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## Variational symmetries and pluri-Lagrangian systems in classical mechanics

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We analyze the relation of the notion of a pluri-Lagrangian system, which recently emerged in the theory of integrable systems, to the classical notion of variational symmetry, due to E. Noether. We treat classical mechanical systems and show that, for any Lagrangian system with  $m$  commuting variational symmetries, one can construct a pluri-Lagrangian 1-form in the  $(m + 1)$ -dimensional time, whose multi-time Euler-Lagrange equations coincide with the original system supplied with  $m$  commuting evolutionary flows corresponding to the variational symmetries. We also give a Hamiltonian counterpart of this construction, leading, for any system of commuting Hamiltonian flows, to a pluri-Lagrangian 1-form with coefficients depending on functions in the phase space.

*Keywords:* Lagrangian system; variational symmetry; Noether theorem; pluri-Lagrangian structure; integrable system.

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### 1. Introduction

This paper investigates some aspects of variational structure of integrable systems. The corresponding theory was initiated in [13], where it was shown that solutions of integrable (multi-dimensionally consistent) quad-equations on any quad-surface  $\Sigma$  in  $\mathbb{Z}^m$  are critical points of a certain action functional  $\int_{\Sigma} \mathcal{L}$  obtained by integration of a suitable discrete Lagrangian 2-form  $\mathcal{L}$ . Moreover, it was observed in [13] that the critical value of the action remains invariant under local changes of the underlying quad-surface, or, in other words, that the 2-form  $\mathcal{L}$  is closed on solutions of quad-equations, and it was suggested to consider this as a defining feature of integrability.

This line of research was developed in several directions, mainly by two teams: by Nijhoff with collaborators (under the name “theory of Lagrangian multi-forms”), see [2, 13–15, 23], and by the authors of the present paper with collaborators, who termed the corresponding structures “pluri-Lagrangian”, see [8–11, 19–22]. As argued in [9], the unconventional idea to consider the action on arbitrary two-dimensional surfaces in the multi-dimensional space of independent variables has significant precursors. These include:

- *Theory of pluriharmonic functions.* By definition, a pluriharmonic function of  $m$  complex variables  $f : \mathbb{C}^m \rightarrow \mathbb{R}$  minimizes the Dirichlet energy  $E_\Gamma$  along any holomorphic curve  $\Gamma : \mathbb{C} \rightarrow \mathbb{C}^m$ ,

$$E_\Gamma = \int_\Gamma |(f \circ \Gamma)_z|^2 dz \wedge d\bar{z}.$$

The similarity of this definition with the above mentioned property of the Lagrangian structure of multi-dimensionally consistent discrete systems motivates the term *pluri-Lagrangian systems*, which was proposed in [8, 9]. Differential equations governing pluriharmonic functions,

$$\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} = 0 \quad \text{for all } i, j = 1, \dots, m,$$

are heavily overdetermined. Therefore it is not surprising that pluriharmonic functions belong to the theory of integrable systems.

- *Baxter's Z-invariance* of solvable models of statistical mechanics [4, 5]. This concept is based on invariance of the partition functions of solvable models under elementary local transformations of the underlying planar graphs. It is well known that one can identify planar graphs underlying these models with quad-surfaces in  $\mathbb{Z}^m$ . On the other hand, the classical mechanical analogue of the partition function is the action functional. This suggests the relation of Z-invariance to the concept of closedness of the Lagrangian 2-form, at least at the heuristic level. This relation has been made mathematically precise for a number of models, through the quasiclassical limit [6, 7].
- The notion of *variational symmetry*, going back to the seminal work of E. Noether [16]. One only rarely finds an adequate account of her result in the modern literature. Even the classical textbooks like the Arnold's one [1] only present very particular and restricted versions of Noether's theorem which deal with point symmetries. Relevant for us is the most general (original) version, dealing with what is nowadays called generalized symmetries. One of the best sources treating Noether's theorem in its full generality is Olver's textbook [17], which we closely follow in our presentation. In the context of integrable partial differential equations, a direct relation between Noether's theorem and closedness of the Lagrangian form in the multi-time has been shown in [20]. Here, we elaborate on this topic further, in the context of classical mechanics.

The structure of the paper is as follows. In Section 2, we recall, following mainly [17, Chapter 5], the notions of generalized vector fields and of variational symmetries, and give a proof of Noether's theorem in the context of classical mechanical systems. In Section 3, we provide the reader with the Hamiltonian counterpart of the theory, and show that variational symmetries of a Lagrangian system are in a one-to-one correspondence with integrals of the corresponding Hamiltonian system. Surprisingly, this fundamental result, well-known in the folklore, is difficult to locate in the literature. Our presentation is pretty close to [18]. In Section 4, we discuss commuting variational symmetries, which are Lagrangian counterparts of commuting integrals of a given Hamiltonian system. Again, even if our results on commuting variational symmetries will be of no surprise for an expert, we were unable to locate a reference which would contain a satisfying presentation, and therefore we include complete details here. In Section 5, we undertake a slight but a fundamentally important change of the viewpoint on variational symmetries, from an algebraic interpretation as derivations

acting on differential polynomials to a geometric interpretation as commuting flows. Of course, this latter interpretation is not less classical, but, amazingly, it seems to be usually suppressed in the literature on symmetries of differential equations. This geometric interpretation is the crucial link to the emerging theory of pluri-Lagrangian systems. In Section 6, we recall, following [20], the main positions of this theory. In Section 7, we establish the pluri-Lagrangian structure for a general Lagrangian system possessing commuting variational symmetries. Section 8 is devoted to some technical computations whose results illuminate the main property of the pluri-Lagrangian structure, namely the almost closedness of the pluri-Lagrangian 1-form on the space of solutions. In Section 9, we demonstrate that considering the pluri-Lagrangian structure in the phase space (rather than in the configuration space) allows us to greatly simplify all the concepts and constructions. The final Section 10 illustrates the concepts and results of the paper with two classical examples of integrable systems, the Kepler system and the Toda lattice.

**2. Variational symmetries and Noether theorem**

Let  $x = (x_1, \dots, x_N)$  be coordinates on an  $N$ -dimensional configuration space  $\mathcal{X}$ . We mainly consider the case  $\mathcal{X} = \mathbb{R}^N$ , but  $\mathcal{X}$  can also be a smooth  $N$ -dimensional manifold with local coordinates  $x$ . Let  $(x, \dot{x})$  be the natural coordinates on the  $2N$ -dimensional tangent bundle  $T\mathcal{X}$ , and let  $L : T\mathcal{X} \rightarrow \mathbb{R}$  be a Lagrange function. The Hamilton’s principle states that motions of the corresponding mechanical system are critical curves of the action functional  $S$ , which assigns to each curve  $x : [t_0, t_1] \rightarrow \mathcal{X}$  (with fixed values of  $x(t_0)$  and  $x(t_1)$ ) the number

$$S[x] = \int_{t_0}^{t_1} L(x(t), \dot{x}(t)) dt. \tag{2.1}$$

Critical curves of the action (2.1) are solutions of the Euler-Lagrange equations given by

$$\mathcal{E}_i = \mathcal{E}_i(x, \dot{x}, \ddot{x}) := \frac{\partial L(x, \dot{x})}{\partial x_i} - D_t \left( \frac{\partial L(x, \dot{x})}{\partial \dot{x}_i} \right) = 0, \quad i = 1, \dots, N. \tag{2.2}$$

Here and everywhere below  $D_t$  denotes the total derivative w.r.t. time  $t$ . We will assume that  $L$  is non-degenerate:

$$\det \left( \frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} \right) \neq 0, \tag{2.3}$$

so that the Euler-Lagrange equations can be solved for  $\ddot{x}_i$ , resulting in explicit second order differential equations.

Consider an evolutionary generalized vector field  $v_1$  on  $\mathcal{X}$ ,

$$v_1 = \sum_{i=1}^N V_i^{(1)}(x, \dot{x}) \frac{\partial}{\partial x_i}, \tag{2.4}$$

where the  $N$ -tuple of functions  $V_i^{(1)}(x, \dot{x})$ ,  $i = 1, \dots, N$ , is called the characteristic of  $v_1$  (the index “1” will become important later, when we consider several generalized vector fields simultaneously). It

generates a linear differential operator acting on differential polynomials, i.e., on functions depending on  $x$  and its time derivatives  $\dot{x}$ ,  $\ddot{x}$ , etc., defined by the formal sum

$$D_{v_1} = \sum_{i=1}^N V_i^{(1)} \frac{\partial}{\partial x_i} + \sum_{i=1}^N \left( D_t V_i^{(1)} \right) \frac{\partial}{\partial \dot{x}_i} + \sum_{i=1}^N \left( D_t^2 V_i^{(1)} \right) \frac{\partial}{\partial \ddot{x}_i} + \dots \tag{2.5}$$

Observe that  $D_t V_i^{(1)}$  is a function of  $x$ ,  $\dot{x}$  and  $\ddot{x}$ ,  $D_t^2 V_i^{(1)}$  is a function of  $x$ ,  $\dot{x}$ ,  $\ddot{x}$  and  $\dddot{x}$ , and so on.

**Definition 2.1. (Variational symmetry)** We say that  $v_1$  is a *variational symmetry* of the Lagrangian problem (2.1) if there exists a function  $F_1 = F_1(x, \dot{x})$ , called the flux of the variational symmetry  $v_1$ , such that

$$D_{v_1} L(x, \dot{x}) - D_t F_1(x, \dot{x}) = 0. \tag{2.6}$$

**Definition 2.2. (Integral and its characteristic)** We say that  $J_1(x, \dot{x})$  is an *integral of the Euler-Lagrange equations*  $\mathcal{E}_i = 0$  (see (2.2)) with the *characteristic*  $(V_i^{(1)}(x, \dot{x}))_{1 \leq i \leq N}$ , if

$$D_t J_1(x, \dot{x}) = - \sum_{i=1}^N V_i^{(1)}(x, \dot{x}) \mathcal{E}_i. \tag{2.7}$$

The famous theorem by Emmy Noether establishes a one-to-one correspondence between variational symmetries and integrals of Lagrangian systems.

**Theorem 2.1. (E. Noether’s theorem)**

a) Let the generalized vector field  $v_1$  given by equation (2.4) be a variational symmetry of the Lagrangian problem (2.1), with the flux  $F_1(x, \dot{x})$ . Then the function

$$J_1(x, \dot{x}) = \sum_{i=1}^N \frac{\partial L(x, \dot{x})}{\partial \dot{x}_i} V_i^{(1)}(x, \dot{x}) - F_1(x, \dot{x}) \tag{2.8}$$

is an integral of Euler-Lagrange equations  $\mathcal{E}_i = 0$  with the characteristic  $(V_i^{(1)}(x, \dot{x}))_{1 \leq i \leq N}$ .

b) Conversely, let  $J_1(x, \dot{x})$  be an integral of motion of Euler-Lagrange equations  $\mathcal{E}_i = 0$ , with the characteristic  $(V_i^{(1)}(x, \dot{x}))_{1 \leq i \leq N}$ . Then the generalized vector field  $v_1$  given by equation (2.4) is a variational symmetry of the Lagrangian problem (2.1), with the flux

$$F_1(x, \dot{x}) = \sum_{i=1}^N \frac{\partial L(x, \dot{x})}{\partial \dot{x}_i} V_i^{(1)}(x, \dot{x}) - J_1(x, \dot{x}). \tag{2.9}$$

**Proof.** Both parts are consequences of the following computation:

$$\begin{aligned} D_{v_1} L &= \sum_{i=1}^N V_i^{(1)} \frac{\partial L}{\partial x_i} + \sum_{i=1}^N \left( D_t V_i^{(1)} \right) \frac{\partial L}{\partial \dot{x}_i} \\ &= \sum_{i=1}^N V_i^{(1)} \left( \mathcal{E}_i + D_t \left( \frac{\partial L}{\partial \dot{x}_i} \right) \right) + \sum_{i=1}^N \left( D_t V_i^{(1)} \right) \frac{\partial L}{\partial \dot{x}_i} \\ &= D_t \left( \sum_{i=1}^N \frac{\partial L}{\partial \dot{x}_i} V_i^{(1)} \right) + \sum_{i=1}^N V_i^{(1)} \mathcal{E}_i. \end{aligned}$$

The theorem is proved. □

**Remark. (Energy integral)** The generalized vector field

$$v = \sum_{i=1}^N \dot{x}_i \frac{\partial}{\partial x_i},$$

which generates the differential operator  $D_v = D_t$  (when acting on functions which do not explicitly depend on  $t$ ), is trivially a variational symmetry of any Lagrange function  $L(x, \dot{x})$ , with the flux  $F(x, \dot{x}) = L(x, \dot{x})$ . For this variational symmetry, the Noether integral turns into the *energy integral*

$$J(x, \dot{x}) = \sum_{i=1}^N \frac{\partial L}{\partial \dot{x}_i} \dot{x}_i - L(x, \dot{x}). \quad (2.10)$$

### 3. Hamiltonian side of the picture

Introduce the conjugate momenta  $p = (p_1, \dots, p_N)$  by the Legendre transformation  $T \mathcal{X} \ni (x, \dot{x}) \mapsto (x, p) \in T^* \mathcal{X}$ ,

$$p_i = p_i(x, \dot{x}) = \frac{\partial L(x, \dot{x})}{\partial \dot{x}_i}, \quad i = 1, \dots, N. \quad (3.1)$$

The cotangent bundle  $T^* \mathcal{X}$  is a  $2N$ -dimensional symplectic manifold equipped with the canonical symplectic 2-form

$$\omega = \sum_{i=1}^N dx_i \wedge dp_i.$$

Under the non-degeneracy condition (2.3), equation (3.1) defines a local diffeomorphism  $T \mathcal{X} \rightarrow T^* \mathcal{X}$ , which can be locally inverted to express the velocities  $\dot{x}_i$  in terms of  $(x, p)$ , i.e.,  $\dot{x}_i = \dot{x}_i(x, p)$ . The dynamics on  $T^* \mathcal{X}$ , equivalent to that governed by Euler-Lagrange equations (2.2) on  $T \mathcal{X}$ , is described in terms of Hamiltonian equations of motion:

$$D_t x_i = \frac{\partial H(x, p)}{\partial p_i}, \quad D_t p_i = -\frac{\partial H(x, p)}{\partial x_i}, \quad i = 1, \dots, N. \quad (3.2)$$

Here  $H : T^* \mathcal{X} \rightarrow \mathbb{R}$  is the Hamilton function corresponding to the Lagrange function  $L$  by means of a Legendre transformation:

$$H(x, p) = \sum_{i=1}^N p_i \dot{x}_i - L(x, \dot{x}) \Big|_{\dot{x}=\dot{x}(x, p)}. \quad (3.3)$$

Derivation of Hamiltonian equations of motion (3.2) is based on the following facts:

$$\frac{\partial H(x, p)}{\partial x_i} = -\frac{\partial L(x, \dot{x})}{\partial x_i}, \quad \frac{\partial H(x, p)}{\partial p_i} = \dot{x}_i, \quad (3.4)$$

which hold true at the points  $(x, p) \in T^* \mathcal{X}$  and  $(x, \dot{x}) \in T \mathcal{X}$  related by the Legendre transformation (3.1) and which follow from the definition (3.3) by differentiation. The Hamilton function  $H(x, p)$  is nothing but the energy integral  $J(x, \dot{x})$  expressed in terms of  $(x, p)$ . The fact that  $H(x, p)$  is an integral of motion of the Hamiltonian equations (3.2) is easily checked by direct computation.

The following two theorems, which are in a sense mutually converse, show that variational symmetries of a Lagrangian system are in a one-to-one correspondence with integrals of the corresponding Hamiltonian system.

**Theorem 3.1. (From a variational symmetry to a commuting Hamiltonian flow)** Let  $v_1$  be a variational symmetry of the Lagrange function  $L(x, \dot{x})$ , with the flux  $F_1(x, \dot{x})$  and with the Noether integral  $J_1(x, \dot{x})$ . Set  $H_1(x, p) = J_1(x, \dot{x})|_{\dot{x}=\dot{x}(x,p)}$ . Then

$$\{H, H_1\} = \sum_{i=1}^N \left( \frac{\partial H}{\partial x_i} \frac{\partial H_1}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial H_1}{\partial x_i} \right) = 0, \tag{3.5}$$

so that  $H_1$  is an integral of motion of the Hamiltonian flow with the Hamilton function  $H$ .

**Proof.** First of all, we prove that

$$\frac{\partial H_1(x, p)}{\partial x_i} = \sum_{j=1}^N p_j \frac{\partial V_j^{(1)}(x, \dot{x})}{\partial x_i} - \frac{\partial F_1(x, \dot{x})}{\partial x_i}, \tag{3.6}$$

$$\frac{\partial H_1(x, p)}{\partial p_i} = V_i^{(1)}(x, \dot{x}). \tag{3.7}$$

Indeed, we compute:

$$\begin{aligned} \frac{\partial H_1(x, p)}{\partial x_i} &= \sum_{j=1}^N p_j \frac{\partial V_j^{(1)}}{\partial x_i} + \sum_{j=1}^N \sum_{k=1}^N p_j \frac{\partial V_j^{(1)}}{\partial \dot{x}_k} \frac{\partial \dot{x}_k}{\partial x_i} - \frac{\partial F_1}{\partial x_i} - \sum_{k=1}^N \frac{\partial F_1}{\partial \dot{x}_k} \frac{\partial \dot{x}_k}{\partial x_i}, \\ \frac{\partial H_1(x, p)}{\partial p_i} &= V_i^{(1)} + \sum_{j=1}^N \sum_{k=1}^N p_j \frac{\partial V_j^{(1)}}{\partial \dot{x}_k} \frac{\partial \dot{x}_k}{\partial p_i} - \sum_{k=1}^N \frac{\partial F_1}{\partial \dot{x}_k} \frac{\partial \dot{x}_k}{\partial p_i}, \end{aligned}$$

and (3.6), (3.7) follow by virtue of the following Lemma:

**Lemma 3.1.** For the flux  $F_1(x, \dot{x})$  of a variational symmetry (2.4), one has:

$$\frac{\partial F_1}{\partial \dot{x}_i} = \sum_{j=1}^N \frac{\partial L}{\partial \dot{x}_j} \frac{\partial V_j^{(1)}}{\partial \dot{x}_i}. \tag{3.8}$$

**Proof of Lemma 3.1.** We have:

$$\begin{aligned} D_t F_1 &= \sum_{i=1}^N \frac{\partial F_1}{\partial x_i} \dot{x}_i + \sum_{i=1}^N \frac{\partial F_1}{\partial \dot{x}_i} \ddot{x}_i, \\ D_{v_1} L &= \sum_{j=1}^N \frac{\partial L}{\partial x_j} V_j^{(1)} + \sum_{j=1}^N \frac{\partial L}{\partial \dot{x}_j} (D_t V_j^{(1)}) \\ &= \sum_{j=1}^N \frac{\partial L}{\partial x_j} V_j^{(1)} + \sum_{j=1}^N \sum_{i=1}^N \frac{\partial L}{\partial \dot{x}_j} \frac{\partial V_j^{(1)}}{\partial x_i} \dot{x}_i + \sum_{j=1}^N \sum_{i=1}^N \frac{\partial L}{\partial \dot{x}_j} \frac{\partial V_j^{(1)}}{\partial \dot{x}_i} \ddot{x}_i. \end{aligned}$$

Equation (3.8) follows by comparison of coefficients by  $\ddot{x}_i$ . □

Continuing the proof of Theorem 3.1, we compute with the help of (3.4), (3.6) and (3.7):

$$\begin{aligned} \{H_1, H\} &= \sum_{i=1}^N \left( -\frac{\partial H}{\partial x_i} \frac{\partial H_1}{\partial p_i} + \frac{\partial H}{\partial p_i} \frac{\partial H_1}{\partial x_i} \right) \\ &= \sum_{i=1}^N \left( \frac{\partial L}{\partial x_i} V_i^{(1)} + \dot{x}_i \left( \sum_{j=1}^N p_j \frac{\partial V_j^{(1)}}{\partial x_i} - \frac{\partial F_1}{\partial x_i} \right) \right) \\ &= \sum_{i=1}^N (\mathcal{E}_i + D_t p_i) V_i^{(1)} + \sum_{j=1}^N p_j (D_t V_j^{(1)}) - D_t F_1 - \sum_{i=1}^N \ddot{x}_i \left( \sum_{j=1}^N p_j \frac{\partial V_j^{(1)}}{\partial \dot{x}_i} - \frac{\partial F_1}{\partial \dot{x}_i} \right). \end{aligned}$$

According to Lemma 3.1, the last sum vanishes, and we find:

$$\{H_1, H\} = \sum_{i=1}^N V_i^{(1)} \mathcal{E}_i + D_t \left( \sum_{i=1}^N p_i V_i^{(1)} - F_1 \right) = \sum_{i=1}^N V_i^{(1)} \mathcal{E}_i + D_t J_1 = 0.$$

Theorem 3.1 is proved. □

**Remark.** Note that equations (3.4) can be recovered from equations (3.6), (3.7), if we replace  $V_i^{(1)}(x, \dot{x})$  and  $F_1(x, \dot{x})$  by  $V_i(x, \dot{x}) = \dot{x}_i$  and  $F(x, \dot{x}) = L(x, \dot{x})$ , respectively.

**Theorem 3.2. (From a commuting Hamiltonian flow to a variational symmetry)** Let  $H_1 : T^* \mathcal{X} \rightarrow \mathbb{R}$  be an integral of motion of the Hamiltonian flow with the Hamilton function  $H$ . Set

$$V_i^{(1)}(x, \dot{x}) = \left. \frac{\partial H_1(x, p)}{\partial p_i} \right|_{p=p(x, \dot{x})},$$

and define a generalized vector field  $v_1 = \sum_{i=1}^N V_i^{(1)} \partial / \partial x_i$ . Then  $v_1$  is a variational symmetry of the Lagrangian problem (2.1), with the Noether integral  $J_1(x, \dot{x})$  given by

$$J_1(x, \dot{x}) = \left. H_1(x, p) \right|_{p=p(x, \dot{x})},$$

and with the flux  $F_1(x, \dot{x})$  given by formula (2.9).

**Proof.** According to Theorem 2.1, part b), it is enough to show that  $J_1$  is an integral of motion of equations  $\mathcal{E}_i = 0$  with the characteristic  $(V_i^{(1)})_{1 \leq i \leq N}$ , so that equation (2.7) is satisfied. But this follows by a direct computation (in the following formulas we always assume that  $(x, p) \in T^* \mathcal{X}$  and  $(x, \dot{x}) \in T \mathcal{X}$  are related by the Legendre transformation (3.1)):

$$\begin{aligned} D_t J_1 &= \sum_{i=1}^N \left( \frac{\partial H_1(x, p)}{\partial x_i} \dot{x}_i + \frac{\partial H_1(x, p)}{\partial p_i} D_t p_i(x, \dot{x}) \right) \\ &= \sum_{i=1}^N \left( \frac{\partial H_1(x, p)}{\partial x_i} \dot{x}_i + \frac{\partial H_1(x, p)}{\partial p_i} \left( \frac{\partial L(x, \dot{x})}{\partial x_i} - \mathcal{E}_i \right) \right) \\ &= \sum_{i=1}^N \left( \frac{\partial H_1}{\partial x_i} \frac{\partial H}{\partial p_i} - \frac{\partial H_1}{\partial p_i} \frac{\partial H}{\partial x_i} \right) - \sum_{i=1}^N \frac{\partial H_1(x, p)}{\partial p_i} \mathcal{E}_i \\ &= \{H_1, H\} - \sum_{i=1}^N V_i^{(1)} \mathcal{E}_i, \end{aligned}$$

where we used (2.2) and (3.4). It remains to use (3.5). □



#### 4. Commuting variational symmetries

We show that commutativity of variational symmetries provides us with an adequate framework to discuss integrability in the Lagrangian context. Recall that usually the notion of integrability is expressed using the Hamiltonian language.

Assume that the Lagrange function  $L(x, \dot{x})$  admits two variational symmetries

$$v_k = \sum_{i=1}^N V_i^{(k)}(x, \dot{x}) \frac{\partial}{\partial x_i}, \quad k = 1, 2,$$

with the corresponding fluxes  $F_k(x, \dot{x})$ , so that

$$D_{v_k} L(x, \dot{x}) - D_t F_k(x, \dot{x}) = 0, \quad k = 1, 2. \tag{4.1}$$

Recall [17] that the commutator  $[D_{v_1}, D_{v_2}]$  is a differential operator which can be represented as  $D_{[v_1, v_2]}$ , that is, the prolongation of the generalized vector field

$$[v_1, v_2] = \sum_{i=1}^N \left( D_{v_1} V_i^{(2)} - D_{v_2} V_i^{(1)} \right) \frac{\partial}{\partial x_i}. \tag{4.2}$$

**Definition 4.1. (Commuting variational symmetries)** We say that variational symmetries  $v_1, v_2$  of the variational problem (2.1) *commute* if

$$D_{v_1} V_i^{(2)} - D_{v_2} V_i^{(1)} = \sum_{j=1}^N r_{ij}(x, \dot{x}) \mathcal{E}_j, \quad i = 1, \dots, N \tag{4.3}$$

with some functions  $r_{ij}(x, \dot{x})$ , so that  $[v_1, v_2] = 0$  on solutions of the Euler-Lagrange equations (2.2).

**Proposition 4.1.** *For two commuting variational symmetries  $v_1, v_2$ , the functions  $r_{ij}(x, \dot{x})$  are skew-symmetric,*

$$r_{ij}(x, \dot{x}) = -r_{ji}(x, \dot{x}).$$

More precisely, setting  $H_k(x, p) = J_k(x, \dot{x})|_{\dot{x}=\dot{x}(x,p)}$ , we have:

$$r_{ij}(x, \dot{x}) = \sum_{k,m=1}^N \left( \frac{\partial^2 H_1}{\partial p_i \partial p_m} \frac{\partial^2 L}{\partial \dot{x}_m \partial \dot{x}_k} \frac{\partial^2 H_2}{\partial p_k \partial p_j} - \frac{\partial^2 H_2}{\partial p_i \partial p_m} \frac{\partial^2 L}{\partial \dot{x}_m \partial \dot{x}_k} \frac{\partial^2 H_1}{\partial p_k \partial p_j} \right). \tag{4.4}$$

**Proof.** We compare the terms with  $\ddot{x}_\ell$  on the both sides of equation (4.3). Such terms in  $D_{v_1} V_i^{(2)} - D_{v_2} V_i^{(1)}$  come from

$$\sum_{k=1}^N \left( (D_t V_k^{(1)}) \frac{\partial V_i^{(2)}}{\partial \dot{x}_k} - (D_t V_k^{(2)}) \frac{\partial V_i^{(1)}}{\partial \dot{x}_k} \right),$$

and are equal to

$$\sum_{k=1}^N \sum_{\ell=1}^N \left( \frac{\partial V_i^{(2)}}{\partial \dot{x}_k} \frac{\partial V_k^{(1)}}{\partial \dot{x}_\ell} - \frac{\partial V_i^{(1)}}{\partial \dot{x}_k} \frac{\partial V_k^{(2)}}{\partial \dot{x}_\ell} \right) \ddot{x}_\ell.$$

From (3.7) we derive:

$$\frac{\partial V_i^{(1)}}{\partial \dot{x}_k} = \sum_{m=1}^N \frac{\partial^2 H_1}{\partial p_i \partial p_m} \frac{\partial p_m}{\partial \dot{x}_k}.$$

Thus, the terms with  $\ddot{x}_\ell$  in  $D_{v_1} V_i^{(2)} - D_{v_2} V_i^{(1)}$  are equal to

$$\sum_{k=1}^N \sum_{\ell=1}^N \sum_{m=1}^N \sum_{j=1}^N \left( \frac{\partial^2 H_2}{\partial p_i \partial p_m} \frac{\partial p_m}{\partial \dot{x}_k} \frac{\partial^2 H_1}{\partial p_k \partial p_j} \frac{\partial p_j}{\partial \dot{x}_\ell} - \frac{\partial^2 H_1}{\partial p_i \partial p_m} \frac{\partial p_m}{\partial \dot{x}_k} \frac{\partial^2 H_2}{\partial p_k \partial p_j} \frac{\partial p_j}{\partial \dot{x}_\ell} \right) \ddot{x}_\ell.$$

This has to be compared with the terms with  $\ddot{x}_\ell$  in  $\sum_{j=1}^N r_{ij} \mathcal{E}_j$ . According to (2.2), we have:

$$\mathcal{E}_j = - \sum_{\ell=1}^N \frac{\partial p_j}{\partial \dot{x}_\ell} \ddot{x}_\ell + \dots$$

With a reference to the non-degeneracy condition (2.3), we arrive at (4.4). □

**Proposition 4.2.** *If two variational symmetries  $v_1, v_2$  commute, then their fluxes  $F_1, F_2$  satisfy:*

$$D_{v_1} F_2 - D_{v_2} F_1 = c_{12} + \sum_{i=1}^N p_i \left( D_{v_1} V_i^{(2)} - D_{v_2} V_i^{(1)} \right), \tag{4.5}$$

where  $c_{12} = \text{const}$ . In particular, on solutions of the Euler-Lagrange equations  $\mathcal{E}_j = 0$  we have:

$$D_{v_1} F_2 - D_{v_2} F_1 = c_{12}. \tag{4.6}$$

**Proof.** Using definition (4.1) of variational symmetries and the fact that evolutionary vector fields  $D_{v_k}$  commute with  $D_t$  (see [17]), we find:

$$\begin{aligned} D_t \left( D_{v_1} F_2 - D_{v_2} F_1 \right) &= [D_{v_1}, D_{v_2}] L = D_{[v_1, v_2]} L \\ &= \sum_{i=1}^N \left( \left( D_{v_1} V_i^{(2)} - D_{v_2} V_i^{(1)} \right) \frac{\partial L}{\partial x_i} + D_t \left( D_{v_1} V_i^{(2)} - D_{v_2} V_i^{(1)} \right) \frac{\partial L}{\partial \dot{x}_i} \right) \\ &= \sum_{i=1}^N \left( \left( D_{v_1} V_i^{(2)} - D_{v_2} V_i^{(1)} \right) (\mathcal{E}_i + D_t p_i) + D_t \left( D_{v_1} V_i^{(2)} - D_{v_2} V_i^{(1)} \right) p_i \right) \\ &= \sum_{i=1}^N \left( D_{v_1} V_i^{(2)} - D_{v_2} V_i^{(1)} \right) \mathcal{E}_i + D_t \left( \sum_{i=1}^N p_i \left( D_{v_1} V_i^{(2)} - D_{v_2} V_i^{(1)} \right) \right). \end{aligned}$$

The first sum on the right-hand side is equal to  $\sum_{i=1}^N \sum_{j=1}^N r_{ij} \mathcal{E}_j \mathcal{E}_i$  and vanishes due to the skew-symmetry of  $r_{ij}$  (see Proposition 4.1). □

**Theorem 4.1. (From commuting variational symmetries to commuting Hamiltonian flows)** *Let  $v_k$  ( $k = 1, 2$ ) be commuting variational symmetries of the Lagrange function  $L(x, \dot{x})$  with the Noether integrals  $J_k(x, \dot{x})$ . Set  $H_k(x, p) = J_k(x, \dot{x})|_{\dot{x}=\dot{x}(x,p)}$ . Then*

$$\{H_1, H_2\} = c_{12}, \tag{4.7}$$

with the constant  $c_{12}$  from Proposition 4.2. Thus, Hamiltonian flows with the Hamilton functions  $H_k$  commute.

**Proof.** We compute:

$$D_{v_1}F_2 - D_{v_2}F_1 = \sum_{i=1}^N \left( V_i^{(1)} \frac{\partial F_2}{\partial x_i} - V_i^{(2)} \frac{\partial F_1}{\partial x_i} \right) + \sum_{i=1}^N \left( (D_t V_i^{(1)}) \frac{\partial F_2}{\partial \dot{x}_i} - (D_t V_i^{(2)}) \frac{\partial F_1}{\partial \dot{x}_i} \right).$$

Using Lemma 3.1, we find:

$$\begin{aligned} D_{v_1}F_2 - D_{v_2}F_1 &= \sum_{i=1}^N \left( V_i^{(1)} \frac{\partial F_2}{\partial x_i} - V_i^{(2)} \frac{\partial F_1}{\partial x_i} \right) \\ &\quad + \sum_{i=1}^N \sum_{j=1}^N p_j \left( (D_t V_i^{(1)}) \frac{\partial V_j^{(2)}}{\partial \dot{x}_i} - (D_t V_i^{(2)}) \frac{\partial V_j^{(1)}}{\partial \dot{x}_i} \right) \\ &= \sum_{i=1}^N \left( V_i^{(1)} \frac{\partial F_2}{\partial x_i} - V_i^{(2)} \frac{\partial F_1}{\partial x_i} \right) - \sum_{i=1}^N \sum_{j=1}^N p_j \left( V_i^{(1)} \frac{\partial V_j^{(2)}}{\partial x_i} - V_i^{(2)} \frac{\partial V_j^{(1)}}{\partial x_i} \right) \\ &\quad + \sum_{j=1}^N p_j (D_{v_1}V_j^{(2)} - D_{v_2}V_j^{(1)}). \end{aligned}$$

Comparing with Proposition 4.2, we find:

$$\sum_{i=1}^N \left( V_i^{(2)} \left( \sum_{j=1}^N p_j \frac{\partial V_j^{(1)}}{\partial x_i} - \frac{\partial F_1}{\partial x_i} \right) - V_i^{(1)} \left( \sum_{j=1}^N p_j \frac{\partial V_j^{(2)}}{\partial x_i} - \frac{\partial F_2}{\partial x_i} \right) \right) = c_{12}.$$

According to (3.6), (3.7), this can be put as

$$\sum_{i=1}^N \left( \frac{\partial H_2}{\partial p_i} \frac{\partial H_1}{\partial x_i} - \frac{\partial H_1}{\partial p_i} \frac{\partial H_2}{\partial x_i} \right) = \{H_1, H_2\} = c_{12},$$

which finishes the proof. □

**Theorem 4.2.** (From commuting Hamiltonian flows to commuting variational symmetries) Let  $H_1, H_2 : T^* \mathcal{X} \rightarrow \mathbb{R}$  be integrals of motion of the Hamiltonian flow with the Hamilton function  $H$ , such that the corresponding Hamiltonian flows commute, so that (4.7) is satisfied. Set

$$V_i^{(k)}(x, \dot{x}) = \left. \frac{\partial H_k(x, p)}{\partial p_i} \right|_{p=p(x, \dot{x})}, \quad k = 1, 2,$$

and define variational symmetries  $v_k = \sum_{i=1}^N V_i^{(k)} \partial / \partial x_i$  ( $k = 1, 2$ ) of the Lagrangian problem (2.1). Then  $v_1, v_2$  commute on solutions of the Lagrangian problem (2.1).

**Proof.** We compute:

$$\begin{aligned} D_{v_1}V_i^{(2)} &= \sum_{k=1}^N \left( \frac{\partial V_i^{(2)}}{\partial x_k} V_k^{(1)} + \frac{\partial V_i^{(2)}}{\partial \dot{x}_k} (D_t V_k^{(1)}) \right) \\ &= \sum_{k=1}^N \frac{\partial V_i^{(2)}}{\partial x_k} V_k^{(1)} + \sum_{k=1}^N \sum_{\ell=1}^N \frac{\partial V_i^{(2)}}{\partial \dot{x}_k} \left( \frac{\partial V_k^{(1)}}{\partial x_\ell} \dot{x}_\ell + \frac{\partial V_k^{(1)}}{\partial \dot{x}_\ell} \ddot{x}_\ell \right). \end{aligned}$$

Differentiating the definition  $V_i^{(k)} = \partial H_k / \partial p_i$ , we find:

$$D_{v_1} V_i^{(2)} = \sum_{k=1}^N \left( \frac{\partial^2 H_2}{\partial p_i \partial x_k} + \sum_{m=1}^N \frac{\partial^2 H_2}{\partial p_i \partial p_m} \frac{\partial p_m}{\partial x_k} \right) \frac{\partial H_1}{\partial p_k} + \sum_{k=1}^N \sum_{m=1}^N \frac{\partial^2 H_2}{\partial p_i \partial p_m} \frac{\partial p_m}{\partial \dot{x}_k} \sum_{\ell=1}^N \left( \frac{\partial^2 H_1}{\partial p_k \partial x_\ell} \dot{x}_\ell + \sum_{j=1}^N \frac{\partial^2 H_1}{\partial p_k \partial p_j} \left( \frac{\partial p_j}{\partial x_\ell} \dot{x}_\ell + \frac{\partial p_j}{\partial \dot{x}_\ell} \ddot{x}_\ell \right) \right).$$

In the last line we use

$$\sum_{\ell=1}^N \left( \frac{\partial p_j}{\partial x_\ell} \dot{x}_\ell + \frac{\partial p_j}{\partial \dot{x}_\ell} \ddot{x}_\ell \right) = D_t p_j = \frac{\partial L}{\partial x_j} - \mathcal{E}_j,$$

and obtain:

$$D_{v_1} V_i^{(2)} = \sum_{k=1}^N \left( \frac{\partial^2 H_2}{\partial p_i \partial x_k} + \sum_{m=1}^N \frac{\partial^2 H_2}{\partial p_i \partial p_m} \frac{\partial p_m}{\partial x_k} \right) \frac{\partial H_1}{\partial p_k} + \sum_{k=1}^N \sum_{m=1}^N \frac{\partial^2 H_2}{\partial p_i \partial p_m} \frac{\partial p_m}{\partial \dot{x}_k} \left( \sum_{\ell=1}^N \frac{\partial^2 H_1}{\partial p_k \partial x_\ell} \dot{x}_\ell + \sum_{j=1}^N \frac{\partial^2 H_1}{\partial p_k \partial p_j} \frac{\partial L}{\partial x_j} - \sum_{j=1}^N \frac{\partial^2 H_1}{\partial p_k \partial p_j} \mathcal{E}_j \right).$$

In the last line we replace  $\dot{x}_\ell$  by  $\partial H / \partial p_\ell$  and  $\partial L / \partial x_j$  by  $-\partial H / \partial x_j$ :

$$D_{v_1} V_i^{(2)} = \sum_{k=1}^N \frac{\partial^2 H_2}{\partial p_i \partial x^k} \frac{\partial H_1}{\partial p_k} + \sum_{k=1}^N \sum_{m=1}^N \frac{\partial^2 H_2}{\partial p_i \partial p_m} \frac{\partial^2 L}{\partial \dot{x}_m \partial x_k} \frac{\partial H_1}{\partial p_k} + \sum_{k=1}^N \sum_{m=1}^N \frac{\partial^2 H_2}{\partial p_i \partial p_m} \frac{\partial^2 L}{\partial \dot{x}_m \partial \dot{x}_k} \sum_{j=1}^N \left( \frac{\partial^2 H_1}{\partial p_k \partial x_j} \frac{\partial H}{\partial p_j} - \frac{\partial^2 H_1}{\partial p_k \partial p_j} \frac{\partial H}{\partial x_j} \right) - \sum_{k=1}^N \sum_{m=1}^N \sum_{j=1}^N \frac{\partial^2 H_2}{\partial p_i \partial p_m} \frac{\partial^2 L}{\partial \dot{x}_m \partial \dot{x}_k} \frac{\partial^2 H_1}{\partial p_k \partial p_j} \mathcal{E}_j.$$

In the second line of the above formula we use the identity

$$\sum_{j=1}^N \left( \frac{\partial^2 H_1}{\partial p_k \partial x_j} \frac{\partial H}{\partial p_j} - \frac{\partial^2 H_1}{\partial p_k \partial p_j} \frac{\partial H}{\partial x_j} \right) = \sum_{j=1}^N \left( \frac{\partial H_1}{\partial p_j} \frac{\partial^2 H}{\partial p_k \partial x_j} - \frac{\partial H_1}{\partial x_j} \frac{\partial^2 H}{\partial p_k \partial p_j} \right),$$

which is obtained from

$$\{H_1, H\} = \sum_{j=1}^N \left( \frac{\partial H_1}{\partial x_j} \frac{\partial H}{\partial p_j} - \frac{\partial H_1}{\partial p_j} \frac{\partial H}{\partial x_j} \right) = 0$$

by differentiation with respect to  $p_k$ . Thus, we find:

$$D_{v_1} V_i^{(2)} = \sum_{k=1}^N \frac{\partial^2 H_2}{\partial p_i \partial x_k} \frac{\partial H_1}{\partial p_k} + \sum_{k=1}^N \sum_{m=1}^N \frac{\partial^2 H_2}{\partial p_i \partial p_m} \frac{\partial^2 L}{\partial \dot{x}_m \partial x_k} \frac{\partial H_1}{\partial p_k} + \sum_{k=1}^N \sum_{m=1}^N \frac{\partial^2 H_2}{\partial p_i \partial p_m} \frac{\partial^2 L}{\partial \dot{x}_m \partial \dot{x}_k} \sum_{j=1}^N \left( \frac{\partial H_1}{\partial p_j} \frac{\partial^2 H}{\partial p_k \partial x_j} - \frac{\partial H_1}{\partial x_j} \frac{\partial^2 H}{\partial p_k \partial p_j} \right) - \sum_{k=1}^N \sum_{m=1}^N \sum_{j=1}^N \frac{\partial^2 H_2}{\partial p_i \partial p_m} \frac{\partial^2 L}{\partial \dot{x}_m \partial \dot{x}_k} \frac{\partial^2 H_1}{\partial p_k \partial p_j} \mathcal{E}_j.$$

In the second line of the above formula we use the following identities:

$$\sum_{k=1}^N \frac{\partial^2 L}{\partial \dot{x}_m \partial \dot{x}_k} \frac{\partial^2 H}{\partial p_k \partial x_j} = - \frac{\partial^2 L}{\partial \dot{x}_m \partial x_j}, \tag{4.8}$$

and

$$\sum_{k=1}^N \frac{\partial^2 L}{\partial \dot{x}_m \partial \dot{x}_k} \frac{\partial^2 H}{\partial p_k \partial p_j} = \delta_{mj}. \tag{4.9}$$

Both are easily derived by differentiating the identity

$$\frac{\partial L}{\partial \dot{x}_m} \left( x, \frac{\partial H(x, p)}{\partial p} \right) = p_m$$

with respect to  $x_j$  and to  $p_j$ . Using (4.8) and (4.9), we finally obtain:

$$\begin{aligned} D_{v_1} V_i^{(2)} &= \sum_{k=1}^N \frac{\partial^2 H_2}{\partial p_i \partial x_k} \frac{\partial H_1}{\partial p_k} - \sum_{k=1}^N \frac{\partial^2 H_2}{\partial p_i \partial p_k} \frac{\partial H_1}{\partial x_k} \\ &\quad - \sum_{k=1}^N \sum_{m=1}^N \sum_{j=1}^N \frac{\partial^2 H_2}{\partial p_i \partial p_m} \frac{\partial^2 L}{\partial \dot{x}_m \partial \dot{x}_k} \frac{\partial^2 H_1}{\partial p_k \partial p_j} \mathcal{E}_j. \end{aligned} \tag{4.10}$$

By interchanging the roles of indices 1 and 2, and by subtracting the resulting formula from the previous one, we come to the final result:

$$D_{v_1} V_i^{(2)} - D_{v_2} V_i^{(1)} = \frac{\partial}{\partial p_i} \{H_1, H_2\} + \sum_{j=1}^N r_{ij} \mathcal{E}_j,$$

with  $r_{ij}$  as given in (4.4). This proves the theorem. □

### 5. From variational symmetries to a multi-time 1-form

Now we would like to promote an alternative point of view. In the preceding presentation, the property of a generalized vector field  $v_1$  to be a variational symmetry was mainly understood as an algebraic property of an operator  $D_{v_1}$  acting on differential polynomials (functions of  $x, \dot{x}, \ddot{x}$  etc.), where  $x$  was interpreted as a function of a single time  $t$ . However, this interpretation does not incorporate the view of a symmetry as a commuting flow. This is the point of view we would like to adopt now.

Suppose that a Lagrangian problem (2.1) admits  $m$  pairwise commuting variational symmetries  $v_1, \dots, v_m$ . We interpret them as  $m$  flows

$$(x_i)_{t_k} = V_i^{(k)}(x, \dot{x}), \quad i = 1, \dots, N, \quad k = 1, \dots, m. \tag{5.1}$$

These flows commute when restricted to solutions of the variational problem (2.1) and therefore govern the evolution of the fields  $x$  which can be now interpreted as functions of  $m + 1$  independent variables,  $x = x(t, t_1, \dots, t_m)$ . Moreover, we consider derivation operators acting on differential polynomials which are now functions of  $x$  and mixed derivatives of  $x$  of all orders with respect to  $t$

and to all  $t_1, \dots, t_m$  (derivatives with respect to  $t$  being still denoted by dots). We will use the standard multi-index notation for such derivatives, with multi-indices  $I = (i_0, i_1, \dots, i_m) \in (\mathbb{Z}_{\geq 0})^{m+1}$ . The operators of full derivatives  $D_t$  and  $D_{t_k}$  are defined as usual:

$$D_t = \sum_{i=1}^N \sum_I (x_i)_{I+e_0} \frac{\partial}{\partial (x_i)_I} = \sum_{i=1}^N \left( \dot{x}_i \frac{\partial}{\partial x_i} + \ddot{x}_i \frac{\partial}{\partial \dot{x}_i} + \sum_{\ell=1}^m (\dot{x}_i)_{t_\ell} \frac{\partial}{\partial (x_i)_{t_\ell}} + \dots \right),$$

$$D_{t_k} = \sum_{i=1}^N \sum_I (x_i)_{I+e_k} \frac{\partial}{\partial (x_i)_I} = \sum_{i=1}^N \left( (x_i)_{t_k} \frac{\partial}{\partial x_i} + (\dot{x}_i)_{t_k} \frac{\partial}{\partial \dot{x}_i} + \sum_{\ell=1}^m (x_i)_{t_\ell t_k} \frac{\partial}{\partial (x_i)_{t_\ell}} + \dots \right).$$

Operators  $D_{t_k}$  come to replace  $D_{v_k}$ . In particular, when acting on a function depending on  $x, \dot{x}$  only, like  $L(x, \dot{x})$  or  $F_\ell(x, \dot{x})$ , we have:

$$D_{t_k} - D_{v_k} = \sum_{i=1}^N ((x_i)_{t_k} - V_i^{(k)}(x, \dot{x})) \frac{\partial}{\partial x_i} + \sum_{i=1}^N D_t ((x_i)_{t_k} - V_i^{(k)}(x, \dot{x})) \frac{\partial}{\partial \dot{x}_i}, \tag{5.2}$$

so that results coincide on solutions of the respective flows (5.1).

Upon the replacement of  $D_{v_k}$  by  $D_{t_k}$ , the defining formula (4.1) of variational symmetries would read

$$D_{t_k} L(x, \dot{x}) - D_t F_k(x, \dot{x}) = 0,$$

while the formulas similar to (4.6) would read

$$D_{t_k} F_\ell(x, \dot{x}) - D_{t_\ell} F_k(x, \dot{x}) = c_{k\ell}.$$

An important insight is that the latter relations are formally nothing but the coefficients of  $d\mathcal{L}$  for the 1-form

$$\mathcal{L} = L(x, \dot{x}) dt + F_1(x, \dot{x}) dt_1 + \dots + F_m(x, \dot{x}) dt_m. \tag{5.3}$$

Of course, all these relations are satisfied not as algebraic identities, but only on solutions of a consistent system consisting of differential equations (2.2) and (5.1).

### 6. One-dimensional pluri-Lagrangian systems

In [19], a theory of the so called *pluri-Lagrangian problems* which consist in finding critical points for action functionals associated with such forms, has been developed. Here are the main positions of this theory. In doing this, we do not single out the time  $t$  playing a special role. Thus, in the present section we consider functions on the  $m$ -dimensional time  $(t_1, \dots, t_m) \in \mathbb{R}^m$ .

**Definition 6.1. (Pluri-Lagrangian problem)** Consider a 1-form on  $\mathbb{R}^m$  given by

$$\mathcal{L} = \mathcal{L}[x, x_{t_1}, \dots, x_{t_m}] = \sum_{k=1}^m L_k(x, x_{t_1}, \dots, x_{t_m}) dt_k, \tag{6.1}$$

depending on (the first jet of) a function  $x : \mathbb{R}^m \rightarrow \mathcal{X}$ . For any smooth curve  $\Gamma : [0, 1] \rightarrow \mathbb{R}^m$ , define the action functional

$$S_\Gamma = \int_\Gamma \mathcal{L}. \tag{6.2}$$

A *pluri-Lagrangian problem* for the 1-form  $\mathcal{L}$  consists in finding functions  $x : \mathbb{R}^m \rightarrow \mathcal{X}$  which deliver critical point of  $S_\Gamma$  for any  $\Gamma$  (for fixed values of  $x$  at the endpoints of  $\Gamma$ ).

**Theorem 6.1. (Multi-time Euler-Lagrange equations)** A function  $x : \mathbb{R} \rightarrow \mathcal{X}$  solves the pluri-Lagrangian problem for the 1-form  $\mathcal{L}$ , if and only if it satisfies the following differential equations (6.3)–(6.5), called the multi-time Euler-Lagrange equations:

$$\frac{\partial L_k}{\partial (x_i)_{t_\ell}} = 0, \quad \forall i = 1, \dots, N, \quad \forall k, \ell \in \{1, \dots, m\} \text{ with } k \neq \ell, \quad (6.3)$$

$$\frac{\partial L_1}{\partial (x_i)_{t_1}} = \dots = \frac{\partial L_m}{\partial (x_i)_{t_m}}, \quad \forall i = 1, \dots, N, \quad (6.4)$$

$$D_{t_k} p_i = \frac{\partial L_k}{\partial x_i} \quad \forall i = 1, \dots, N, \quad \forall k = 1, \dots, m, \quad (6.5)$$

where  $p_i : \mathbb{R}^m \rightarrow \mathbb{R}$  is the common value of the functions in (6.4).

**Theorem 6.2. (Almost closedness of the pluri-Lagrangian form)** On solutions of multi-time Euler-Lagrange equations (6.3)–(6.5), we have:

$$D_{t_k} L_\ell - D_{t_\ell} L_k = c_{k\ell} = \text{const.}$$

In particular, if all  $c_{k\ell} = 0$ , then the 1-form (6.1) is closed on solutions of multi-time Euler-Lagrange equations, so that the action functional  $S_\Gamma$  does not depend on the choice of the curve  $\Gamma$  connecting two given points in  $\mathbb{R}^m$ .

### 7. From variational symmetries to a pluri-Lagrangian problem

It is easy to check that for the 1-form (5.3) with  $m + 1$  times  $t, t_1, \dots, t_m$  the pluri-Lagrangian problem is inconsistent (for instance, equation (6.4) is not satisfied, since  $\partial F_k / \partial (x_i)_{t_k} = 0 \neq \partial L / \partial \dot{x}_i$ ). However, it turns out to be possible to modify this 1-form to

$$\mathcal{L} = L(x, \dot{x}) dt + L_1(x, \dot{x}, x_{t_1}) dt_1 + \dots + L_m(x, \dot{x}, x_{t_m}) dt_m, \quad (7.1)$$

for which the pluri-Lagrangian problem is well-posed, with multi-time Euler-Lagrange equations being exactly the desired ones. Moreover, on solutions of these equations, forms (5.3) and (7.1) coincide. To achieve this, the functions  $L_k(x, \dot{x}, x_{t_k})$  are obtained by replacing in formula (2.9) for the flux  $F_k$  the quantities  $V_i^{(k)}(x, \dot{x})$  by  $(x_i)_{t_k}$ .

**Theorem 7.1.** Let  $v_1, \dots, v_m$  be commuting variational symmetries for the Lagrange function  $L(x, \dot{x})$ , with the fluxes  $F_1(x, \dot{x}), \dots, F_m(x, \dot{x})$ , and with the Noether integrals  $J_1(x, \dot{x}), \dots, J_m(x, \dot{x})$ . Define for  $k = 1, \dots, m$ :

$$L_k(x, \dot{x}, x_{t_k}) = \sum_{i=1}^N \frac{\partial L(x, \dot{x})}{\partial \dot{x}_i} (x_i)_{t_k} - J_k(x, \dot{x}) \quad (7.2)$$

$$= \sum_{i=1}^N \frac{\partial L(x, \dot{x})}{\partial \dot{x}_i} ((x_i)_{t_k} - V_i^{(k)}(x, \dot{x})) + F_k(x, \dot{x}), \quad (7.3)$$

so that  $L_k(x, \dot{x}, x_{t_k})$  coincides with  $F_k(x, \dot{x})$  on solutions of differential equations (5.1). Then for the 1-form (7.1), the multi-time Euler-Lagrange equations are equivalent to a consistent system of the standard Euler-Lagrange equations  $\mathcal{E}_i = 0$  coupled with the evolution equations (5.1).

**Proof.** The multi-time Euler-Lagrange equations for the 1-form  $\mathcal{L}$  consist of:

- Equations (6.3), namely

$$\frac{\partial L}{\partial(x_i)_{t_k}} = 0, \quad \frac{\partial L_k}{\partial \dot{x}_i} = 0, \quad \frac{\partial L_k}{\partial(x_i)_{t_\ell}} = 0, \quad i = 1, \dots, N, \quad k \neq \ell \in \{1, \dots, m\}. \quad (7.4)$$

The first and the third ones are trivially satisfied, while the second is equivalent to (5.1). Indeed, from definition (7.3) of  $L_k$ , we compute the second equation in (7.4) to be :

$$\frac{\partial L_k}{\partial \dot{x}_i} = \sum_{j=1}^N \frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} \left( (x_j)_{t_k} - V_j^{(k)} \right) - \sum_{j=1}^N \frac{\partial L}{\partial \dot{x}_j} \frac{\partial V_j^{(k)}}{\partial \dot{x}_i} + \frac{\partial F_k}{\partial \dot{x}_i} = 0,$$

with  $i = 1, \dots, N$ . According to Lemma 3.1, this results in

$$\frac{\partial L_k}{\partial \dot{x}_i} = \sum_{j=1}^N \frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} \left( (x_j)_{t_k} - V_j^{(k)} \right) = 0, \quad i = 1, \dots, N.$$

Due to the non-degeneracy condition (2.3), this is equivalent to equations (5.1).

- Equations (6.4) read:

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L_1}{\partial(x_i)_{t_1}} = \dots = \frac{\partial L_m}{\partial(x_i)_{t_m}}, \quad i = 1, \dots, N,$$

and are automatically satisfied by construction.

- Equations (6.5) consist of the standard Euler-Lagrange equations  $\mathcal{E}_i = 0$  given in (2.2), and of the Euler-Lagrange equations associated with  $L_k$ :

$$\mathcal{E}_i^{(k)} = \frac{\partial L_k(x, \dot{x}, x_{t_k})}{\partial x_i} - D_{t_k} p_i(x, \dot{x}) = 0, \quad (7.5)$$

with  $i = 1, \dots, N, k = 1, \dots, m$ . The latter will be shown to be differential consequences of  $\mathcal{E}_i = 0$  and equations (5.1), see Proposition 7.1 below.

This finishes the proof of Theorem 7.1. □

**Remark.** It is a highly nontrivial feature of the pluri-Lagrangian theory that the multi-time Euler-Lagrange equations include the evolutionary first order differential equations (5.1).

**Proposition 7.1.** For  $L_k(x, \dot{x}, x_{t_k})$  defined as in (7.3), Euler-Lagrange equations (7.5) are differential consequences of  $\mathcal{E}_i = 0$  and equations (5.1).

**Proof.** Both parts of equation  $\mathcal{E}_i^{(k)} = 0$ , that is, of equation  $D_{t_k} p_i = \partial L_k / \partial x_i$ , written in length, read:

$$D_{t_k} p_i = \sum_{j=1}^N \frac{\partial^2 L}{\partial \dot{x}_i \partial x_j} (x_j)_{t_k} + \sum_{j=1}^N \frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} (\dot{x}_j)_{t_k}, \quad (7.6)$$

and

$$\frac{\partial L_k}{\partial x_i} = \sum_{j=1}^N \frac{\partial^2 L}{\partial x_i \partial \dot{x}_j} \left( (x_j)_{t_k} - V_j^{(k)} \right) - \sum_{j=1}^N \frac{\partial L}{\partial \dot{x}_j} \frac{\partial V_j^{(k)}}{\partial x_i} + \frac{\partial F_k}{\partial x_i}. \quad (7.7)$$



On solutions of  $(x_j)_{t_k} = V_j^{(k)}$  they reduce respectively to

$$D_{t_k} p_i = \sum_{j=1}^N \frac{\partial^2 L}{\partial \dot{x}_i \partial x_j} V_j^{(k)} + \sum_{j=1}^N \frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} (D_t V_j^{(k)}), \tag{7.8}$$

and

$$\frac{\partial L_k}{\partial x_i} = - \sum_{j=1}^N \frac{\partial L}{\partial \dot{x}_j} \frac{\partial V_j^{(k)}}{\partial x_i} + \frac{\partial F_k}{\partial x_i}. \tag{7.9}$$

We transform expression (7.8) as follows. First, we integrate by parts and use equation (4.1) for variational symmetries:

$$\begin{aligned} D_{t_k} p_i &= \frac{\partial}{\partial \dot{x}_i} \left( \sum_{j=1}^N \frac{\partial L}{\partial x_j} V_j^{(k)} + \sum_{j=1}^N \frac{\partial L}{\partial \dot{x}_j} (D_t V_j^{(k)}) \right) \\ &\quad - \sum_{j=1}^N \frac{\partial L}{\partial x_j} \frac{\partial V_j^{(k)}}{\partial \dot{x}_i} - \sum_{j=1}^N \frac{\partial L}{\partial \dot{x}_j} \frac{\partial}{\partial \dot{x}_i} (D_t V_j^{(k)}) \\ &= \frac{\partial}{\partial \dot{x}_i} (D_t F_k) - \sum_{j=1}^N \frac{\partial L}{\partial x_j} \frac{\partial V_j^{(k)}}{\partial \dot{x}_i} - \sum_{j=1}^N \frac{\partial L}{\partial \dot{x}_j} \frac{\partial}{\partial \dot{x}_i} (D_t V_j^{(k)}). \end{aligned}$$

In the first sum of the last line, we make use of Euler-Lagrange equations (2.2), to get:

$$D_{t_k} p_i = \frac{\partial}{\partial \dot{x}_i} (D_t F_k) - \sum_{j=1}^N \mathcal{E}_i \frac{\partial V_j^{(k)}}{\partial \dot{x}_i} - \sum_{j=1}^N D_t \left( \frac{\partial L}{\partial \dot{x}_j} \right) \frac{\partial V_j^{(k)}}{\partial \dot{x}_i} - \sum_{j=1}^N \frac{\partial L}{\partial \dot{x}_j} \frac{\partial}{\partial \dot{x}_i} (D_t V_j^{(k)}).$$

Integrating by parts once again, we get:

$$\begin{aligned} D_{t_k} p_i &= \frac{\partial}{\partial \dot{x}_i} (D_t F_k) - \sum_{j=1}^N \mathcal{E}_i \frac{\partial V_j^{(k)}}{\partial \dot{x}_i} - D_t \left( \sum_{j=1}^N \frac{\partial L}{\partial \dot{x}_j} \frac{\partial V_j^{(k)}}{\partial \dot{x}_i} \right) \\ &\quad + \sum_{j=1}^N \frac{\partial L}{\partial \dot{x}_j} \left( D_t \left( \frac{\partial V_j^{(k)}}{\partial \dot{x}_i} \right) - \frac{\partial}{\partial \dot{x}_i} (D_t V_j^{(k)}) \right). \end{aligned} \tag{7.10}$$

But for any function  $f(x, \dot{x})$ , we have the following identity:

$$\begin{aligned} &D_t \left( \frac{\partial f}{\partial \dot{x}_i} \right) - \frac{\partial}{\partial \dot{x}_i} (D_t f) \\ &= \sum_{j=1}^N \frac{\partial^2 f}{\partial \dot{x}_i \partial x_j} \dot{x}_j + \sum_{j=1}^N \frac{\partial^2 f}{\partial \dot{x}_i \partial \dot{x}_j} \ddot{x}_j - \frac{\partial}{\partial \dot{x}_i} \left( \sum_{j=1}^N \frac{\partial f}{\partial x_j} \dot{x}_j + \sum_{j=1}^N \frac{\partial f}{\partial \dot{x}_j} \ddot{x}_j \right) \\ &= - \frac{\partial f}{\partial x_i}. \end{aligned}$$

Using this identity twice (for  $F_k$  and for  $V_j^{(k)}$ ), we put (7.10) in the form

$$D_{t_k} p_i = - \sum_{j=1}^N \mathcal{E}_i \frac{\partial V_j^{(k)}}{\partial \dot{x}_i} + D_t \left( \frac{\partial F_k}{\partial \dot{x}_i} - \sum_{j=1}^N \frac{\partial L}{\partial \dot{x}_j} \frac{\partial V_j^{(k)}}{\partial \dot{x}_i} \right) + \frac{\partial F_k}{\partial x_i} - \sum_{j=1}^N \frac{\partial L}{\partial \dot{x}_j} \frac{\partial V_j^{(k)}}{\partial x_i}.$$

According to Lemma 3.1, this equals to

$$D_{t_k} p_i = - \sum_{j=1}^N \mathcal{E}_i \frac{\partial V_j^{(k)}}{\partial \dot{x}_i} + \frac{\partial F_k}{\partial x_i} - \sum_{j=1}^N \frac{\partial L}{\partial \dot{x}_j} \frac{\partial V_j^{(k)}}{\partial x_i},$$

which on solutions of  $\mathcal{E}_i = 0$  coincides with the right-hand side of (7.9). □

### 8. Almost closedness of the pluri-Lagrangian 1-form

As a corollary of Theorem 6.2, we can conclude that

$$D_{t_k} L(x, \dot{x}) - D_t L_k(x, \dot{x}, x_{t_k}) = 0, \tag{8.1}$$

$$D_{t_k} L_\ell(x, \dot{x}, x_{t_\ell}) - D_{t_\ell} L_k(x, \dot{x}, x_{t_k}) = c_{k\ell}, \tag{8.2}$$

on solutions of the Euler-Lagrange equations  $\mathcal{E}_i = 0$  and of the evolution equations of variational symmetries  $(x_i)_{t_k} = V_i^{(k)}$ . However, a more detailed information is available.

**Proposition 8.1.** *The following identities hold:*

$$D_{t_k} L(x, \dot{x}) - D_t L_k(x, \dot{x}, x_{t_k}) = \sum_{i=1}^N ((x_i)_{t_k} - V_i^{(k)}) \mathcal{E}_i, \tag{8.3}$$

$$\begin{aligned} D_{t_k} L_\ell(x, \dot{x}, x_{t_\ell}) - D_{t_\ell} L_k(x, \dot{x}, x_{t_k}) &= c_{k\ell} + \sum_{i=1}^N \left( ((x_i)_{t_\ell} - V_i^{(\ell)}) \mathcal{E}_i^{(k)} - ((x_i)_{t_k} - V_i^{(k)}) \mathcal{E}_i^{(\ell)} \right) \\ &+ \sum_{i=1}^N \sum_{j=1}^N \left( \frac{\partial^2 L}{\partial x_i \partial \dot{x}_j} - \frac{\partial^2 L}{\partial x_j \partial \dot{x}_i} \right) ((x_i)_{t_\ell} - V_i^{(\ell)}) ((x_j)_{t_k} - V_j^{(k)}), \end{aligned} \tag{8.4}$$

where  $c_{k\ell}$  are constants analogous to those from Proposition 4.2. In particular, equations (8.1), (8.2) are satisfied as soon as any  $m$  out of  $m + 1$  systems  $\mathcal{E}_i = 0$  and  $(x_i)_{t_k} = V_i^{(k)}$  ( $k = 1, \dots, m$ ) are satisfied.

**Proof.** For the proof of equation (8.3), we compute:

$$D_{t_k} L(x, \dot{x}) - D_{v_k} L(x, \dot{x}) = \sum_{i=1}^N ((x_i)_{t_k} - V_i^{(k)}) \frac{\partial L}{\partial x_i} + \sum_{i=1}^N D_t ((x_i)_{t_k} - V_i^{(k)}) \frac{\partial L}{\partial \dot{x}_i},$$

and

$$D_t L_k(x, \dot{x}, x_{t_k}) = \sum_{i=1}^N D_t ((x_i)_{t_k} - V_i^{(k)}) \frac{\partial L}{\partial \dot{x}_i} + \sum_{i=1}^N ((x_i)_{t_k} - V_i^{(k)}) D_t \left( \frac{\partial L}{\partial \dot{x}_i} \right) + D_t F_k.$$

Taking the difference and using (2.2), we find:

$$D_{t_k} L(x, \dot{x}) - D_t L_k(x, \dot{x}, x_{t_k}) = D_{v_k} L - D_t F_k + \sum_{i=1}^N ((x_i)_{t_k} - V_i^{(k)}) \mathcal{E}_i.$$

Taking into account equation (4.1), we arrive at (8.3).

For the proof of equation (8.4), we compute:

$$\begin{aligned} D_{t_k} L_\ell(x, \dot{x}, x_{t_\ell}) &= \sum_{i=1}^N (D_{t_k} p_i) ((x_i)_{t_\ell} - V_i^{(\ell)}) + \sum_{i=1}^N p_i (x_i)_{t_k t_\ell} - \sum_{i=1}^N p_i (D_{t_k} V_i^{(\ell)}) + D_{t_k} F_\ell \\ &= \sum_{i=1}^N \left( -\mathcal{E}_i^{(k)} + \frac{\partial L_k}{\partial x_i} \right) ((x_i)_{t_\ell} - V_i^{(\ell)}) \\ &\quad + \sum_{i=1}^N p_i (x_i)_{t_k t_\ell} - \sum_{i=1}^N p_i (D_{t_k} V_i^{(\ell)}) + D_{t_k} F_\ell. \end{aligned}$$

Here

$$D_{t_k} F_\ell = D_{v_k} F_\ell + \sum_{i=1}^N ((x_i)_{t_k} - V_i^{(k)}) \frac{\partial F_\ell}{\partial x_i} + \sum_{i=1}^N D_t ((x_i)_{t_k} - V_i^{(k)}) \frac{\partial F_\ell}{\partial \dot{x}_i}.$$

Thus, we find:

$$\begin{aligned} D_{t_k} L_\ell(x, \dot{x}, x_{t_\ell}) - D_\ell L_k(x, \dot{x}, x_{t_k}) &= D_{v_k} F_\ell - D_{v_\ell} F_k \\ &\quad + \sum_{i=1}^N \left( ((x_i)_{t_k} - V_i^{(k)}) \mathcal{E}_i^{(\ell)} - ((x_i)_{t_\ell} - V_i^{(\ell)}) \mathcal{E}_i^{(k)} \right) \\ &\quad + \sum_{i=1}^N \frac{\partial L_k}{\partial x_i} ((x_i)_{t_\ell} - V_i^{(\ell)}) - \sum_{i=1}^N ((x_i)_{t_\ell} - V_i^{(\ell)}) \frac{\partial F_k}{\partial x_i} - \sum_{i=1}^N D_t ((x_i)_{t_\ell} - V_i^{(\ell)}) \frac{\partial F_k}{\partial \dot{x}_i} \\ &\quad - \sum_{i=1}^N \frac{\partial L_\ell}{\partial x_i} ((x_i)_{t_k} - V_i^{(k)}) + \sum_{i=1}^N ((x_i)_{t_k} - V_i^{(k)}) \frac{\partial F_\ell}{\partial x_i} + \sum_{i=1}^N D_t ((x_i)_{t_k} - V_i^{(k)}) \frac{\partial F_\ell}{\partial \dot{x}_i} \\ &\quad + \sum_{i=1}^N p_i (D_{t_\ell} V_i^{(k)} - D_{t_k} V_i^{(\ell)}). \end{aligned} \tag{8.5}$$

Here the last line (with the index  $i$  replaced by  $j$ ) is transformed as follows:

$$\begin{aligned} \sum_{j=1}^N p_j (D_{t_\ell} V_j^{(k)} - D_{t_k} V_j^{(\ell)}) &= \sum_{j=1}^N p_j (D_{v_\ell} V_j^{(k)} - D_{v_k} V_j^{(\ell)}) \\ &\quad + \sum_{j=1}^N p_j \sum_{i=1}^N ((x_i)_{t_\ell} - V_i^{(\ell)}) \frac{\partial V_j^{(k)}}{\partial x_i} + \sum_{j=1}^N p_j \sum_{i=1}^N D_t ((x_i)_{t_\ell} - V_i^{(\ell)}) \frac{\partial V_j^{(k)}}{\partial \dot{x}_i} \\ &\quad - \sum_{j=1}^N p_j \sum_{i=1}^N ((x_i)_{t_k} - V_i^{(k)}) \frac{\partial V_j^{(\ell)}}{\partial x_i} - \sum_{j=1}^N p_j \sum_{i=1}^N D_t ((x_i)_{t_k} - V_i^{(k)}) \frac{\partial V_j^{(\ell)}}{\partial \dot{x}_i}. \end{aligned}$$

Since the symmetries  $v_k$  and  $v_\ell$  commute, we have from Proposition 4.2:

$$D_{v_k} F_\ell - D_{v_\ell} F_k = c_{k\ell} + \sum_{j=1}^N p_j (D_{v_k} V_j^{(\ell)} - D_{v_\ell} V_j^{(k)}).$$

Collecting everything together and using Lemma 3.1, we see that in (8.5) all terms with  $D_t((x_i)_{t_k} - V_i^{(k)})$  cancel away, and we obtain:

$$\begin{aligned} D_{t_k}L_\ell(x, \dot{x}, x_{t_\ell}) - D_\ell L_k(x, \dot{x}, x_{t_k}) &= c_{k\ell} + \sum_{i=1}^N \left( ((x_i)_{t_k} - V_i^{(k)}) \mathcal{E}_i^{(\ell)} - ((x_i)_{t_\ell} - V_i^{(\ell)}) \mathcal{E}_i^{(k)} \right) \\ &+ \sum_{i=1}^N \frac{\partial L_k}{\partial x_i} ((x_i)_{t_\ell} - V_i^{(\ell)}) - \sum_{i=1}^N ((x_i)_{t_\ell} - V_i^{(\ell)}) \frac{\partial F_k}{\partial x_i} + \sum_{j=1}^N p_j \sum_{i=1}^N ((x_i)_{t_\ell} - V_i^{(\ell)}) \frac{\partial V_j^{(k)}}{\partial x_i} \\ &- \sum_{i=1}^N \frac{\partial L_\ell}{\partial x_i} ((x_i)_{t_k} - V_i^{(k)}) + \sum_{i=1}^N ((x_i)_{t_k} - V_i^{(k)}) \frac{\partial F_\ell}{\partial x_i} - \sum_{j=1}^N p_j \sum_{i=1}^N ((x_i)_{t_k} - V_i^{(k)}) \frac{\partial V_j^{(\ell)}}{\partial x_i}. \end{aligned}$$

It remains to substitute expressions for  $\partial L_k/\partial x_i$  from (7.7). □

### 9. Pluri-Lagrangian problems in the phase space

We recall that motions of mechanical systems can be described as extremals of a more general variational principle that the Hamilton’s principle of the least action, namely that they are critical points of the following *action functional in the phase space*:

$$S[x, p] = \int_{t_0}^{t_1} \Lambda(x(t), p(t), \dot{x}(t)) dt \tag{9.1}$$

where  $(x, p) : [t_0, t_1] \rightarrow T^*\mathcal{X}$  is an arbitrary curve in the phase space with the fixed values of  $x(t_0)$  and  $x(t_1)$ , and the “Lagrange function”  $\Lambda : T(T^*\mathcal{X}) \rightarrow \mathbb{R}$  is given by

$$\Lambda(x, p, \dot{x}) = \sum_{i=1}^N p_i \dot{x}_i - H(x, p). \tag{9.2}$$

Note that this “Lagrange function” is degenerate in two ways. First, it depends on the “velocities”  $(\dot{x}, \dot{p})$  linearly. Second, it actually does not depend on one half of velocities, namely on  $\dot{p}$ . This last feature forces us to consider admissible variations fixing  $x(t_0)$  and  $x(t_1)$ , but not  $p(t_0)$  and  $p(t_1)$ . However, Euler-Lagrange equations for the “Lagrange function”  $\Lambda(x, p, \dot{x})$  are computed in the standard way, using

$$\frac{\partial \Lambda}{\partial x_i} = -\frac{\partial H}{\partial x_i}, \quad \frac{\partial \Lambda}{\partial p_i} = \dot{x}_i - \frac{\partial H}{\partial p_i}, \quad \frac{\partial \Lambda}{\partial \dot{x}_i} = p_i, \quad \frac{\partial \Lambda}{\partial \dot{p}_i} = 0,$$

and read:

$$-\frac{\partial H}{\partial x_i} - D_t p_i = 0, \quad \dot{x}_i - \frac{\partial H}{\partial p_i} = 0, \tag{9.3}$$

which coincides with the Hamiltonian equation of motion (3.2). An important insight is that the usual Lagrange function  $L(x, \dot{x})$  is the critical value of  $\Lambda(x, p, \dot{x})$  with respect to  $p$ , which is achieved at  $p$  given by the Legendre transformation  $p_i = \partial L(x, \dot{x})/\partial \dot{x}_i$ . This explains why the critical curves of the action (2.1) in the configuration space  $\mathcal{X}$ , in the class of variations fixing  $x(t_0)$  and  $x(t_1)$ , are obtained by substitution  $p_i = \partial L(x, \dot{x})/\partial \dot{x}_i$  from the critical curves of the action (9.1) in the phase space  $T^*\mathcal{X}$ , in the class of variations fixing  $x(t_0)$  and  $x(t_1)$ , but not  $p(t_0)$  and  $p(t_1)$ .

We now observe that the coefficients  $L_k(x, \dot{x}, x_{t_k})$  given by equation (7.2) are, in a similar way, the values of the functions

$$\Lambda_k(x, p, x_{t_k}) = \sum_{i=1}^N p_i(x_i)_{t_k} - H_k(x, p), \tag{9.4}$$

evaluated at the point  $p$  given by the Legendre transformation  $p_i = \partial L(x, \dot{x}) / \partial x_i$ . Here  $H_k : T^* \mathcal{X} \rightarrow \mathbb{R}$  are the Hamilton functions related, via the Legendre transformation, to the Noether integrals  $J_k : T \mathcal{X} \rightarrow \mathbb{R}$  of the commuting variational symmetries  $v_k$ , as in Section 3. Comparing formulas (9.2) and (9.4), we see that the time  $t$  associated with the Lagrange function  $L$  is on absolutely the same footing as the times  $t_k$  associated with the commuting variational symmetries.

This leads us naturally to consider the following *pluri-Lagrangian problem in the phase space* (which we formulate without choosing one time playing a special role, like we did in Section 6). Consider the 1-form

$$\mathcal{L} = \Lambda_1(x, p, x_{t_1}) dt_1 + \dots + \Lambda_m(x, p, x_{t_m}) dt_m, \tag{9.5}$$

whose coefficients depend on (the first jet of) a function  $(x, p) : \mathbb{R}^m \rightarrow T^* \mathcal{X}$  according to formulas (9.4). Find functions  $(x, p) : \mathbb{R}^m \rightarrow T^* \mathcal{X}$  which deliver critical points to functionals

$$S_\Gamma = \int_\Gamma \mathcal{L}$$

for any curve  $\Gamma : [0, 1] \rightarrow \mathbb{R}^m$  in the multi-dimensional space (for fixed values of  $x$  at the endpoints of  $\Gamma$ ).

**Theorem 9.1. (Multi-time Euler-Lagrange equations in the phase space)** *A function  $(x, p) : \mathbb{R}^m \rightarrow T^* \mathcal{X}$  solves the pluri-Lagrangian problem for the 1-form (9.5) with coefficients (9.4), where  $H_1, \dots, H_m : T^* \mathcal{X} \rightarrow \mathbb{R}$  are some functions, if and only if it satisfies Hamiltonian equations for the Hamilton functions  $H_1, \dots, H_m$ :*

$$(x_i)_{t_k} = \frac{\partial H_k}{\partial p_i}, \quad (p_i)_{t_k} = -\frac{\partial H_k}{\partial x_i}, \quad k = 1, \dots, m. \tag{9.6}$$

*This system is compatible if and only if these  $m$  Hamiltonian flows pairwise commute.*

**Proof.** One easily computes the multi-time Euler-Lagrange equations for the 1-form (9.5) with coefficients (9.4):

- equations (6.3) are satisfied trivially by construction;
- equations (6.4) are also satisfied by construction, since  $\partial \Lambda_k / \partial (x_i)_{t_k} = p_i$  for all  $k$ ;
- equations (6.5) read:  $D_{t_k} p_i = -\partial H_k / \partial x_i$  and  $0 = (x_i)_{t_k} - \partial H_k / \partial x_i$ . These are derived literally as equations (9.3).

The last statement of the theorem is obvious. □

Thus, we established the property of joint solutions of a system of pairwise commuting Hamiltonian flows to be critical for the action functionals in the phase space defined along any curve in the multi-dimensional time.

## 10. Examples

We illustrate the results of the present paper with a couple of well-known examples.

### 10.1. Kepler system

We start with the Kepler system, an example with a “hidden symmetry” described by the so called Laplace-Runge-Lenz vector, a non-obvious integral whose presence ensures that the system is super-integrable, or integrable in the non-commutative sense, with one-dimensional invariant tori (which in this case are just periodic orbits foliating the phase space).

- The Lagrange function of the Kepler system (with unit mass) reads:

$$L(x, \dot{x}) = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{\alpha}{\|x\|}, \quad (10.1)$$

where  $\alpha > 0$  is a constant and  $\|x\| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ . The corresponding Euler-Lagrange equations are

$$\mathcal{E}_i = \frac{\alpha x_i}{\|x\|^3} + \ddot{x}_i = 0, \quad i = 1, 2, 3. \quad (10.2)$$

They admit the energy integral

$$J(x, \dot{x}) = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) + \frac{\alpha}{\|x\|}, \quad (10.3)$$

as well as three angular momenta integrals  $M_i = x_j \dot{x}_k - x_k \dot{x}_j$ . The latter can be considered as Noether integrals due to the point symmetries  $x_j \partial / \partial x_k - x_k \partial / \partial x_j$ .

- The Kepler system admits a variational symmetry:

$$v_1 = V_1^{(1)}(x, \dot{x}) \frac{\partial}{\partial x_1} + V_2^{(1)}(x, \dot{x}) \frac{\partial}{\partial x_2} + V_3^{(1)}(x, \dot{x}) \frac{\partial}{\partial x_3}, \quad (10.4)$$

with

$$\begin{aligned} V_1^{(1)}(x, \dot{x}) &= x_2 \dot{x}_2 + x_3 \dot{x}_3, \\ V_2^{(1)}(x, \dot{x}) &= x_2 \dot{x}_1 - 2x_1 \dot{x}_2, \\ V_3^{(1)}(x, \dot{x}) &= x_3 \dot{x}_1 - 2x_1 \dot{x}_3. \end{aligned}$$

Indeed, a straightforward computation confirms that

$$D_{v_1} L(x, \dot{x}) - D_t F_1(x, \dot{x}) = 0,$$

where the flux  $F_1$  is given by

$$F_1(x, \dot{x}) = \dot{x}_1(x_2 \dot{x}_2 + x_3 \dot{x}_3) - x_1(\dot{x}_2^2 + \dot{x}_3^2) - \frac{\alpha x_1}{\|x\|}.$$

The corresponding Noether integral is

$$J_1(x, \dot{x}) = \dot{x}_1(x_2 \dot{x}_2 + x_3 \dot{x}_3) - x_1(\dot{x}_2^2 + \dot{x}_3^2) + \frac{\alpha x_1}{\|x\|}. \quad (10.5)$$

- To the variational symmetry  $v_1$  given by (10.4) we can associate a flow with the time  $t_1$ :

$$\begin{aligned}(x_1)_{t_1} &= x_2\dot{x}_2 + x_3\dot{x}_3, \\(x_2)_{t_1} &= x_2\dot{x}_1 - 2x_1\dot{x}_2, \\(x_3)_{t_1} &= x_3\dot{x}_1 - 2x_1\dot{x}_3.\end{aligned}\tag{10.6}$$

Set

$$L_1(x, \dot{x}, x_{t_1}) = \sum_{i=1}^3 \dot{x}_i(x_i)_{t_1} - J_1(x, \dot{x}).$$

Then the Euler-Lagrange equations associated with  $L_1$  (see (7.5)) are computed to be

$$\begin{aligned}\mathcal{E}_1^{(1)} &= \dot{x}_2^2 + \dot{x}_3^2 - \frac{\alpha(x_2^2 + x_3^2)}{\|x\|^3} - (\dot{x}_1)_{t_1} = 0, \\ \mathcal{E}_2^{(1)} &= -\dot{x}_1\dot{x}_2 + \frac{\alpha x_1 x_2}{\|x\|^3} - (\dot{x}_2)_{t_1} = 0, \\ \mathcal{E}_3^{(1)} &= -\dot{x}_1\dot{x}_3 + \frac{\alpha x_1 x_3}{\|x\|^3} - (\dot{x}_3)_{t_1} = 0.\end{aligned}\tag{10.7}$$

As ensured by Proposition 7.1, equations (10.7) are differential consequences of (10.2) and (10.6). Indeed, differentiating (10.6) with respect to  $t$  and then substituting  $\ddot{x}_i = -\alpha x_i / \|x\|^3$  (coming from (10.2)) in the resulting expression, we recover (10.7).

- Actually, Kepler system possesses two further variational symmetries, say  $v_2$  and  $v_3$ , which are obtained by permuting coordinates  $x_1, x_2, x_3$  in (10.4). These permutations give also the fluxes and Noether integrals corresponding to symmetries  $v_2$  and  $v_3$ . One can consider the integrals  $J_1, J_2, J_3$  as the components of the famous Laplace-Runge-Lenz vector:

$$J(x, \dot{x}) = \dot{x} \times (x \times \dot{x}) + \frac{\alpha x}{\|x\|}.$$

However, symmetries  $v_1, v_2, v_3$  do not commute on solutions of Euler-Lagrange equations (10.2). For example, one finds:

$$\begin{aligned}D_{v_1} V_1^{(2)}(x, \dot{x}) - D_{v_2} V_1^{(1)}(x, \dot{x}) &= -\ddot{x}_2(2x_1^2 - 2x_2^2 - x_3^2) - x_2 x_3 \ddot{x}_3 \\ &\quad - x_2(3\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) + 2x_1 \dot{x}_1 \dot{x}_2,\end{aligned}$$

which does not vanish modulo  $\mathcal{E}_i = 0$ . On the Hamiltonian side (that is, upon setting  $\dot{x}_i = p_i$ ), the Poisson brackets of the integrals  $M_i$  and  $J_i$  are known to reproduce the commutation relations of the Lie algebra  $so(4)$  or  $so(3, 1)$  for negative resp. positive values of energy, see [12], [3].

## 10.2. Toda lattice

- The Lagrangian of the Toda lattice reads

$$L(x, \dot{x}) = \sum_{i=1}^N \left( \frac{1}{2} \dot{x}_i^2 - e^{x_{i+1} - x_i} \right).\tag{10.8}$$

One usually imposes one of the two types of boundary conditions: periodic,  $x_0 \equiv x_N, x_{N+1} \equiv x_1$ , or open-end,  $x_0 = +\infty, x_{N+1} = -\infty$ . The corresponding Euler-Lagrange equations are

$$\mathcal{E}_i = e^{x_{i+1}-x_i} - e^{x_i-x_{i-1}} - \ddot{x}_i = 0, \quad i = 1, \dots, N. \quad (10.9)$$

- The Toda lattice possesses  $N$  commuting variational symmetries. The two simplest ones are:

$$v_1 = \sum_{i=1}^N V_i^{(1)}(x, \dot{x}) \frac{\partial}{\partial x_i}, \quad v_2 = \sum_{i=1}^N V_i^{(2)}(x, \dot{x}) \frac{\partial}{\partial x_i}, \quad (10.10)$$

with characteristics

$$\begin{aligned} V_i^{(1)}(x, \dot{x}) &= \dot{x}_i^2 + e^{x_{i+1}-x_i} + e^{x_i-x_{i-1}}, \\ V_i^{(2)}(x, \dot{x}) &= \dot{x}_i^3 + (\dot{x}_{i+1} + 2\dot{x}_i)e^{x_{i+1}-x_i} + (2\dot{x}_i + \dot{x}_{i-1})e^{x_i-x_{i-1}}. \end{aligned}$$

Indeed, a simple computation confirms that

$$D_{v_k} L(x, \dot{x}) - D_t F_k(x, \dot{x}) = 0, \quad k = 1, 2,$$

where the fluxes  $F_1$  and  $F_2$  are given by

$$\begin{aligned} F_1(x, \dot{x}) &= \frac{2}{3} \sum_{i=1}^N \dot{x}_i^3, \\ F_2(x, \dot{x}) &= \sum_{i=1}^N \left( \frac{3}{4} \dot{x}_i^4 + (\dot{x}_i^2 + \dot{x}_i \dot{x}_{i+1} + \dot{x}_{i+1}^2) e^{x_{i+1}-x_i} \right) \\ &\quad - \sum_{i=1}^N \left( \frac{1}{2} e^{2(x_{i+1}-x_i)} - e^{x_{i+2}-x_i} \right). \end{aligned}$$

The Noether integrals corresponding to the characteristics  $V_i^{(1)}$  and  $V_i^{(2)}$  are:

$$J_1(x, \dot{x}) = \sum_{i=1}^N \left( \frac{1}{3} \dot{x}_i^3 + (\dot{x}_i + \dot{x}_{i+1}) e^{x_{i+1}-x_i} \right), \quad (10.11)$$

$$\begin{aligned} J_2(x, \dot{x}) &= \sum_{i=1}^N \left( \frac{1}{4} \dot{x}_i^4 + (\dot{x}_i^2 + \dot{x}_i \dot{x}_{i+1} + \dot{x}_{i+1}^2) e^{x_{i+1}-x_i} \right) \\ &\quad + \sum_{i=1}^N \left( \frac{1}{2} e^{2(x_{i+1}-x_i)} + e^{x_{i+2}-x_i} \right). \end{aligned} \quad (10.12)$$

The two symmetries  $v_1$  and  $v_2$  commute on solutions of Euler-Lagrange equations (10.9), since

$$D_{v_1} V_i^{(2)}(x, \dot{x}) - D_{v_2} V_i^{(1)}(x, \dot{x}) = \sum_{j=1}^N r_{ij}(x, \dot{x}) \mathcal{E}_j, \quad i = 1, \dots, N,$$

with

$$r_{ij}(x, \dot{x}) = -2(\dot{x}_{i+1} - \dot{x}_i) e^{x_{i+1}-x_i} \delta_{i+1,j} + 2(\dot{x}_i - \dot{x}_{i-1}) e^{x_i-x_{i-1}} \delta_{i-1,j}.$$



- The two flows corresponding to the commuting variational symmetries  $v_1$  and  $v_2$  are:

$$(x_i)_{t_1} = \dot{x}_i^2 + e^{x_{i+1}-x_i} + e^{x_i-x_{i-1}}, \quad (10.13)$$

$$(x_i)_{t_2} = \dot{x}_i^3 + (2\dot{x}_i + \dot{x}_{i+1})e^{x_{i+1}-x_i} + (2\dot{x}_i + \dot{x}_{i-1})e^{x_i-x_{i-1}}, \quad (10.14)$$

with  $i = 1, \dots, N$ . For  $k = 1, 2$ , we define

$$L_k(x, \dot{x}, x_{t_k}) = \sum_{i=1}^N \dot{x}_i (x_i)_{t_k} - J_k(x, \dot{x}),$$

with  $J_1, J_2$  from (10.11), (10.12). Euler-Lagrange equations (7.5) associated with  $L_1$  and  $L_2$  read respectively:

$$\mathcal{E}_i^{(1)} = (\dot{x}_i + \dot{x}_{i+1})e^{x_{i+1}-x_i} - (\dot{x}_i + \dot{x}_{i-1})e^{x_i-x_{i-1}} - (\dot{x}_i)_{t_1} = 0, \quad (10.15)$$

and

$$\begin{aligned} \mathcal{E}_i^{(2)} = & (\dot{x}_i^2 + \dot{x}_i \dot{x}_{i+1} + \dot{x}_{i+1}^2)e^{x_{i+1}-x_i} - (\dot{x}_{i-1}^2 + \dot{x}_{i-1} \dot{x}_i + \dot{x}_i^2)e^{x_i-x_{i-1}} \\ & + e^{2(x_{i+1}-x_i)} - e^{2(x_i-x_{i-1})} + e^{x_{i+2}-x_i} - e^{x_i-x_{i-2}} - (\dot{x}_i)_{t_2} = 0, \end{aligned} \quad (10.16)$$

with  $i = 1, \dots, N$ . As stated in Proposition 7.1, equations (10.15) and (10.16) are differential consequences of (10.9) and (10.13)–(10.14). Indeed, to derive (10.15), (10.16), one can differentiate (10.13), resp. (10.14), with respect to  $t$  and then substitute  $\ddot{x}_i = e^{x_{i+1}-x_i} - e^{x_i-x_{i-1}}$  (coming from (10.9)) into the resulting expressions.

## 11. Conclusions

In hindsight, the Hamiltonian picture of the pluri-Lagrangian structure on the phase space, generated by commuting Hamiltonian flows, as stated in Theorem 9.1, looks to be the most simple and natural and to provide a genuine explanation to the corresponding results in the configuration space, as given in Sections 7, 8. However, the purely Lagrangian point of view seems to be more universal and directly applicable in many contexts, where the Hamiltonian point of view is tricky or just unavailable, like discrete time and/or higher dimensional problems described by partial differential or partial difference equations [9–11, 20, 21]. Extending results of the present paper to these contexts is under current investigation.

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