrable systems as a compatible condition with $\Phi_x = M(u,\eta)\Phi$ [19,20]. Moreover, Darboux transformations with different parameters, i.e.

$$\widetilde{\Phi} = D(u, \widetilde{u}, \lambda_1)\Phi, \ \ \widehat{\Phi} = D(u, \widehat{u}, \lambda_2)\Phi$$

can be used as a Lax pair to generate fully discrete integrable systems. As examples one can refer to [5,6,16,24]. It is natural that the semidiscrete equations generated from a Darboux transformation (as a discrete spectral problem) are related to the original continuous spectral problem via suitable continuum limits.

The paper is organized as follows. Sec.2 contains necessary notions and notations. In Sec.3 we construct a squared-eigenfunction symmetry of the scalar D$^2\Delta$KP hierarchy. Through the corresponding symmetry constraint, Lax triad of the D$^2\Delta$KP hierarchy is reduced to the discrete spectral problem (1.4) and a semidiscrete AKNS (sdAKNS) hierarchy. In Sec.4 we construct possible sdAKNS hierarchies related to (1.4). It is interesting that isospectral and nonisospectral flows derived from (1.4) compose a Lie algebra which is as same as the one composed by the AL flows. By considering infinite dimensional subalgebras of the algebra and continuum limit of recursion operator, we construct three semi-discrete AKNS hierarchies. Sec.5 contributes conclusions and discussions. There are two sections in Appendix. One is, as a comparison, gives the sdAKNS hierarchies derived from the AL spectral problem. The other gives several discrete spectral problems that are gauge equivalent to (1.4).

2. Basic notions

Let us shortly describe some notions and notations that we will use in the paper. (We mainly follow [13, 14]).

For functions Q_n and R_n defined on $\mathbb{Z}$ and vanishing rapidly as $n \to \pm\infty$, let $U_n \doteq (Q_n, R_n)^T$. Consider a differential-difference evolution equation

$$U_{n,t} = K(U_n), \quad U_n \in \mathscr{M}, \tag{2.1}$$

where by $\mathscr{M}$ we denote the infinite dimensional linear manifold of functions U_n. The solution $U_n = U(n,t)$ is usually asked to depend in a $\mathbb{C}^\infty$-way on the time parameter t. Let S be the fiber of the tangent bundle $T\mathscr{M}$ at any point $U_n \in M$. In principle there is an identification between the linear spaces $\mathscr{M}$ and S, but it is convenient to regard them as different objects for a better geometrical understanding (i.e. $\mathscr{M}$ is the manifold under examination, S is the tangent space at any point $U_n \in \mathscr{M}$). Let S^* be the dual space of S w.r.t. the bilinear form $\langle \cdot,\cdot \rangle : S^* \times S \to \mathbb{R}$ defined as

$$\langle f_n, g_n \rangle = \sum_{n=-\infty}^{+\infty} f_n g_n, \quad f_n \in S^*, g_n \in S. \tag{2.2}$$

The Gâteaux derivative of a function (or an operator or a functional) $F(U_n)$ on $\mathscr{M}$ in the direction $g_n \in S$ is defined as

$$F'[g_n] = \frac{\partial}{\partial \varepsilon} F(U_n + \varepsilon g_n)\Big|_{\varepsilon=0}, \quad U_n \in \mathscr{M}, g_n \in S. \tag{2.3}$$

The above definition is valid as well for the case $F = F(U_n,t)$, where F depends explicitly on the time parameter t and we treat U_n and t as independent variables (cf. [14]). For the sake of a more generic sense, in the following definitions are given for the time-dependent cases. They are valid as well when we remove the independent time variable t.

For two vector fields $F(U_n,t), G(U_n,t) : \mathscr{M} \times \mathbb{R} \to S$, their standard commutator is defined as

$$[\![F, G]\!] = F'[G] - G'[F]. \tag{2.4}$$

Vector field $G(U_n,t) : \mathscr{M} \times \mathbb{R} \to S$ is called a symmetry of equation (2.1) if

$$\partial_t G(U_n,t) + [\![G(U_n,t), K(U_n)]\!] = 0 \tag{2.5}$$

holds everywhere in $\mathscr{M}\times\mathbb{R}$.

A linear operator $L(U_n,t): S\to S$ is called a strong symmetry operator of equation (2.1) if

$$\partial_t L + L'[K] = [K', L] \tag{2.6}$$

holds everywhere on $\mathscr{M}$, where $[A,B]=AB-BA$. A linear operator $L(U_n,t): S\to S$ is called to be hereditary (or a hereditary operator) if

$$L'[LF]G - L'[LG]F = L(L'[F]G - L'[G]F),\ \ \forall F,G\in S. \tag{2.7}$$

If L is a hereditary operator, so is L^{-1}. If L is a hereditary operator and is a strong symmetry of equation (2.1), then L is also a strong symmetry of equation $U_{n,t}=LK(U_n)$.

3. Symmetry constraint of the D$^2\Delta$KP system

In this section we investigate in detail a squared-eigenfunction symmetry constraint of the scalar D$^2\Delta$KP hierarchy. As a result, the spectral problem (1.4) and a semidiscrete AKNS hierarchy are obtained from Lax triad of the D$^2\Delta$KP hierarchy. Let us first recall Lax triad of the D$^2\Delta$KP hierarchy [11].

3.1. *The scalar D$^2\Delta$KP hierarchy and Lax triad [11]*

Consider the pseudo-difference operator $\mathfrak{L}$ defined in (1.3), i.e.,

$$\mathfrak{L} = \Delta + u_{0,n} + u_{1,n}\Delta^{-1} + u_{2,n}\Delta^{-2} + \cdots, \tag{3.1}$$

where $u_{i,n}=u_i(n,x,\mathbf{t})$ and $\mathbf{t}=(t_1,t_2,\cdots)$. The difference operator Δ obeys the discrete Leibniz rule

$$\Delta^s g(n) = \sum_{i=0}^{\infty} \mathrm{C}_s^i (\Delta^i g(n+s-i))\Delta^{s-i},\ \ s\in\mathbb{Z}, \tag{3.2}$$

where

$$\mathrm{C}_s^i = \frac{s(s-1)(s-2)\cdots(s-i+1)}{i!},\ \ \mathrm{C}_0^0 = 1. \tag{3.3}$$

The scalar D$^2\Delta$KP hierarchy can be derived from the following Lax triad [11],

$$\mathfrak{L}\varphi_n = \xi\varphi_n, \tag{3.4a}$$
$$\varphi_{n,x} = A_1\varphi_n,\ \ A_1 = \Delta + u_{0,n}, \tag{3.4b}$$
$$\varphi_{n,t_j} = A_j\varphi_n,\ \ (j=1,2,\cdots), \tag{3.4c}$$

where $A_j=(\mathfrak{L}^j)_+$ denotes the difference part of $\mathfrak{L}^j$, the first two of which are

$$A_1 = \Delta + u_{0,n}, \tag{3.5a}$$
$$A_2 = \Delta^2 + ((\Delta u_{0,n}) + 2u_{0,n})\Delta + (\Delta u_{0,n}) + u_{0,n}^2 + (\Delta u_{1,n}) + 2u_{1,n}. \tag{3.5b}$$

Through the compatibility of (3.4), i.e.

$$\mathfrak{L}_x = [A_1, \mathfrak{L}], \tag{3.6a}$$
$$\mathfrak{L}_{t_j} = [A_j, \mathfrak{L}],\ \ (j=1,2,\cdots), \tag{3.6b}$$
$$A_{1,t_j} - A_{j,x} + [A_1, A_j] = 0,\ \ (j=1,2,\cdots), \tag{3.6c}$$

where $[A,B]=AB-BA$. the first equation (3.6a) provides expressions of $u_{j,n}$ in terms of $u_{0,n}$, which are

$$\Delta u_{1,n}=u_{0,n,x}, \tag{3.7a}$$

$$\Delta u_{k+1,n}=u_{k,n,x}-\Delta u_{k,n}-u_{0,n}u_{k,n}+\sum_{j=0}^{k-1}(-1)^j\mathrm{C}_{k-1}^{j}u_{k-j,n}\Delta^j u_{0,n-k},\quad k\geq 1; \tag{3.7b}$$

the second equation (3.6b) is actually to determine $A_j=(\mathfrak{L}^j)_+$; the third one (3.6c), rewritten as

$$u_{0,n,t_j}=\mathscr{K}_j=A_{j,x}-[A_1,A_j],\ \ (j=1,2,\cdots), \tag{3.8}$$

provides zero curvature representations of a hierarchy of the scalar D$^2\Delta$KP equation if substituting $u_{j,n}$ with $u_{0,n}$ by using (3.7). Particularly, when $j=2$ we get the D$^2\Delta$KP equation (1.2).

3.2. *Squared-eigenfunction symmetry of the D$^2\Delta$KP hieraechy*

For the eigenfunction φ_n defined by the Lax triad (3.4), we introduce its adjoint form $\bar{\varphi}_n$ which satisfies

$$\bar{\varphi}_{n,x}=-A_1^*\bar{\varphi}_n, \tag{3.9a}$$

$$\bar{\varphi}_{n,t_j}=-A_j^*\bar{\varphi}_n,\ \ (j=1,2,\cdots). \tag{3.9b}$$

Here A_j^* stands for the formal adjoint operator of A_j w.r.t. the bilinear form (2.2), e.g.

$$A_1^*=-\Delta E^{-1}+u_{0,n},$$
$$A_2^*=\Delta^2E^{-2}-\Delta E^{-1}((\Delta u_{0,n})+2u_{0,n})+(\Delta u_{0,n})+u_{0,n}^2+u_{0,n,x}+2(\Delta^{-1}u_{0,n,x}),$$

where we have replaced $u_{1,n}$ with $\Delta^{-1}u_{0,n,x}$. In [28] it is verified that $(\varphi_n\bar{\varphi}_n)_x$ is a symmetry of the D$^2\Delta$KP equation (1.2) if φ_n and $\bar{\varphi}_n$ satisfy (3.4b), (3.4c) and (3.9) with $j=2$. Following the name used in [25] for the continuous case, here we call $(\varphi_n\bar{\varphi}_n)_x$ to be a squared-eigenfunction symmetry. In the following we prove that $(\varphi_n\bar{\varphi}_n)_x$ is a symmetry of the whole scalar D$^2\Delta$KP hierarchy (3.8).

Theorem 3.1. $(\varphi_n\bar{\varphi}_n)_x$ *is a symmetry of the D$^2\Delta$KP hierarchy* (3.8) *if* φ_n *and* $\bar{\varphi}_n$ *satisfy* (3.4b), (3.4c) *and* (3.9).

Proof. We can partially follow the idea in [25] for continuous case. We introduce an additional symmetry

$$\mathfrak{L}_z=-[\varphi_n\Delta^{-1}\bar{\varphi}_n,\mathfrak{L}], \tag{3.10}$$

where $\mathfrak{L}$ is the pseudo-difference operator (3.1), z is an auxiliary variable, φ_n and $\bar{\varphi}_n$ satisfy (3.4b), (3.4c) and (3.9). The coefficient of Δ^0 term in (3.10) yields

$$u_{0,n,z}=\varphi_{n+1}\bar{\varphi}_n-\varphi_n\bar{\varphi}_{n-1},$$

from which, together with (3.4b) and (3.9a), one immediately find

$$u_{0,n,z}=(\varphi_n\bar{\varphi}_n)_x. \tag{3.11}$$

Then, making use of the result $[\partial_{t_j},\partial_z]\mathfrak{L}=0$ (see Proposition 3.1 in [21]) for $u_{0,n}$ we have $\partial_{t_j}\partial_z u_{0,n}=\partial_z\partial_{t_j}u_{0,n}$, which means $(\varphi_n\bar{\varphi}_n)_x$ and the D$^2\Delta$KP flows $\{\mathscr{K}_j\}$ defined in (3.8) commute, i.e. $[\![\mathscr{K}_j,(\varphi_n\bar{\varphi}_n)_x]\!]=0$. Thus, $(\varphi_n\bar{\varphi}_n)_x$ is a symmetry of the whole scalar D$^2\Delta$KP hierarchy.

□

3.3. *Spectral problem* (1.4) *from the symmetry constraint*

Now we introduce a symmetry constraint under which the spectral problem (3.4a) will be reduced to (1.4).

Noting that $u_{0,n,x}$ is a symmetry of the D$^2\Delta$KP equation (1.2) (cf. [11]), We consider a symmetry $\sigma = u_{0,n,x} + (\varphi_n \bar{\varphi}_n)_x$. Taking $\sigma = 0$ leads to a group invariant solution to (1.2) as well as a symmetry constraint $u_{0,n} = -\varphi_n \bar{\varphi}_n$. For convenience, we write $\varphi_n = Q_n,\ \bar{\varphi}_n = R_n$, and then we have

$$u_{0,n} = -Q_n R_n \tag{3.12}$$

and it follows from (3.4b) and (3.9a) that

$$Q_{n,x} = Q_{n+1} - Q_n - Q_n^2 R_n, \ \ R_{n,x} = R_n - R_{n-1} + Q_n R_n^2. \tag{3.13}$$

Note that (3.13) is the condition exerted on Q_n and R_n. If we replace R_n with R_{n+1}, (3.13) provides a Bäcklund transformation for the nonlinear Schrödinger equations (cf. [4]), and through suitable continuum limit (3.13) yields the nonlinear Schrödinger equations.

Let us investigate the change of $\mathfrak{L}$ under the symmetry constraint (3.12) and (3.13).

Lemma 3.1. *If $u_{0,n}$ is given by* (3.12) *where Q_n and R_n obey* (3.13)*, then $u_{k,n}$ defined in* (3.7) *can be expressed in terms of Q_n and R_n as*

$$u_{k+1,n} = (-1)^{k+1} Q_n \Delta^k R_{n-k-1}, \ \ k = 0, 1, 2, \cdots. \tag{3.14}$$

Proof. From (3.7), (3.12) and (3.13) and by direct calculation one can find

$$u_{1,n} = -Q_n R_{n-1}, \ \ u_{2,n} = Q_n \Delta R_{n-2}, \ \ u_{3,n} = -Q_n \Delta^2 R_{n-3}.$$

Now we suppose (3.14) is valid up to $u_{k+1,n}$. Then for $u_{k+2,n}$, from (3.7) we have

$$\Delta u_{k+2,n} = u_{k+1,n,x} - \Delta u_{k+1,n} - u_{0,n} u_{k+1,n} + \sum_{j=0}^{k} (-1)^j \mathrm{C}_k^j u_{k+1-j,n} \Delta^j u_{0,n-k-1}. \tag{3.15}$$

For the first three terms on the right hand side, substituting (3.14) into them and making use of (3.12) and (3.13), we find

$$\begin{aligned} & u_{k+1,n,x} - \Delta u_{k+1,n} - u_{0,n} u_{k+1,n} \\ =& \Delta((-1)^{k+2} Q_n \Delta^{k+1} R_{n-k-2}) + (-1)^{k+1} Q_n \Delta^k (Q_{n-k-1} R_{n-k-1}^2). \end{aligned}$$

in which, by using formula (3.2), the last term yields

$$\begin{aligned} & (-1)^{k+1} Q_n \Delta^k (Q_{n-k-1} R_{n-k-1}^2) \\ =& (-1)^{k+2} Q_n \Delta^k (R_{n-k-1} u_{0,n-k-1}) \\ =& (-1)^{k+2} \sum_{j=0}^{k} \mathrm{C}_k^j Q_n (\Delta^{k-j} R_{n-k-j-1})(\Delta^j u_{0,n-k-1}) \\ =& -\sum_{j=0}^{k} (-1)^j \mathrm{C}_k^j u_{k+1-j,n} \Delta^j u_{0,n-k-1}. \end{aligned}$$

It is just canceled by the last term on the right hand side of (3.15). Thus we reach

$$\Delta u_{k+2,n} = \Delta((-1)^{k+2} Q_n \Delta^{k+1} R_{n-k-2}),$$

i.e.,

$$u_{k+2,n} = (-1)^{k+2} Q_n \Delta^{k+1} R_{n-k-2}.$$

Based on mathematical induction, we complete the proof. □

With Lemma 3.1 in hand and making use of formula (3.2) (for $s = -1$), we immediately find

$$(\mathfrak{L})_- = \sum_{j=1}^{\infty} u_{j,n} \Delta^{-j} = -Q_n \Delta^{-1} R_n. \tag{3.16}$$

As a result we have the following theorem.

Theorem 3.2. *Under symmetry constraint* (3.12) *together with* (3.13)*, the spectral problem* (3.4a) *is written as*

$$\mathfrak{L}\varphi_n = \xi \varphi_n, \quad \mathfrak{L} = \Delta - Q_n R_n - Q_n \Delta^{-1} R_n, \tag{3.17}$$

which is nothing but the spectral problem (1.4) *by introducing*

$$\theta_{1,n} = \varphi_n, \;\; \theta_{2,n} = \Delta^{-1} R_n \varphi_n, \;\; \lambda^2 = \xi + 1. \tag{3.18}$$

(3.17) *can also be written as the form* (B.6) *(see Appendix Appendix B).*

3.4. *The sdAKNS hierarchy from the symmetry constraint*

There is a sdAKNS hierarchy coming from (3.4c) and its adjoint form (3.9b) under the constraint (3.12) and (3.13). This agrees with the continuum limit of the $\mathrm{D}^2\Delta$KP hierarchy and symmetry constraint of the continuous KP hierarchy.

In the following we prove

Theorem 3.3. *For the pseudo-difference operator*

$$\mathfrak{L} = \Delta - Q_n R_n - Q_n \Delta^{-1} R_n, \tag{3.19}$$

we define $A_m = (\mathfrak{L}^m)_+$ and A_m^* to be the adjoint operator of A_m w.r.t. the bilinear form (2.2)*. Then* (3.4c) *and* (3.9b)*, i.e.*

$$Q_{n,t_m} = A_m Q_n, \tag{3.20a}$$

$$R_{n,t_m} = -A_m^* R_n \tag{3.20b}$$

generate a recursive relation

$$\begin{pmatrix} Q_n \\ R_n \end{pmatrix}_{t_{m+1}} = \mathscr{L}^{(+)} \begin{pmatrix} Q_n \\ R_n \end{pmatrix}_{t_m}, \tag{3.21}$$

where

$$\mathscr{L}^{(+)} = \begin{pmatrix} \Delta - Q_n(E+1)\Delta^{-1}R_n - Q_nR_n & -Q_n(E+1)\Delta^{-1}Q_n \\ R_n(E+1)\Delta^{-1}R_n & -\Delta E^{-1} + R_n(E+1)\Delta^{-1}Q_n - Q_nR_n \end{pmatrix}. \tag{3.22}$$

(3.21) *provides a sdAKNS hierarchy (see* (4.20)*).*

To prove the theorem we need the following two lemmas.

Lemma 3.2. *Suppose that* $p^{(m)}_{-1,n} = \mathrm{Res}_\Delta \mathfrak{L}^m$*, i.e.* $p^{(m)}_{-1,n}$ *is the coefficient of* Δ^{-1} *term in* $\mathfrak{L}^m$*, where* $\mathfrak{L}$ *is given in* (3.19). *Then we have*

$$\Delta p^{(m)}_{-1,n} = -(Q_nR_n)_{t_m}. \tag{3.23}$$

Proof. Comparing the constant terms of the left and right hand sides of the Lax equation

$$\mathfrak{L}_{t_m} = [A_m, \mathfrak{L}] = -[(\mathfrak{L}^m)_-, \mathfrak{L}],$$

one immediately obtains (3.23). Here $(\mathfrak{L}^m)_- = \mathfrak{L}^m - A_m$. □

Lemma 3.3. *The following relations hold,*

$$(Q_n\Delta^{-1}R_nA_m)_+ = Q_n\Delta^{-1}R_nA_m - Q_n\Delta^{-1}(A_m^*R_n), \tag{3.24}$$

$$(A_mQ_n\Delta^{-1}R_n)_+ = A_mQ_n\Delta^{-1}R_n - (A_mQ_n)\Delta^{-1}R_n. \tag{3.25}$$

Proof. We prove them one by one. For (3.24) it is equivalent to to prove

$$(Q_n\Delta^{-1}R_nA_m)_- = Q_n\Delta^{-1}(A_m^*R_n). \tag{3.26}$$

Supposing that $A_m = \sum_{j=0}^m a_{j,n}\Delta^{m-j}$ and noting that R_n tends to 0 as $|n| \to \infty$, we have

$$\begin{aligned}
(Q_n\Delta^{-1}R_nA_m)_- =& \Big(Q_n\Delta^{-1}R_n\sum_{j=0}^m a_{j,n}\Delta^{m-j}\Big)_- \\
=& \Big[Q_n\sum_{j=0}^m\sum_{s=1}^\infty(-1)^{s-1}(\Delta^{s-1}E^{-s}R_na_{j,n})\Delta^{m-j-s}\Big]_- \\
=& Q_n\sum_{l=1}^\infty\Big[(-1)^{l-1}\Delta^{l-1}E^{-l}\sum_{j=0}^m(-1)^{m-j}(\Delta^{m-j}E^{-(m-j)}a_{j,n}R_n)\Big]\Delta^{-l} \\
=& Q_n\sum_{l=1}^\infty\Big[(-1)^{l-1}\Delta^{l-1}E^{-l}(A_m^*R_n)\Big]\Delta^{-l} \\
=& Q_n\Delta^{-1}\Delta\sum_{l=1}^\infty\Big[(-1)^{l-1}\Delta^{l-1}E^{-l}(A_m^*R_n)\Big]\Delta^{-l} \\
=& Q_n\Delta^{-1}\sum_{l=1}^\infty\Big\{[(\Delta^*)^{l-1}(A_m^*R_n)]\Delta^{-l+1} - [(\Delta^*)^l(A_m^*R_n)]\Delta^{-l}\Big\} \\
=& Q_n\Delta^{-1}(A_m^*R_n),
\end{aligned}$$

i.e. (3.26).

Next, we prove (3.25). To calculate $(A_mQ_n\Delta^{-1}R_n)_-$ we rewrite the operator A_mQ_n as a form of pure difference operator, i.e. $A_mQ_n = \sum_{j=0}^m b_{j,n}\Delta^{m-j}$, in which only the constant term $b_{m,n}$ makes sense in $(A_mQ_n\Delta^{-1}R_n)_-$. Since $b_{m,n} = (A_mQ_n)$ we immediately find $(A_mQ_n\Delta^{-1}R_n)_- = (A_mQ_n)\Delta^{-1}R_n$, which leads to the relation (3.25). □

Now we come to the proof of Theorem 3.3.

Proof of Theorem 3.3. First,

$$\begin{aligned}A_{m+1} =&(\mathfrak{L}^{m+1})_+ = \Big[(\Delta - Q_nR_n - Q_n\Delta^{-1}R_n)(A_m + p^{(m)}_{-1,n}\Delta^{-1})\Big]_+\\ =&\Delta A_m - Q_nR_nA_m - [E\Delta^{-1}(Q_nR_{n,t_m} + R_nQ_{n,t_m})] - (Q_n\Delta^{-1}R_nA_m)_+,\end{aligned}$$

where we have made use of Lemma 3.2. Substituting (3.24) into the above we find

$$A_{m+1} = \Delta A_m - Q_nR_nA_m - Q_n\Delta^{-1}R_nA_m - Q_n\Delta^{-1}R_{n,t_m} - [E\Delta^{-1}(Q_nR_{n,t_m} + R_nQ_{n,t_m})]. \tag{3.27}$$

Note that the last term is a scalar.

Next, we calculate A_{m+1} in another way:

$$\begin{aligned}A_{m+1} =&\Big[(A_m + p^{(m)}_{-1,n}\Delta^{-1})(\Delta - Q_nR_n - Q_n\Delta^{-1}R_n)\Big]_+\\ =&A_m\Delta - A_mQ_nR_n + p^{(m)}_{-1,n} - (A_mQ_n\Delta^{-1}R_n)_+\\ =&A_m\Delta - A_mQ_nR_n - A_mQ_n\Delta^{-1}R_n + Q_{n,t_m}\Delta^{-1}R_n - [\Delta^{-1}(Q_nR_{n,t_m} + R_nQ_{n,t_m})],\end{aligned}$$

where we have made use of Lemma 3.2 and (3.25). Its adjoint form reads

$$A^*_{m+1} = \Delta^*A^*_m - Q_nR_nA^*_m + R_nE\Delta^{-1}Q_nA^*_m - R_nE\Delta^{-1}Q_{n,t_m} - [\Delta^{-1}(Q_nR_{n,t_m} + R_nQ_{n,t_m})]. \tag{3.28}$$

Now, exerting (3.27) on Q_n and (3.28) on R_n, respectively, and making use of (3.20), we arrive at the recursive relation (3.21). Thus we complete the proof.

□

According to Theorem 3.3, from (3.20) the first equation ($m = 1$) is

$$Q_{n,t_1} = Q_{n+1} - Q_n - Q_n^2R_n,\ \ R_{n,t_1} = R_n - R_{n-1} + Q_nR_n^2; \tag{3.29}$$

and starting from it,

$$\begin{pmatrix} Q_n \\ R_n \end{pmatrix}_{t_{m+1}} = (\mathscr{L}^{(+)})^m \begin{pmatrix} Q_n \\ R_n \end{pmatrix}_{t_1} \tag{3.30}$$

provides a hierarchy, which is the sdAKNS hierarchy (see (4.20)).

4. The sdAKNS hierarchies related to (1.4)

In this section we will first list isospectral and nonisospectral flows together with their Lie algebra derived from (1.4). As new results, from Lie algebraic structures of these flows we construct new symmetries of nonisospectral equations. In addition, by considering infinite dimensional subalgebras and continuum limit of recursion operators, we construct three types of isospectral and nonisospectral sdAKNS hierarchies, including one isospectral hierarchy found in [23].

4.1. *Flows related to* (1.4) *and their Lie algebra*

From spectral problem (1.4), following the procedure in [31], we can derive isospectral hierarchy $U_{n,t_s} = K_s$ and nonisospectral hierarchy $U_{n,t_s} = \sigma_s$:

$$U_{n,t_s} = K_s = L^s K_0, \;\; K_0 = \begin{pmatrix} Q_n \\ -R_n \end{pmatrix}, \quad s \in \mathbb{Z}, \tag{4.1}$$

$$U_{n,t_s} = \sigma_s = L^s \sigma_0, \;\; \sigma_0 = \begin{pmatrix} (n+\frac{1}{2})Q_n \\ -(n-\frac{1}{2})R_n \end{pmatrix}, \quad s \in \mathbb{Z}, \tag{4.2}$$

where $U_n = (Q_n, R_n)^T$, L is a recursion operator

$$L = \begin{pmatrix} 1 & 0 \\ 0 & E^{-1} \end{pmatrix} \left[\mu_n I - \begin{pmatrix} Q_n \\ -R_{n+1} \end{pmatrix} (E+1)\Delta^{-1}(R_{n+1}, Q_n) \right] \begin{pmatrix} E & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.3}$$

and its inverse L^{-1} is

$$L^{-1} = \begin{pmatrix} E^{-1}\mu_n^{-1} & 0 \\ 0 & \mu_n^{-1}E \end{pmatrix} + \begin{pmatrix} E^{-1}Q_n\mu_n^{-1} \\ -R_{n+1}\mu_n^{-1} \end{pmatrix} (E+1)\Delta^{-1}(\mu_n^{-1}R_{n+1}, \mu_n^{-1}Q_n E) \tag{4.4}$$

with $\mu_n = 1 + Q_n R_{n+1}$ and I to ba the 2×2 identity matrix. L is a hereditary operator. The simplest ispospectral flows and nonisospectral flows are

$$K_{-1} = \begin{pmatrix} Q_{n-1}/\mu_{n-1} \\ -R_{n+1}/\mu_n \end{pmatrix}, \;\; K_0 = \begin{pmatrix} Q_n \\ -R_n \end{pmatrix}, \;\; K_1 = \begin{pmatrix} Q_{n+1} - Q_n^2 R_n \\ -R_{n-1} + R_n^2 Q_n \end{pmatrix}, \tag{4.5a}$$

$$K_2 = \begin{pmatrix} Q_{n+2} - Q_{n+1}^2 R_{n+1} - Q_n^2 R_{n-1} - 2Q_n Q_{n+1} R_n + Q_n^3 R_n^2 \\ -R_{n-2} + R_{n-1}^2 Q_{n-1} + R_n^2 Q_{n+1} + 2R_{n-1} R_n Q_n - R_n^3 Q_n^2 \end{pmatrix}, \tag{4.5b}$$

and

$$\sigma_{-1} = \begin{pmatrix} (n-1/2)Q_{n-1}/\mu_{n-1} \\ -(n+1/2)R_{n+1}/\mu_n \end{pmatrix}, \;\; \sigma_0 = \begin{pmatrix} (n+1/2)Q_n \\ -(n-1/2)R_n \end{pmatrix}, \tag{4.6a}$$

$$\sigma_1 = \begin{pmatrix} (n+3/2)Q_{n+1} - (n+3/2)Q_n^2 R_n - 2Q_n\Delta^{-1}Q_n R_n \\ -(n-3/2)R_{n-1} + (n+1/2)R_n^2 Q_n + 2R_n\Delta^{-1}R_n Q_n \end{pmatrix}. \tag{4.6b}$$

By the procedure used in [31], we find flows K_s and σ_k defined in (4.1) and (4.2) constitute a centerless Virasoro algebra,

$$[\![K_m, K_s]\!] = 0, \tag{4.7a}$$

$$[\![K_m, \sigma_s]\!] = mK_{m+s}, \tag{4.7b}$$

$$[\![\sigma_m, \sigma_s]\!] = (m-s)\sigma_{m+s}, \quad m, s \in \mathbb{Z}. \tag{4.7c}$$

Here we point out that the results of this subsection for $s, m \geq 0$ have been listed in [34] where σ_0 defers from the present one in (4.6a) by K_0 or by a shift $n \to n - \frac{1}{2}$, which brings differences for other σ_s but does not change algebraic structure (4.7). Besides, an equivalent form of L^{-1} was given in [16].

4.2. *New symmetries of nonisospectral equations* (4.2)

Note that algebraic structure (4.7) is as same as the one generated by the AL flows (see [31] and also [32, 34]). This means the hierarchies (4.1) and (4.2) and the AL hierarchies can share those results obtained based on the algebraic structure (4.7). One remarkable result is that the nonisospectral hierarchy (4.2) posses symmetries. In fact, it is very rare for a nonisospectral equation to have infinitely many symmetries. However, making use of minus indices in the structure (4.7), infinitely many symmetries for the nonisospectral hierarchy (4.2) can be constructed.

Theorem 4.1. *Any given equation $U_{n,t_m} = \sigma_m$ in the nonisospectral hierarchy* (4.2) *possesses two sets of symmetries,*

$$\eta_s^{(m)} = \sum_{j=0}^{s} \mathrm{C}_s^j (mt_m)^{s-j} \sigma_{m-jm}, \quad (s = 0,1,2\cdots.), \tag{4.8a}$$

$$\gamma_s^{(m)} = \sum_{j=0}^{s} \mathrm{C}_s^j (mt_m)^{s-j} K_{-jm}, \quad (s = 0,1,2\cdots.), \tag{4.8b}$$

and these symmetries form a centerless Virasoro algebra with structure

$$[\![\gamma_l^{(m)}, \gamma_s^{(m)}]\!] = 0, \tag{4.9a}$$

$$[\![\gamma_l^{(m)}, \eta_s^{(m)}]\!] = -ml\gamma_{l+s-1}^{(m)}, \tag{4.9b}$$

$$[\![\eta_l^{(m)}, \eta_s^{(m)}]\!] = -m(l-s)\eta_{l+s-1}^{(m)}. \tag{4.9c}$$

Here the suffix (m) corresponds to equation $U_{n,t_m} = \sigma_m$.

We skip the proof, for which one can refer to Proposition 5.1 in [31] for the AL flows.

For an isospectral equation $U_{n,t_m} = K_m$ in the hierarchy (4.1), it also has two sets of symmetries,

$$\{K_s\} \text{ and } \{\tau_s^{(m)} = mt_m K_{m+s} + \sigma_s\}, \quad s \in \mathbb{Z},$$

and they form a centerless Virosoro algebra as well, with structure

$$[\![K_l, K_s]\!] = 0,$$
$$[\![K_l, \tau_s^{(m)}]\!] = lK_{l+s},$$
$$[\![\tau_l^{(m)}, \tau_s^{(m)}]\!] = (l-s)\tau_{l+s}^{(m)}.$$

4.3. *Infinite dimensional subalgebras and new sdAKNS hierarchies*

Equations (4.1) and (4.2) are not the sdAKNS hierarchies. It is interesting to consider infinite dimensional subalgebras of the algebra (4.7). These subalgebras, together with continuum limits of the recursion operator (4.3), can be used to construct and to identify the sdAKNS hierarchies.

4.3.1. *Infinite dimensional subalgebras*

Define

$$\bar{K}_s^{(+)} = (\mathscr{L}^{(+)})^s K_0, \;\; \bar{\sigma}_s^{(+)} = (\mathscr{L}^{(+)})^s \sigma_0, \;\; s = 0,1,2,\cdots, \tag{4.10}$$

$$\bar{K}_s^{(-)} = (\mathscr{L}^{(-)})^s K_0, \;\; \bar{\sigma}_s^{(-)} = (\mathscr{L}^{(-)})^s \sigma_0, \;\; s = 0,1,2,\cdots, \tag{4.11}$$

and

$$\bar{K}_{2s+1} = \mathscr{L}^s(K_1 - K_{-1})/2, \ \ \bar{K}_{2s} = \mathscr{L}^s K_0, \ \ s = 0,1,2,\cdots, \tag{4.12a}$$

$$\bar{\sigma}_{2s+1} = \mathscr{L}^s(\sigma_1 - \sigma_{-1})/2, \ \ \bar{\sigma}_{2s} = \mathscr{L}^s \sigma_0, \ \ s = 0,1,2,\cdots, \tag{4.12b}$$

where

$$\mathscr{L} = L - 2I + L^{-1}, \ \ \mathscr{L}^{(+)} = L - I, \ \ \mathscr{L}^{(-)} = I - L^{-1} \tag{4.13}$$

and I is the 2×2 unit matrix. Note that $\mathscr{L}^{(+)}$ defined above is the same as (3.22). These new flows can generate infinite dimensional subalgebras of (4.7).

Lemma 4.1. *The flows*

$$(\mathrm{I}): \{\bar{K}_s^{(+)}, \bar{\sigma}_l^{(+)}\}, \quad (\mathrm{II}): \{\bar{K}_s^{(-)}, \bar{\sigma}_l^{(-)}\}, \quad (\mathrm{III}): \{\bar{K}_{2m+j}, \bar{\sigma}_{2s+k}\}, \tag{4.14}$$

generate three infinite dimensional subalgebras of (4.7)*, respectively, with structures*

$$\begin{aligned}
(\mathrm{I}): &[\![\bar{K}_s^{(+)}, \bar{K}_l^{(+)}]\!] = 0,\\
&[\![\bar{K}_s^{(+)}, \bar{\sigma}_l^{(+)}]\!] = s(\bar{K}_{s+l}^{(+)} + \bar{K}_{s+l-1}^{(+)}),\\
&[\![\bar{\sigma}_s^{(+)}, \bar{\sigma}_l^{(+)}]\!] = (s-l)(\bar{\sigma}_{s+l}^{(+)} + \bar{\sigma}_{s+l-1}^{(+)}),
\end{aligned}$$

$$\begin{aligned}
(\mathrm{II}): &[\![\bar{K}_s^{(-)}, \bar{K}_l^{(-)}]\!] = 0,\\
&[\![\bar{K}_s^{(-)}, \bar{\sigma}_l^{(-)}]\!] = -s(\bar{K}_{s+l}^{(-)} - \bar{K}_{s+l-1}^{(-)}),\\
&[\![\bar{\sigma}_s^{(-)}, \bar{\sigma}_l^{(-)}]\!] = -(s-l)(\bar{\sigma}_{s+l}^{(-)} - \bar{\sigma}_{s+l-1}^{(-)}),
\end{aligned}$$

$$\begin{aligned}
(\mathrm{III}): &[\![\bar{K}_{2m+j}, \bar{K}_{2s+k}]\!] = 0,\\
&[\![\bar{K}_{2m}, \bar{\sigma}_{2s}]\!] = 2m\bar{K}_{2(m+s)-1},\\
&[\![\bar{K}_{2m}, \bar{\sigma}_{2s+1}]\!] = 2m\bar{K}_{2(m+s)} + \frac{m}{2}\bar{K}_{2(m+s+1)},\\
&[\![\bar{K}_{2m+1}, \bar{\sigma}_{2s+k}]\!] = (2m+1)\bar{K}_{2(m+s)+k} + \frac{m+1}{2}\bar{K}_{2(m+s+1)+k},\\
&[\![\bar{\sigma}_{2m}, \bar{\sigma}_{2s}]\!] = 2(m-s)\bar{\sigma}_{2(m+s)-1},\\
&[\![\bar{\sigma}_{2m+j}, \bar{\sigma}_{2s+1}]\!] = [2(m-s)-1+j]\bar{\sigma}_{2(m+s)+j} + \frac{m-s-1+j}{2}\bar{\sigma}_{2(m+s+1)+j},
\end{aligned}$$

where $j,k \in \{0,1\}$*,* $m,s \geq 0$ *and we define* $\bar{K}_{-1}^{(\pm)} = \bar{\sigma}_{-1}^{(\pm)} = \bar{K}_{-1} = \bar{\sigma}_{-1} = 0$*. Obviously, the set (III) has a subalgebra* $\{\bar{K}_{2m+1}, \bar{\sigma}_{2s+1}\}$*.*

This lemma can be verified directly. Same structure as (III) has been proved for the AL case in [33]. Note that operator $L - L^{-1}$ does not generate a subalgebra of (4.7).

4.3.2. *Three sdAKNS hierarchies*

Let us construct the sdAKNS hierarchies through considering possible combinations of the flows $\{K_s\}$ and $\{\sigma_s\}$ defined in (4.1) and (4.2). The criteria is that these combined flows should be closed as a subalgebra of (4.7) and meanwhile, they must yield their counterparts in the continuous AKNS flows in reasonable continuum limits. For this purpose we investigate continuum limits of initial flows, L and L^{-1} under a uniform scheme[a]

$$U_{n+j} = \varepsilon(q(x+j\varepsilon,t), r(x+j\varepsilon,t))^T, \quad n\varepsilon = x, \quad (n\to\infty, \varepsilon\to 0). \tag{4.15}$$

We find

$$K_0 = \varepsilon(q,-r)^T, \quad (K_1 - K_{-1})/2 = \varepsilon^2(q,r)_x^T + O(\varepsilon^3), \tag{4.16a}$$

$$\sigma_0 = x(q,-r)^T + O(\varepsilon), \quad (\sigma_1 - \sigma_{-1})/2 = \varepsilon(xq_x + q, xr_x + r)^T + O(\varepsilon^2), \tag{4.16b}$$

and

$$L = I + \varepsilon L_{AKNS} + \frac{\varepsilon^2}{2} L_{NLS} + O(\varepsilon^3), \tag{4.17a}$$

$$L^{-1} = I - \varepsilon L_{AKNS} + \frac{\varepsilon^2}{2}(2L_{AKNS}^2 - L_{NLS}) + O(\varepsilon^3), \tag{4.17b}$$

where

$$L_{AKNS} = \begin{pmatrix} \partial_x - 2q\partial_x^{-1}r & -2q\partial_x^{-1}q \\ 2r\partial_x^{-1}r & -\partial_x + 2r\partial_x^{-1}q \end{pmatrix} \tag{4.18}$$

is the recursion operator of the continuous AKNS hierarchy, and

$$L_{NLS} = \begin{pmatrix} \partial_x^2 - 2qr & 0 \\ 0 & \partial_x^2 - 2qr \end{pmatrix},$$

which yields the nonlinear Schrödinger system by acting on $(q,-r)^T$. (4.17) indicates

$$\mathscr{L}^{(+)} = L - I = \varepsilon L_{AKNS} + O(\varepsilon^2), \tag{4.19a}$$

$$\mathscr{L}^{(-)} = I - L^{-1} = \varepsilon L_{AKNS} + O(\varepsilon^2), \tag{4.19b}$$

$$\mathscr{L} = L - 2I + L^{-1} = \varepsilon^2 L_{AKNS}^2 + O(\varepsilon^3). \tag{4.19c}$$

Thus, based on the continuum limits of initial flows given in (4.16), the definition of the flows (4.10,4.11,4.12) and Lemma 4.1, we obtain three sets of sdAKNS hierarchies.

Theorem 4.2. *The flows defined in* (4.10), (4.11) *and* (4.12)*) generate three sets of the sdAKNS hierarchies*

$$(\mathrm{I}): U_{t_s} = \bar{K}_s^{(+)}, \quad U_{t_s} = \bar{\sigma}_s^{(+)}, \tag{4.20}$$

$$(\mathrm{II}): U_{t_s} = \bar{K}_s^{(-)}, \quad U_{t_s} = \bar{\sigma}_s^{(-)}, \tag{4.21}$$

$$(\mathrm{III}): U_{t_s} = \bar{K}_s, \quad U_{t_s} = \bar{\sigma}_s, \tag{4.22}$$

where $s = 0,1,\cdots$. *They all correspond to the continuous isospectral and nonisospectral AKNS hierarchies under the continuum limit* (4.15)[b]

Here we remark that $\{U_{t_s} = \bar{K}_s^{(+)}\}$ was already found in [23], which is nothing but (3.21), the result of symmetry constraint of the $\mathrm{D}^2\Delta$KP hierarchy.

[a] The correspondence between x and n is $x = x_0 + n\varepsilon$ where ε is viewed as a spacing parameter. Here we take $x_0 = 0$ for convenience.

[b] In principle we need to suitably rescale t_s by $\varepsilon^j t_s$. For example, for $U_{t_s} = \bar{K}_s$, rescale t_s by $\varepsilon^{-s} t_s$.

5. Conclusions

In the paper we presented a connection between differential-difference (2+1)-dimensional systems and (1+1)-dimensional systems. We constructed a squared-eigenfunction symmetry $(\varphi_n\bar{\varphi}_n)_x$ of the scalar $\mathrm{D}^2\Delta$KP hierarchy. The constraint of this symmetry converts Lax triad (3.4) of the $\mathrm{D}^2\Delta$KP hierarchy to the discrete spectral problem (1.4) and a sdAKNS hierarchy (3.30). It is well known that the AL spectral problem (1.7) is a discrete ZS-AKNS spectral problem, while (1.4) corresponds to a bidirectional discretization of the derivatives $\phi_{1,x}$ and $\phi_{2,x}$ in (1.1) and therefore it presents a second discretization of the ZS-AKNS spectral problem. (1.4) is also related to the ZS-AKNS spectral problem (1.1) as a Darboux transformation of the later. In addition to these results, three sdAKNS hierarchies (4.20), (4.21) and (4.22) are obtained with a criteria that the corresponding flows are closed w.r.t. Lie product (2.4) and in continuum limit they approach to the continuous AKNS hierarchies. Among these sdAKNS hierarchies, $\{U_{n,t_j}=\bar{K}_j^{(+)}\}$ is the one that is derived from Lax triad of the $\mathrm{D}^2\Delta$KP hierarchy via the symmetry constraint.

The paper, on one side, provides a differential-difference counterpart of the connection of continuous AKNS and KP systems via squared-eigenfunction symmetry constraint (cf. [8, 18]). On the other side, it presents a good example that there are different discrete spectral problems and integrable equations associated to a continuous system. Besides, there would be more interactions between the (1+1) and (2+1)-dimensional differential-difference systems due to the symmetry constraint. Some known results related to the pseudo-difference operator (3.1) (e.g. [15, 22]) could be used to investigate the sdAKNS hierarchies, and vice versa.

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Appendix A. The sdAKNS hierarchies from the AL spectral problem

From the AL spectral problem (1.7) one can derive the AL hierarchy (cf. [12]):

$$U_{t_s}=K_s=\bar{L}^sK_0,\ \ s\in\mathbb{Z}, \tag{A.1}$$

where K_0 and the first few flows are

$$K_0=\begin{pmatrix}Q_n\\-R_n\end{pmatrix},\ \ K_1=\bar{\mu}_n\begin{pmatrix}Q_{n+1}\\-R_{n-1}\end{pmatrix},\ \ K_{-1}=\bar{\mu}_n\begin{pmatrix}Q_{n-1}\\-R_{n+1}\end{pmatrix},$$

the recursion operator reads

$$\begin{aligned}\bar{L}=&\begin{pmatrix}E&0\\0&E^{-1}\end{pmatrix}+\begin{pmatrix}-Q_nE\\R_n\end{pmatrix}\Delta^{-1}(R_nE,Q_nE^{-1})\\&+\bar{\mu}_n\begin{pmatrix}-EQ_n\\R_{n-1}\end{pmatrix}\Delta^{-1}(R_n,Q_n)\frac{1}{\bar{\mu}_n},\end{aligned} \tag{A.2}$$

with its inverse

$$\begin{aligned}\bar{L}^{-1}=&\begin{pmatrix}E^{-1}&0\\0&E\end{pmatrix}+\begin{pmatrix}Q_n\\-R_nE\end{pmatrix}\Delta^{-1}(R_nE^{-1},Q_nE)\\&+\bar{\mu}_n\begin{pmatrix}Q_{n-1}\\-ER_n\end{pmatrix}\Delta^{-1}(R_n,Q_n)\frac{1}{\bar{\mu}_n},\end{aligned}$$

and here $\bar{\mu}_n = 1 - Q_n R_n$. Under the continuum limit scheme (4.15), one can find

$$\bar{L} = I + \varepsilon L_{AKNS} + \frac{\varepsilon^2}{2} L_{AKNS}^2 + O(\varepsilon^3),$$

where L_{AKNS} is given in (4.18).

There are also three sdAKNS hierarchies related to the AL spectral problem:

$$\mathrm{I}: \{U_{n,t_s} = \bar{K}_s^{(+)}\}, \ \ \mathrm{II}: \{U_{n,t_s} = \bar{K}_s^{(-)}\}, \ \ \mathrm{III}: \{U_{n,t_s} = \bar{K}_s\}, \tag{A.3}$$

where

$$\bar{K}_s^{(+)} = (\bar{\mathscr{L}}^{(+)})^s K_0, \tag{A.4}$$

$$\bar{K}_s^{(-)} = (\bar{\mathscr{L}}^{(-)})^s K_0, \tag{A.5}$$

$$\bar{K}_{2s+1} = \bar{\mathscr{L}}^s (K_1 - K_{-1})/2, \ \ \bar{K}_{2s} = \mathscr{L}^s K_0, \ \ s = 0, 1, 2, \cdots, \tag{A.6}$$

$$\bar{\mathscr{L}} = \bar{L} - 2I + \bar{L}^{-1}, \ \ \bar{\mathscr{L}}^{(+)} = \bar{L} - I, \ \ \bar{\mathscr{L}}^{(-)} = I - \bar{L}^{-1}. \tag{A.7}$$

The third sdAKNS hierarchy has been well studied. As a review one can refer to [12].

Appendix B. Gauge equivalent forms of (1.4)

Note that after the early work in [23, 34], the spectral problem (1.4) has been reinvestigated in several gauge equivalent forms (e.g. [10,26,29]). Here we list out two of them,

$$\Phi_{n+1} = \begin{pmatrix} \lambda & Q_n \\ R_{n+1} & (1 + Q_n R_{n+1})/\lambda \end{pmatrix} \Phi_n, \quad \Phi_n = (\phi_{1,n}, \phi_{2,n})^T, \tag{B.1}$$

$$\Psi_{n+1} = \begin{pmatrix} \lambda^2 & Q_n \\ \lambda^2 R_{n+1} & 1 + Q_n R_{n+1} \end{pmatrix} \Psi_n, \quad \Psi_n = (\psi_{1,n}, \psi_{2,n})^T, \tag{B.2}$$

The connection with (1.4) are the gauge transformations

$$\Phi_n = T_{1,n} \Theta_n, \ \ \Psi_n = T_{2,n} \Theta_n, \tag{B.3a}$$

where

$$T_{1,n} = \lambda^{-n-1} \begin{pmatrix} \lambda & 0 \\ R_n & 1 \end{pmatrix}, \ \ T_{2,n} = \begin{pmatrix} 1 & 0 \\ R_n & 1 \end{pmatrix}. \tag{B.3b}$$

In fact, when Θ_n is formulated by the spectral problem (1.4), by direct calculation, one can find the above Φ_n and Ψ_n satisfy (B.1) and (B.2) respectively.

In the following we show that the spectral problem (1.5) is equivalent to (B.1). First, we rewrite (B.1) as

$$\phi_{1,n+1} = \lambda \phi_{1,n} + Q_n \phi_{2,n}, \tag{B.4a}$$

$$\lambda \phi_{2,n+1} = \lambda R_{n+1} \phi_{1,n} + \phi_{2,n} + Q_n R_{n+1} \phi_{2,n}. \tag{B.4b}$$

Substituting (B.4a) multiplied by R_{n+1} into (B.4b) and after a backward shift of n, we reach

$$\lambda \phi_{2,n} = R_n \phi_{1,n} + \phi_{2,n-1}. \tag{B.5}$$

(B.4a) and (B.5) compose the spectral problem (1.5).

(1.5) is gauge equivalent to the following spectral problem

$$\begin{pmatrix} \gamma_{1,n+1} \\ -\gamma_{2,n-1} \end{pmatrix} = \begin{pmatrix} \lambda^2 & Q_n \\ R_n & -1 \end{pmatrix} \begin{pmatrix} \gamma_{1,n} \\ \gamma_{2,n} \end{pmatrix} \tag{B.6}$$

by the transformation

$$\Gamma_n = T_{3,n}\Phi_n, \;\; T_{3,n} = \lambda^{-n} \begin{pmatrix} 1 & 0 \\ 0 & 1/\lambda \end{pmatrix}, \;\; \Gamma_n = (\gamma_{1,n}, \gamma_{2,n})^T. \tag{B.7}$$

(B.6) can be derived from the scalar spectral problem (3.17). In fact, making use of relation $R_n + \Delta^{-1}R_n = \Delta^{-1}R_{n+1}E$, we can rewrite (3.17) as

$$\mathfrak{L}\varphi_n = \xi\varphi_n, \quad \mathfrak{L} = \Delta - Q_n\Delta^{-1}R_{n+1}E. \tag{B.8}$$

Then, introducing

$$\gamma_{1,n} = \varphi_n, \;\; \gamma_{2,n} = \Delta^{-1}R_{n+1}\varphi_{n+1}, \;\; \lambda = \xi, \tag{B.9}$$

we arrive at (B.6).

Since in these gauge transformations only eigenfunctions are involved (without any changes of potentials), the evolution equations derived from all these gauge equivalent spectral problems are the same (up to linear combinations of flows). In fact, for two evolution equations which are derived respectively as compatibilities of linear problems

$$\Phi_{n+1} = M_n(u_n, \lambda)\Phi_n, \;\; \Phi_{n,t} = N_n\Phi_n, \tag{B.10}$$

and

$$\Psi_{n+1} = U_n(u_n, \lambda)\Psi_n, \;\; \Psi_{n,t} = V_n\Psi_n, \tag{B.11}$$

if they are gauge equivalent via transformation $\Phi_n = T_n\Psi_n$, $|T_n| \neq 0$, then there are relations

$$M_n = T_{n+1}U_nT_n^{-1}, \;\; N_n = T_{n,t}T_n^{-1} + T_nV_nT_n^{-1},$$

and consequently their compatibilities are related by

$$M_{n,t} - N_{n+1}M_n + M_nN_n = T_{n+1}(U_{n,t_m} - V_{n+1}U_n + U_nV_n)T_n^{-1},$$

which means the two equations derived from (B.10) and (B.11) are same.

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