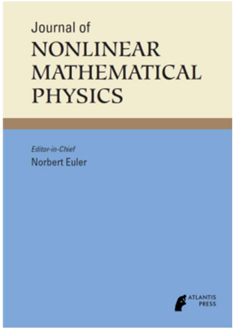

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Ideals generated by traces or by supertraces in the symplectic reflection algebra $H_{1,\nu}(I_2(2m+1))$

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For each complex number ν, an associative symplectic reflection algebra $\mathscr{H} := H_{1,\nu}(I_2(2m+1))$, based on the group generated by root system $I_2(2m+1)$, has an m-dimensional space of traces and an $(m+1)$-dimensional space of supertraces. A (super)trace sp is said to be degenerate if the corresponding bilinear (super)symmetric form $B_{sp}(x,y) = sp(xy)$ is degenerate. We find all values of the parameter ν for which either the space of traces contains a degenerate nonzero trace or the space of supertraces contains a degenerate nonzero supertrace and, as a consequence, the algebra $\mathscr{H}$ has a two-sided ideal of null-vectors. The analogous results for the case $H_{1,\nu_1,\nu_2}(I_2(2m))$ are also presented.

1. Introduction

1.1. *Definitions*

Let $\mathscr{A}$ be an associative $\mathbb{Z}_2$-graded algebra with unit and with a parity ε. All expressions of linear algebra are given for homogenous elements only and are supposed to be extended to inhomogeneous elements via linearity.

A linear complex-valued function tr on $\mathscr{A}$ is called a *trace* if $tr(fg-gf)=0$ for all $f,g\in\mathscr{A}$. A linear complex-valued function str on $\mathscr{A}$ is called a *supertrace* if $str(fg-(-1)^{\varepsilon(f)\varepsilon(g)}gf)=0$ for all $f,g\in\mathscr{A}$. These two definitions can be unified as follows.

Let $\varkappa=\pm1$. A linear complex-valued function sp on $\mathscr{A}$ is called *$\varkappa$-trace* if $sp(fg-\varkappa^{\varepsilon(f)\varepsilon(g)}gf)=0$ for all $f,g\in\mathscr{A}$.

The element $K\in\mathscr{A}$ is called a *Klein operator* if $K^2=1$, $\varepsilon(K)=0$, and $Kf=(-1)^{\varepsilon(f)}fK$ for any $f\in\mathscr{A}$. If the algebra $\mathscr{A}$ contains Klein operator K, then the linear function $f\mapsto tr(fK^{\varepsilon(f)+1})$ is a supertrace, and the linear function $f\mapsto str(fK^{\varepsilon(f)+1})$ is a trace.

Each nonzero (super)trace sp defines the nonzero (super)symmetric bilinear form $B_{sp}(f,g):=sp(fg)$.

If this bilinear form is degenerate, then the set of its null-vectors is a proper ideal in $\mathscr{A}$. We say that the (super)trace sp is *degenerate* if the bilinear form B_{sp} is degenerate.

1.2. *The goal and structure of the paper*

In a number of papers the simplicity (or, alternatively, existence of ideals) of Symplectic Reflection Algebras or. briefly, SRA (for definition, see [1]) was investigated, see, e.g., [2], [3]. In particular, it is shown that all SRA with zero parameters of deformation are simple (see [4], [2]).

It follows from [9] and [8] that an associative algebra of observables of the Calogero model with harmonic term in the potential and with coupling constant ν based on the root system $I_2(2m+1)$ (this algebra is SRA $H_{1,\nu}(I_2(2m+1))$) has an m-dimensional space of the traces and an $(m+1)$-dimensionsl space of supertraces.

We say that the parameter ν is *singular*, if the algebra $H_{1,\nu}(I_2(n))$ has a degenerate trace or supertrace.

The goal of this paper is to find all singular values of ν for the algebras $\mathscr{H} := H_{1,\nu}(I_2(n))$ and to find corresponding degenerate traces and supertraces for n odd, $n = 2m+1$; the result is formulated in Theorem 9.1.

The consideration generalizes [5], where $H_{1,\nu}(A_2) \cong H_{1,\nu}(I_2(3))$ was considered.

The case of n even is considered in [6] and reproduced here in Appendix. It is shown there that the set of singular values of ν-s consists of four families of parallel complex lines on the complex plane (ν_1, ν_2). The results for $H_{1,\nu}(I_2(2m))$ generalize the simplest case $H_{1,\nu_1,\nu_2}(I_2(2)) \cong H_{1,\nu_1}(A_1) \otimes H_{1,\nu_2}(A_1)$; the singular values of ν and ideals in $H_{1,\nu}(A_1)$ were found in [7].

In Sections 2-5 we recall the necessary definitions and prove the preliminary facts. In Section 6, we show that if the $\varkappa$-trace is degenerate, then its generating functions are integer. In Section 7, we derive the equations for the generating functions of the $\varkappa$-trace and solve these equations. The solutions are meromorphic functions of their parameter t for every value of ν, except degenerate values, which are found in Section 9.

2. The group $I_2(n)$

Definition 2.1. The group $I_2(n)$ is a finite subgroup of the orthogonal group $O(2,\mathbb{R})$ generated by the root system $I_2(n)$.

In this subsection we consider $\mathbb{C}$ instead of $\mathbb{R}^2$ for convenience.

The root system $I_2(n)$ consists of $2n$ vectors $v_k = \exp(\frac{i\pi k}{n})$, where $k = 0, 1, ..., 2n-1$. The group $I_2(n)$ has $2n$ elements: n reflections R_k and n rotations S_k. The reflection R_k acts on $z \in \mathbb{C}$ as follows:

$$\begin{aligned} R_k :\ & z \mapsto -z^* v_k^2, \\ R_k :\ & z^* \mapsto -z v_k^{-2} \ \text{ for } k = 0, 1, \dots, n-1, \end{aligned} \tag{2.1}$$

where the generators $z := e_x + ie_y$ and $z^* := e_x - ie_y$ are used instead of basis unit vectors e_x and e_y, the sign $*$ means complex conjugation. The rotations S_k have the form $S_k := R_k R_0$ and act on the generators z and z^* as follows

$$\begin{aligned} S_k :\ & z \mapsto z v_k^{-2}, \\ S_k :\ & z^* \mapsto z^* v_k^2 \ \text{ for } k = 0, 1, \dots, n-1. \end{aligned} \tag{2.2}$$

The element S_0 is the unit in the group $I_2(n)$

It is easy to see from the formulas (2.1), (2.2) that these elements satisfy the relations

$$R_k R_l = S_{k-l}, \qquad S_k S_l = S_{k+l}, \qquad R_k S_l = R_{k-l}, \qquad S_k R_l = R_{k+l}.$$

Obviously, if n is even, then the reflections R_{2k}, where $k = 0, 2, ..., \frac{n}{2} - 1$, constitute one conjugacy class, and the R_{2k+1} constitute another class. If n is odd, then all the reflections R_k are in the same conjugacy class.

The rotations S_k and S_l constitute a conjugacy class if $k + l = n$.

Let

$$\lambda := \exp\left(\frac{2\pi i}{n}\right). \tag{2.3}$$

In the basis z, z^*, matrices R_k and S_k have the form

$$R_k = \begin{pmatrix} 0 & -\lambda^{-k} \\ -\lambda^k & 0 \end{pmatrix}, \; S_k = \begin{pmatrix} \lambda^{-k} & 0 \\ 0 & \lambda^k \end{pmatrix}. \tag{2.4}$$

Let

$$G := \mathbb{C}[I_2(n)] \tag{2.5}$$

be the group algebra of the group $I_2(n)$. In G, it is convenient to introduce the following basis

$$L_p := \frac{1}{n}\sum_{k=0}^{n-1} \lambda^{kp} R_k, \qquad Q_p := \frac{1}{n}\sum_{k=0}^{n-1} \lambda^{-kp} S_k. \tag{2.6}$$

In what follows, we consider only the case of n odd, $n = 2m + 1$.

The result for n even from [6] is reproduced in Appendix. The main differences between odd and even n are as follows:

<u>n even</u>: the algebra $H_{1,\nu}(I_2(2m))$ depends on two complex parameters ν; the algebra $H_{1,\nu}(I_2(2m))$ contains the Klein operator and so the space of traces and the space of supertraces are isomorphic;

<u>n odd</u>: the algebra $H_{1,\nu}(I_2(2m+1))$ depends on one complex parameter ν, the space of traces is an m-dimensional, the space of supertraces is an $(m+1)$-dimensional.

3. Symplectic reflection algebra $H_{1,\nu}(I_2(2m+1))$

Let n be odd, $n = 2m + 1$, and $\lambda = \exp\left(\frac{2\pi i}{n}\right)$.

Let $V := \mathbb{C}^4$ with symplectic form ω:

$$\omega = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

The Coxeter group $I_2(n)$ may be embedded in the group $Sp(4, \mathbb{C})$, which preserves the form ω, as follows: let $x \in V$, $x = \sum_i e_i x^i$, where the vectors e_i constitute a basis in V, $x^i \in \mathbb{C}$, and let $g \in I_2(n)$.

Then

$$g(x) = \sum_{i,j} e_i g^i{}_j x^j \tag{3.2}$$

and

$$(R_k)^i{}_j = \begin{pmatrix} \begin{matrix} 0 & -\lambda^{-k} \\ -\lambda^k & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} 0 & -\lambda^k \\ -\lambda^{-k} & 0 \end{matrix} \end{pmatrix}, \quad (S_k)^i{}_j = \begin{pmatrix} \begin{matrix} \lambda^{-k} & 0 \\ 0 & \lambda^k \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} \lambda^k & 0 \\ 0 & \lambda^{-k} \end{matrix} \end{pmatrix}. \tag{3.3}$$

In this representation, $rk(R_k - 1) = 2$, so the elements R_k are symplectic reflections. They generate the group $I_2(n)$.

Let

$$\omega_k := \omega|_{Im(R_k-1)}. \tag{3.4}$$

Following [1], we define $H_{1,\nu}(I_2(2m+1))$:

Definition 3.1. *The symplectic reflection algebra* $\mathscr{H} := H_{1,\nu}(I_2(2m+1))$ is the associative algebra $TV \rtimes \mathbb{C}[I_2(2m+1)]$ of polynomials in the generators e_i, where $i = 1,...,4$, with coefficients in G satisfying the relations

$$R_k x = R_k(x) R_k, \qquad S_k x = S_k(x) S_k,$$
$$[x,\, y] = \omega(x,y) + 2\sum_{k=0}^{n-1} \nu\omega_k(x,y) R_k$$

for any $x, y \in V$.

In what follows, we set

$$a^0 := e_1,\;\; b^0 := e_2,\;\; b^1 := e_3,\;\; a^1 := e_4, \tag{3.5}$$

and Definition 3.1 reads as

Definition 3.2. *The symplectic reflection algebra* $\mathscr{H} := H_{1,\nu}(I_2(2m+1))$ is the associative algebra of polynomials in the generators a^α and b^α, where $\alpha = 0,1$, with coefficients in G (see Eq. (2.5)), satisfying the relations

$$R_k a^\alpha = -\lambda^k b^\alpha R_k, \qquad R_k b^\alpha = -\lambda^{-k} a^\alpha R_k,$$
$$S_k a^\alpha = \lambda^{-k} a^\alpha S_k, \qquad S_k b^\alpha = \lambda^k b^\alpha S_k,$$

$$\left[a^\alpha, b^\beta\right] = \varepsilon^{\alpha\beta}\left(1 + n\nu L_0\right),$$
$$\left[a^\alpha, a^\beta\right] = \varepsilon^{\alpha\beta} n\nu L_1,$$
$$\left[b^\alpha, b^\beta\right] = \varepsilon^{\alpha\beta} n\nu L_{-1}, \tag{3.6}$$

where $\varepsilon^{\alpha\beta}$ is the skew-symmetric tensor with $\varepsilon^{01} = 1$.

Defining the parity on $\mathscr{H}$ by setting

$$\varepsilon(a^{\alpha}) = \varepsilon(b^{\alpha}) = 1, \quad \varepsilon(R_k) = \varepsilon(S_k) = 0,$$

we turn this algebra into a superalgebra.

Introduce a new parameter of the algebra $\mathscr{H}$:

$$\mu := n\nu, \tag{3.7}$$

and rewrite the relations between the generating elements of $\mathscr{H}$ and elements of the group algebra G:

$$\begin{aligned} L_p a^{\alpha} &= -b^{\alpha} L_{p+1}, & L_p b^{\alpha} &= -a^{\alpha} L_{p-1}, \\ Q_p a^{\alpha} &= a^{\alpha} Q_{p+1}, & Q_p b^{\alpha} &= b^{\alpha} Q_{p-1}, \\ L_k L_l &= \delta_{k+l} Q_l, & L_k Q_l &= \delta_{k-l} L_l, \\ Q_k L_l &= \delta_{k+l} L_l, & Q_k Q_l &= \delta_{k-l} Q_l, \ \text{ where } \delta_k := \delta_{k0}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \left[a^{\alpha}, b^{\beta}\right] &= \varepsilon^{\alpha\beta}(1 + \mu L_0), \\ \left[a^{\alpha}, a^{\beta}\right] &= \varepsilon^{\alpha\beta} \mu L_1, \\ \left[b^{\alpha}, b^{\beta}\right] &= \varepsilon^{\alpha\beta} \mu L_{-1}. \end{aligned}$$

4. Subalgebra of singlets

Consider the elements[a] $T^{\alpha\beta} := \frac{1}{2}(\{a^{\alpha}, b^{\beta}\} + \{b^{\alpha}, a^{\beta}\})$ of the algebra $\mathscr{H}$ and the inner derivations they generate:

$$D^{\alpha\beta}: \ f \mapsto \left[f, T^{\alpha\beta}\right] \ \text{ for any } f \in \mathscr{H}.$$

It is easy to verify that the linear span of these derivations is isomorphic to the Lie algebra sl_2.

Definition 4.1. A *singlet* is any element $f \in \mathscr{H}$ such that $[f, T^{\alpha\beta}] = 0$ for all α, β. The subalgebra $H^0 \subset \mathscr{H}$ consisting of all singlets of the algebra $\mathscr{H}$ is called the *subalgebra of singlets*.

One can consider the algebra $\mathscr{H}$ as an sl_2-module and decompose it into the direct sum of irreducible submodules.

Observe, that any $\varkappa$-trace is identically zero on all irreducible sl_2-submodules of $\mathscr{H}$ except singlets.

Let the skew-symmetric tensor $\varepsilon_{\alpha\beta}$ be normalized so that $\varepsilon_{01} = 1$. We set

$$\mathfrak{s} := \frac{1}{4i} \sum_{\alpha,\beta=0,1} \left(\{a^{\alpha}, b^{\beta}\} - \{b^{\alpha}, a^{\beta}\}\right) \varepsilon_{\alpha\beta}.$$

One can prove the following fact:

Proposition 4.1. *The subalgebra of singlets H^0 of the algebra $\mathscr{H}$ is the algebra of polynomials in the element $\mathfrak{s}$ with coefficients in the group algebra $\mathbb{C}[I_2(2m+1)]$.*

[a]Here the brackets $\{\cdot,\cdot\}$ denote anticommutator.

In what follows we need the commutation relations of the singlet $\mathfrak{s}$ with generators of the algebra $\mathscr{H}$:

$$\begin{aligned} &[\mathfrak{s}, Q_p] = [\mathfrak{s}, S_k] = [T^{\alpha\beta}, \mathfrak{s}] = 0, \\ &\mathfrak{s} L_p = -L_p \mathfrak{s}, \qquad \mathfrak{s} R_k = -R_k \mathfrak{s}, \\ &(\mathfrak{s} - i\mu L_0) a^\alpha = a^\alpha (\mathfrak{s} + i + i\mu L_0). \end{aligned} \tag{4.1}$$

Theorem 4.1. *Let $\mathscr{I}$ be a proper ideal in the algebra $\mathscr{H}$, $\mathscr{I}_0 := \mathscr{I} \bigcap H^0$. Then there exist nonzero polynomials $\phi_k^0 \in \mathbb{C}[\mathfrak{s}]$, where $k = 0, ..., n-1$, such that $\mathscr{I}_0$ is the span over $\mathbb{C}[\mathfrak{s}]$ of the elements*

$$\phi_k^0(\mathfrak{s}) Q_k, \qquad \phi_{n-k}^0 L_k, \text{ where } k = 0, \dots, n-1 \text{ and } \phi_n^0 := \phi_0^0. \tag{4.2}$$

Before proving Theorem 4.1, we formulate and prove several propositions.

Proposition 4.2. *If $\mathscr{I} \subset \mathscr{H}$ is a proper ideal, then $\mathscr{I}_0 = \mathscr{I} \bigcap H^0$ is a proper ideal in H^0.*

Proof. First, note that $\mathscr{I}_0 \neq H^0$ because $\mathscr{I}$ does not contain unit.

Second, to prove that $\mathscr{I}_0 \neq 0$, we consider a nonzero element $g \in \mathscr{I}$. The sl_2-action on g generates an invariant subspace $\mathscr{F} \subset \mathscr{I}$, which can be decomposed into sum of invariant subspaces, $\mathscr{F} = \bigoplus_s \mathscr{F}^s$, where $\mathscr{F}^s \subset \mathscr{I}$ is a direct sum of irreducible sl_2-modules of spin s (and dimension $2s+1$).

We further consider the highest-weight vector $f \in \mathscr{F}^s$ and the set of elements $\{fQ_p \mid p = 0, ..., n-1\}$, belonging to the ideal $\mathscr{I}$. Not all these elements are equal to zero because $\sum_p fQ_p = f$. Let $fQ_p \neq 0$ and let it be of degree N. We consider the highest-degree part of the polynomial fQ_p, which has the form $f_Q Q_p + f_L L_p$, where f_Q and f_L are homogeneous polynomials in a^α, b^α of degree N. We can assume that $f_Q \neq 0$ (otherwise we can take an element $(f_L L_p) L_{-p} = f_L Q_{-p} \neq 0$) and consider the polynomial $\mathfrak{s} fQ_p + fQ_p \mathfrak{s} \simeq 2\mathfrak{s} f_Q Q_p$, where the sign $\simeq$ is used to denote the equality up to polynomials of lesser degrees.

The highest-degree terms of this polynomial have the form

$$h := \sum_{l=0}^{2s} c_l (a^1)^l (b^1)^{2s-l} \mathfrak{s}^{(N+2)/2-s} Q_p, \tag{4.3}$$

where $N+2$ is the degree of the homogeneous polynomial h.

Let $c_k \neq 0$ for some k in Eq. (4.3). Consider the element $\tilde{f} := (b^0)^k (a^0)^{2s-k} (\mathfrak{s} fQ_p + fQ_p \mathfrak{s})$ in the ideal $\mathscr{I}$ and the invariant subspace that it generates under the sl_2-action. This subspace contains a nonvanishing subspace of singlets.

Indeed, let the subscript 0 single out the sl_2-singlet part $(g)_0$ from the polynomial g. Then

$$\begin{aligned} \tilde{f}_0 &= ((b^0)^k (a^0)^{2s-k} h)_0 \simeq ((b^0)^k (a^0)^{2s-k} \sum_{l=0}^{2s} c_l (a^1)^l (b^1)^{2s-l} \mathfrak{s}^{(N+2)/2-s} Q_p)_0 \simeq \\ &\simeq c_k ((b^0)^k (a^0)^{2s-k} (a^1)^k (b^1)^{2s-k} \mathfrak{s}^{(N+2)/2-s} Q_p)_0 \simeq c_k ((b^0 a^1)^k (a^0 b^1)^{2s-k} \mathfrak{s}^{(N+2)/2-s} Q_p)_0 \simeq \\ &\simeq (-1)^k c_k (\mathfrak{s} - \mathfrak{t})^k (\mathfrak{s} + \mathfrak{t})^{2s-k} \mathfrak{s}^{(N+2)/2-s} Q_p)_0, \end{aligned}$$

where $\mathfrak{t} := \frac{1}{2}(a^0 b^1 + a^1 b^0)$. Next, we use the formula (see also [5]) proved in Proposition 4.3:

$$\tilde{f}(\mathfrak{s}, \mathfrak{t})_0 \simeq \frac{1}{2} \int_0^1 (\tilde{f}(\mathfrak{s}, \tau\mathfrak{s}) + \tilde{f}(\mathfrak{s}, -\tau\mathfrak{s})) d\tau, \tag{4.4}$$

which implies (since the integrand is positive)

$$\tilde{f}(\mathfrak{s},\mathfrak{t})_0 \simeq (-1)^k \frac{c_k}{2} \mathfrak{s}^{\frac{N+2}{2}+s} Q_p \int_0^1 ((1-\tau)^k (1+\tau)^{2s-k} + (1+\tau)^k (1-\tau)^{2s-k}) dt \neq 0. \tag{4.5}$$

□

Proposition 4.3. *Let* $\mathfrak{t} := \frac{1}{2}(a^0 b^1 + a^1 b^0)$ *and let* $f(\mathfrak{s}, t)$ *be an arbitrary homogeneous polynomial, then*

$$(f(\mathfrak{s},\, \mathfrak{t}))_0 \simeq \frac{1}{2} \int_0^1 (f(\mathfrak{s},\, \tau\mathfrak{s}) + f(\mathfrak{s},\, -\tau\mathfrak{s}))\, d\tau. \tag{4.6}$$

Proof. To prove (4.6), it is sufficient to consider the case $f(\mathfrak{s},\, \mathfrak{t}) = t^k$.

Consider the following sequence of obvious equalities

$$\begin{aligned} 0 &= \left(\left[T^{11},\, \left[T^{00},\, \mathfrak{t}^k\right]\right]\right)_0 \simeq \left(\left[T^{11},\, 2kT^{00}\mathfrak{t}^{k-1}\right]\right)_0 \simeq \\ &\simeq \left(-k(k-1)\mathfrak{t}^{k-2} T^{00} T^{11} - 8k\mathfrak{t}^k\right)_0 \simeq \left(-4k(k-1)\mathfrak{t}^{k-2}(t^2 - \mathfrak{s}^2) - 8k\mathfrak{t}^k\right)_0 \end{aligned}$$

which implies

$$\left(\mathfrak{t}^k\right)_0 \simeq \mathfrak{s}^2 \frac{k-1}{k+1} \left(\mathfrak{t}^{k-2}\right)_0 \simeq \begin{cases} \dfrac{1}{k+1}\mathfrak{s}^k = \displaystyle\int_0^1 (\mathfrak{s}\tau)^k\, d\tau & \text{if } k \text{ is even,} \\ \\ 0 & \text{if } k \text{ is odd,} \end{cases}$$

□

Definition 4.2. For each $p = 0, ..., 2m$, we define the ideals $\mathscr{J}_p$ and $\mathscr{J}^p$ in the algebra $\mathbb{C}[\mathfrak{s}]$, by setting

$$\mathscr{J}_p := \{f \in \mathbb{C}[\mathfrak{s}] \mid f(\mathfrak{s}) Q_p \in \mathscr{I}\}, \qquad \mathscr{J}^p := \{f \in \mathbb{C}[\mathfrak{s}] \mid f(\mathfrak{s}) L_p \in \mathscr{I}\}.$$

Proposition 4.4. *We have* $\mathscr{J}_p = \mathscr{J}^{-p}$.

Proof. It follows from the identities $f(\mathfrak{s}) Q_p L_{-p} = f(\mathfrak{s}) L_{-p}$ and $f(\mathfrak{s}) L_{-p} L_p = f(\mathfrak{s}) Q_p$. □

Proposition 4.5. *We have* $\mathscr{J}_p \neq 0$ *for any* $p = 0, ..., 2m$.

Proof. Let us consider a nonzero element $f \in \mathscr{I}_0$.

By Proposition 4.1, $f = \sum_p (\phi_p(\mathfrak{s}) Q_p + \psi_p(\mathfrak{s}) L_{-p})$. Obviously, there exists a p such that either $\phi_p \neq 0$ or $\psi_p \neq 0$. So, at least one of the elements $\mathfrak{s} Q_p f + Q_p f \mathfrak{s} = 2\mathfrak{s}\phi_p(\mathfrak{s}) Q_p \in \mathscr{I}_0$ and $\mathfrak{s} Q_p f - Q_p f \mathfrak{s} = 2\mathfrak{s}\psi_p(\mathfrak{s}) L_{-p} \in \mathscr{I}_0$ is nonzero. Hence, $\mathscr{J}_p \neq 0$.

Further, we prove that if $\mathscr{J}_p \neq 0$, then $\mathscr{J}_{p+1} \neq 0$, and therefore $\mathscr{J}_k \neq 0$ for $k = 0, 1, ... n-1$.

Let $g \in \mathscr{J}_p$, $g \neq 0$. Then $gQ_p \in \mathscr{I}$, and the element $\tilde{g} := \varepsilon_{\alpha\beta} b^\alpha g Q_p a^\beta \in \mathscr{I}$ is also nonzero.

By relation (3.8), $\tilde{g} = \varepsilon_{\alpha\beta} b^\alpha g a^\beta Q_{p+1}$, with $\tilde{g} \in \mathscr{I}_0$, and hence $\tilde{g} = \sum_k (\phi_k(\mathfrak{s}) Q_k + \psi_k(\mathfrak{s}) L_{-k})$ by Proposition 4.1. Because $0 \neq \mathfrak{s}\tilde{g} Q_{p+1} + \tilde{g} Q_{p+1}\mathfrak{s} \in \mathscr{I}_0$, as can be verified, we have $\mathfrak{s}\phi_{p+1}(\mathfrak{s}) \neq 0$, and $\mathfrak{s}\phi_{p+1}(\mathfrak{s}) Q_{p+1} \in \mathscr{I}_0$, i.e., $\mathfrak{s}\phi_{p+1}(\mathfrak{s}) \in \mathscr{J}_{p+1} \neq 0$. □

Since $\mathbb{C}[\mathfrak{s}]$ is a principal ideal ring, we have the following statement:

Corollary 4.1. *For any $p = 0,...,2m$, there exists a nonzero polynomial $\phi_p^0 \in \mathbb{C}[\mathfrak{s}]$ such that $\mathscr{J}_p = \phi_p^0 \mathbb{C}[\mathfrak{s}]$.*

Theorem 4.1 evidently follows from Corollary 4.1.

5. Generating functions of $\varkappa$-traces

For each $\varkappa$-trace sp on $\mathscr{H}$, one can define the following set of generating functions, which allows one to calculate the $\varkappa$-trace of arbitrary element in H^0 via finding the derivatives of these functions with respect to parameter t at zero:

$$\begin{aligned} F_p^{sp}(t) &:= sp(\exp(t(\mathfrak{s} - i\mu L_0))Q_p), \\ \Psi_p^{sp}(t) &:= sp(\exp(t\mathfrak{s})L_p), \end{aligned} \tag{5.1}$$

where $p = 0,...,2m$.

Since $L_0 Q_p = 0$ for any $p \neq 0$, it follows from the definition Eq. (5.1) that

$$\begin{aligned} F_p^{sp}(t) &= sp(\exp(t\mathfrak{s})Q_p) \text{ if } p \neq 0, \\ F_0^{sp}(t) &= sp(\exp(t(\mathfrak{s} - i\mu L_0))Q_0). \end{aligned}$$

It is easy to find Ψ_p^{sp} for $p \neq 0$. Since $\mathfrak{s}L_q = -L_q\mathfrak{s}$ for any $q = 0,...,2m$, we have

$$\Psi_q^{sp}(t) = sp(\exp(t\mathfrak{s})L_q) = sp(L_q). \tag{5.2}$$

Next, since $sp(R_k)$ does not depend on k, we have $sp(L_p) = 0$ for any $p \neq 0$ and

$$\Psi_p^{sp}(t) \equiv 0 \text{ for any } p \neq 0. \tag{5.3}$$

The value of $sp(L_0)$ will be calculated later, in Section 8.

We consider also the functions $\Phi_p^{sp}(t) := sp(\exp(t(\mathfrak{s} + i\mu L_0))Q_p)$. It is easily verified, by expanding the exponential in a series, that these functions are related with the functions F_p^{sp} by the formula

$$\Phi_p^{sp}(t) = F_p^{sp}(t) + 2i\Delta_p^{sp}(t), \text{where } \Delta_p(t)^{sp} = \delta_p \sin(\mu t) sp(L_0).$$

The form of generating functions is related with (non)degeneracy of bilinear form B_{sp} by Proposition 6.1 below.

6. Degeneracy conditions for the $\varkappa$-trace

Proposition 6.1. *The $\varkappa$-trace on the algebra $\mathscr{H}$ is degenerate if and only if the generating functions F_p^{sp} defined by formula (5.1) have the following form*

$$F_p^{sp}(t) = \sum_{j=1}^{j_p} \exp(t\omega_{j,p})\varphi_{j,p}(t), \tag{6.1}$$

where $\omega_{j,p} \in \mathbb{C}$ and $\varphi_{j,p} \in \mathbb{C}[t]$.

Proof. *Sufficiency.* Let the functions F_p^{sp} defined by Eq. (5.1) have the form (6.1).

We introduce the polynomials $D_p \in \mathbb{C}[x]$ by the formulas

$$D_p(x) := \prod_{j=1}^{j_p} (x - \omega_{j,p})^{1+\deg \varphi_{j,p}} \qquad \text{for } p \neq 0,$$
$$D_0(x) := \prod_{j=1}^{j_0} (x^2 - \omega_{j,0}^2)^{1+\deg \varphi_{j,0}}.$$

By definition, these polynomials satisfy the conditions $D_p(\frac{d}{dt})F_p^{sp}(t) = 0$ for any p. Besides, introduce the polynomial $\tilde{D}_0$ by setting

$$\tilde{D}_0(x^2) = D_0(x).$$

Since the $\varkappa$-trace sp we consider is non-zero, there exists a p such that $F_p^{sp} \neq 0$.

Now, we see that if $F_p^{sp} \neq 0$ for some $p \neq 0$, then the element $D_p(\mathfrak{s})Q_p \in \mathscr{H}$ is a null-vector of the bilinear form B_{sp}; we also see that if $F_0^{sp} \neq 0$, then the element $\widehat{D}(\mathfrak{s})Q_0 := \mathfrak{s}^2\tilde{D}_0(\mathfrak{s}^2 - \mu^2)Q_0 \in \mathscr{H}$ is a null-vector of the bilinear form B_{sp}.

Indeed, if $f \in \mathscr{H}$ belongs to a nonsinglet irreducible sl_2-module, then $sp(D_p(\mathfrak{s})Q_p f) = 0$ for any $p \neq 0$ and $sp(\widehat{D}_0(\mathfrak{s})Q_0 f) = 0$. If $f \in H^0$, then $f = \sum_q (f_q(\mathfrak{s})Q_q + g_q(\mathfrak{s})L_q)$ and, taking in account Eq. (5.3),

$$sp(D_p(\mathfrak{s})Q_p f) = sp(D_p(\mathfrak{s})Q_p f_p) = f_p\left(\frac{d}{dt}\right) D_p\left(\frac{d}{dt}\right) F_p^{sp}|_{t=0} = 0 \text{ for } p \neq 0.$$

Further, let us decompose the polynomial f_0 in the sum of even and odd polynomials:

$$f_0(\mathfrak{s}) = f_0^+(\mathfrak{s}^2) + \mathfrak{s} f_0^-(\mathfrak{s}^2).$$

Since $sp(\mathfrak{s}^k Q_0) = 0$ when k is odd[b], since

$$sp(\mathfrak{s}^2\tilde{D}_0(\mathfrak{s}^2 - \mu^2)Q_0 g_0 L_0) = 0$$

and

$$\frac{d^2}{dt^2}\exp(t(\mathfrak{s} - i\mu L_0)) = \exp(t(\mathfrak{s} - i\mu L_0))(\mathfrak{s}^2 - \mu^2 Q_0),$$

it follows that

$$\begin{aligned} sp(\mathfrak{s}^2\tilde{D}_0(\mathfrak{s}^2 - \mu^2)Q_0 f) &= sp(\mathfrak{s}^2\tilde{D}_0(\mathfrak{s}^2 - \mu^2)Q_0 f_0^+(\mathfrak{s}^2)) = \\ &= (\frac{d^2}{dt^2} + \mu^2)\, f_0^+(\frac{d^2}{dt^2} + \mu^2)\,\tilde{D}_0(\frac{d^2}{dt^2})\, F_0(t)|_{t=0} = \\ &= (\frac{d^2}{dt^2} + \mu^2)\, f_0^+(\frac{d^2}{dt^2} + \mu^2)\, D_0(\frac{d}{dt})\, F_0(t)|_{t=0} = 0. \end{aligned}$$

Thus, the sufficiency of Proposition 6.1 is proved.

[b]Indeed,

$$sp(\mathfrak{s}^k Q_0) = sp(\mathfrak{s}^k L_0 L_0) = sp(L_0 \mathfrak{s}^k L_0) = sp((-1)^k \mathfrak{s}^k L_0 L_0) = sp((-1)^k \mathfrak{s}^k Q_0).$$

.

Necessity. We now prove that if the $\varkappa$-trace is degenerate, then there exist polynomials $D_p \in \mathbb{C}[x]$ such that $D_p(\frac{d}{dt})F_p(t) = 0$ for $p = 0, ..., 2m$, and therefore the generating functions F_p have the form (6.1).

Let an ideal $\mathscr{I} \subset \mathscr{H}$ consist of null-vectors of the bilinear form B_{sp}. Then $\mathscr{I}_0$ consists of singlet null-vectors, and the vectors $\phi_k^0(\mathfrak{s})Q_k$ and $\phi_k^0(\mathfrak{s})L_{2m+1-k}$ defined by the conditions of Theorem 4.1 generate an ideal $\mathscr{I}_0$ in $\mathscr{H}_0$.

Let $p \neq 0$. Then

$$0 \equiv sp(\phi_p^0(\mathfrak{s})Q_p e^{t\mathfrak{s}} Q_p) = \phi_p^0\left(\frac{d}{dt}\right) F_p(t)$$

and, therefore, the function F_p has the form (6.1).

Further,
we consider the null-vector $\phi(\mathfrak{s}^2)Q_0$ of the bilinear form B_{sp}, where $\phi(\mathfrak{s}^2) := \phi_0^0(\mathfrak{s})\phi_0^0(-\mathfrak{s})$. We note that

$$\frac{d^2}{dt^2}F_0 = sp\left(e^{t(\mathfrak{s}-i\mu L_0)}(\mathfrak{s}-i\mu L_0)^2 Q_0\right) = sp\left(e^{t(\mathfrak{s}-i\mu L_0)}(\mathfrak{s}^2-\mu^2)Q_0\right),$$

hence

$$sp\left(e^{t(\mathfrak{s}-i\mu L_0)}\mathfrak{s}^2 Q_0\right) = \left(\frac{d^2}{dt^2}+\mu^2\right)F_0$$

and

$$0 \equiv sp\left(e^{t(\mathfrak{s}-i\mu L_0)}\phi_0^0(\mathfrak{s})\phi_0^0(-\mathfrak{s})Q_0\right) = sp\left(e^{t(\mathfrak{s}-i\mu L_0)}\phi(\mathfrak{s}^2)Q_0\right) = \phi\left(\frac{d^2}{dt^2}+\mu^2\right)F_0(t),$$

i.e., the function F_0 also has the form (6.1). $\square$

7. Equations for the generating functions F_p^{sp}

Let us differentiate the generating function F_p^{sp}:

$$\frac{d}{dt}F_p^{sp}(t) = sp\left(e^{t(\mathfrak{s}-i\mu L_0)}(\mathfrak{s}-i\mu L_0)Q_p\right) = sp\left(e^{t(\mathfrak{s}-i\mu L_0)}(-ia^\alpha\varepsilon_{\alpha\beta}b^\beta+i)Q_p\right).$$

The second equality here holds because

$$\mathfrak{s} = -ia^\alpha\varepsilon_{\alpha\beta}b^\beta + i(1+\mu L_0).$$

Next,

$$\begin{aligned} & sp\left(e^{t(\mathfrak{s}-i\mu L_0)}(-ia^\alpha\varepsilon_{\alpha\beta}b^\beta)Q_p\right) = sp\left(a^\alpha e^{t(\mathfrak{s}+i\mu L_0)}(-i\varepsilon_{\alpha\beta}b^\beta)Q_p\right) = \\ & = \varkappa sp\left(e^{t(\mathfrak{s}+i+i\mu L_0)}(-i\varepsilon_{\alpha\beta}b^\beta a^\alpha)Q_{p+1}\right) = \varkappa sp\left(e^{t(\mathfrak{s}+i+i\mu L_0)}(\mathfrak{s}+i+i\mu L_0)Q_{p+1}\right) = \\ & = \varkappa\frac{d}{dt}\left(e^{it}\Phi_{p+1}(t)\right). \end{aligned}$$

Thus, we obtain a system of differential equations for the generating functions:

$$\frac{d}{dt}F_p^{sp} - \varkappa e^{it}\frac{d}{dt}F_{p+1}^{sp} = iF_p^{sp} + \varkappa ie^{it}F_{p+1}^{sp} + 2\varkappa i\frac{d}{dt}\left(e^{it}\Delta_{p+1}^{sp}\right). \tag{7.1}$$

The initial conditions for this system are:

$$F_p^{sp}(0) = sp(Q_p).$$

To solve the system (7.1), we consider its Fourier transform. Let

$$\begin{aligned} &\lambda := e^{2\pi i/(2m+1)}, \\ &G_k^{sp} := \sum_{p=0}^{2m} \lambda^{kp} F_p^{sp}, \text{ where } k = 0,...,2m, \\ &\widetilde{\Delta}_k^{sp} := \sum_{p=0}^{2m} \lambda^{kp} \Delta_{p+1}^{sp} = \lambda^{-k} \left(\sin(\mu t) sp(L_0)\right), \text{ where } k = 0,...,2m. \end{aligned} \tag{7.2}$$

For the functions G_k^{sp}, we then obtain the equations

$$\frac{d}{dt} G_k^{sp} = i \frac{\lambda^k + \varkappa e^{it}}{\lambda^k - \varkappa e^{it}} G_k^{sp} + \frac{2i\varkappa\lambda^k}{\lambda^k - \varkappa e^{it}} \frac{d}{dt} \left(e^{it} \widetilde{\Delta}_k^{sp} \right) \tag{7.3}$$

with the initial conditions

$$G_k^{sp}(0) = sp(S_k). \tag{7.4}$$

We choose the solution of the system (7.3) in the form:

$$G_k^{sp}(t) = \frac{\varkappa e^{it}}{(\varkappa e^{it} - \lambda^k)^2} \lambda^k g_k^{sp}(t), \tag{7.5}$$

where

$$g_k^{sp}(t) = \left(\frac{2}{\mu} (\cos(t\mu) - 1) + 2i\lambda^{-k}(\lambda^k - \varkappa e^{it}) \sin(t\mu) \right) sp(L_0) + \varkappa \lambda^{-k} (\varkappa - \lambda^k)^2 sp(S_k). \tag{7.6}$$

Evidently, this solution satisfies initial condition (7.4) for each $\varkappa$ and k, except for the case $\varkappa = +1$ and $k = 0$.

If $\varkappa = +1$ and $k = 0$, then the expression Eq. (7.5) for G_0^{tr} has a removable singularity at $t = 0$. In this case, we consider the condition $\lim_{t\to 0} G_0^{tr}(t) = tr(S_0)$ instead of $G_0^{tr}(0) = tr(S_0)$.

When $\varkappa = +1$ the solution (7.5) – (7.6) gives

$$G_0^{tr}(t) = \frac{e^{it}}{(e^{it} - 1)^2} \left(\frac{2}{\mu} (\cos(t\mu) - 1) + 2i(1 - e^{it}) \sin(t\mu) \right) tr(L_0) \tag{7.7}$$

and one can easily see that

$$\lim_{t\to 0} G_0^{tr}(t) = -\mu tr(L_0). \tag{7.8}$$

It is shown in Subsection 8.1 using Ground Level Conditions, that if $\varkappa = +1$, then

$$tr(S_0) = -\mu tr(L_0) \tag{7.9}$$

for any trace tr on $\mathscr{H}$.

So, $G_0^{tr}(t)$ satisfies the initial conditions (7.4) also.

For the case $\varkappa = -1$, the $\varkappa$-trace is a supertrace (see [9]). In this case, the $m+1$ values $str(S_k) = str(S_{2m+1-k})$ for $k = 0,...,m$ completely define the supertrace on $\mathscr{H}$ (see [8]).

For the case $\varkappa = +1$, the $\varkappa$-trace is a trace (see [9]). In this case, the m values $tr(S_k) = tr(S_{2m+1-k})$ for $k = 1,...,m$ completely define the trace on $\mathscr{H}$ (see [8]). The value $tr(S_0)$ linearly depends on parameters $tr(S_k)$, where $k = 1,...,m$, and it is found in Subsection 8.1 (see Eqs. (8.7) – (8.8)).

8. Values of the $\varkappa$-trace on $\mathbb{C}[I_2(2m+1)]$

To use the generating functions (7.5), we need to express the values $sp(S_k)$ and $sp(L_0)$ via some independent parameters which completely define $\varkappa$-trace.

The results are different for traces ($\varkappa = +1$) and for supertraces ($\varkappa = -1$). First, we express $sp(L_0)$ via $sp(S_k) = sp(S_{2m+1-k})$, where $k = 1,\dots,m$ if $\varkappa = +1$ and $k = 0, 1,\dots,m$ if $\varkappa = -1$.

Let

$$c_k^\alpha := a^\alpha - \varkappa\lambda^k b, \text{ so } R_k c_k^\alpha = \varkappa c_k^\alpha R_k. \tag{8.1}$$

We consider the chain of equalities

$$sp(c_k^0 c_k^1 R_k) = \varkappa sp(c_k^1 R_k c_k^0) = \varkappa^2 sp(c_k^1 c_k^0 R_k), \tag{8.2}$$

which results in

$$sp([c_k^0, c_k^1] R_k) = 0. \tag{8.3}$$

The conditions like (8.3) are called *Ground Level Conditions* in [10], [9]. It follows from (8.3) that

$$-2\lambda^k \varkappa sp\left(R_k - \frac{\mu}{2}\varkappa(\lambda^{-k}L_1 - 2\varkappa L_0 + \lambda^k L_{-1})R_k\right)\right) = 0,$$

which gives

$$sp(R_k) = -\frac{2\mu}{2m+1}\left(\frac{1+\varkappa}{2}X^{sp} + \frac{1-\varkappa}{2}Y^{sp}\right), \tag{8.4}$$

where

$$X^{sp} := \sum_{r=1}^{2m} \sin^2\left(\frac{\pi r}{2m+1}\right) sp(S_r), \tag{8.5}$$

$$Y^{sp} := \sum_{r=0}^{2m} \cos^2\left(\frac{\pi r}{2m+1}\right) sp(S_r). \tag{8.6}$$

Below we consider these values for the traces and supertraces separately.

8.1. *Values of the traces on* $\mathbb{C}[I_2(2m+1)]$

The group $I_2(2m+1)$ has m conjugacy classes without the eigenvalue +1 in the spectrum:

$$\{S_p, S_{2m+1-p}\}, \text{ where } p = 1,...,m.$$

By Theorem 2.3 in [9], the values of the trace on these conjugacy classes

$$s_k := tr(S_k), \quad \text{where } s_{2m+1-k} = s_k, \;\; k = 1,...,m,$$

are arbitrary and completely define the trace on the algebra $\mathscr{H}$, and therefore the dimension of the space of traces is m.

Further, the group $I_2(2m+1)$ has one conjugacy class with one eigenvalue +1 in its spectrum:

$$\{R_1, \ldots, R_{2m+1}\}.$$

The value of $tr(R_k)$ is expressed via s_k by formula (8.4).

Besides, the group $I_2(2m+1)$ has one conjugacy class with two eigenvalues +1 in its spectrum: $\{S_0\}$.

The traces on conjugacy classes with two eigenvalues +1 in the spectrum can also be calculated using Ground Level Conditions (see [9]):

$$tr([a^0, b^1]S_0) = 0,$$

which gives

$$tr(S_0) = 2\nu^2(2m+1)X^{tr}, \tag{8.7}$$

where

$$X^{tr} := \sum_{l=1}^{2m} s_l \sin^2\left(\frac{2\pi l}{2m+1}\right). \tag{8.8}$$

We also note that

$$tr(L_0) = -\frac{2\mu}{2m+1}X^{tr}, \quad tr(L_p) = 0 \text{ for } p \neq 0, \quad tr(S_0) = -\mu tr(L_0).$$

8.2. *Values of the supertraces on* $\mathbb{C}[I_2(2m+1)]$

The group $I_2(2m+1)$ has $m+1$ conjugacy classes without the eigenvalue -1 in the spectrum:

$$\{S_0\},\ \{S_p, S_{2m+1-p}\}, \text{ where } p = 1, \ldots, m.$$

By Theorem 2.3 in [9], the values of the supertrace on these conjugacy classes

$$u_k := str(S_k) = str(S_{2m+1-k}), \quad \text{where} \quad k = 0, \ldots, m,$$

are arbitrary parameters that completely define the supertrace str on the algebra $\mathscr{H}$, and therefore the dimension of the space of supertraces is $m+1$.

Besides, the group $I_2(2m+1)$ has one conjugacy class with one eigenvalue -1 in the spectrum:

$$\{R_1, \ldots, R_{2m+1}\}.$$

The supertraces of the conjugacy class with eigenvalue -1 in its spectrum are calculated via Ground Level Conditions in Section 8. These conditions give

$$str(R_k) = -2\nu Y^{str}, \quad k = 0, 1, \ldots, 2m,$$

where

$$Y^{str} := \sum_{r=0}^{2m} u_r \cos^2\left(\frac{\pi r}{2m+1}\right).$$

9. Singular values of the parameter μ

We now find the values of the parameter μ for which there exists a nonzero $\varkappa$-trace sp, i.e., the values $sp(S_k)$ such that the the generating functions F_p (5.1) have the form (6.1). Since the functions G_k (7.2) are linear combinations of the functions F_p, and vice versa, the algebra $\mathscr{H}$ has a degenerate $\varkappa$-trace if and only if the functions G_k (7.2) have the form (6.1) also.

In particular, it is necessary that the numerator of the expression (7.5) contains all the zeros of the denominator of the expression.

The denominator of the function G_k is equal to

$$(e^{it} - \varkappa\lambda^k)^2$$

and has doubled zeros at

$$t_{k,l} = \frac{2\pi}{n}k + 2\pi l + \pi\theta, \text{ where } \quad l = 0, \pm 1, \pm 2, ...$$

and

$$\theta = \begin{cases} 0 \text{ if } \varkappa = 1 \\ 1 \text{ if } \varkappa = -1. \end{cases} \tag{9.1}$$

It is easy to check that $\frac{d}{dt} g_k^{sp}(t_{k,l_k}) = 0$ for each $k = 0, ..., 2m$ and each integer l_k.

The equalities $g_k^{sp}(t_{k,l_k}) = 0$ can be considered as a system of linear equations for the values $tr(S_k) = tr(S_{n-k})$, where $k = 1, ..., 2m$ if $\varkappa = 1$, and for the values $str(S_k) = str(S_{n-k})$, where $k = 0, \dots, n$ if $\varkappa = -1$:

$$g_k^{sp}(t_{k,l_k}) = \frac{2}{\mu}\left(\cos(t_{k,l_k}\mu) - 1\right) sp(L_0) + \varkappa\lambda^{-k}(\varkappa - \lambda^k)^2 sp(S_k) = 0. \tag{9.2}$$

Our goal is to find the μ for which the system (9.2) has nonzero solutions.

Note that $sp(L_0) \neq 0$ otherwise the $\varkappa$-trace would be zero. We consider the subsystem of two equations with $l_k = 0$:

$$\frac{2}{\mu}(\cos((\frac{2\pi k}{n} + \pi\theta)\mu) - 1)sp(L_0) + \varkappa\lambda^{-k}(\varkappa - \lambda^k)^2 sp(S_k) = 0, \tag{9.3}$$

$$\frac{2}{\mu}(\cos((\frac{2\pi(n-k)}{n} + \pi\theta)\mu) - 1)sp(L_0) + \varkappa\lambda^{k-n}(\varkappa - \lambda^{n-k})^2 sp(S_{n-k}) = 0. \tag{9.4}$$

Since $\frac{\varkappa(\varkappa-\lambda^k)^2}{\lambda^k} = \frac{\varkappa(\varkappa-\lambda^{n-k})^2}{\lambda^{n-k}}$ and $sp(S_{n-k}) = sp(S_k)$, it follows that Eqs. (9.3) – (9.4) imply that

$$\cos((\frac{2\pi k}{n} + \pi\theta)\mu) - \cos((\frac{2\pi(n-k)}{n} + \pi\theta)\mu) = 0 \tag{9.5}$$

or

$$\sin(\pi\mu(1+\theta))\sin(\frac{2k-n}{n}\pi\mu) = 0. \tag{9.6}$$

Eq. (9.6) implies that

$$\mu = \frac{z}{1+\theta}, \text{ where } z \in \mathbb{Z}. \tag{9.7}$$

Next, we consider the two cases separately:

A) $\mu \in \mathbb{Z}$, $\varkappa = \pm 1$,

B) $\mu = z + \frac{1}{2}$, where $z \in \mathbb{Z}$, $\varkappa = -1$.

In the case A), we note that Eq. (9.2) gives for μ integer:

$$0 = \sum_{k=0}^{n-1} g_k^{sp}(t_{k,l_k}) = \frac{2}{\mu} \sum_{k=0}^{n-1} \cos(\frac{2k\pi}{n}\mu)(-1)^{\theta\mu} sp(L_0). \tag{9.8}$$

Since $sp(L_0) \neq 0$, Eq. (9.8) gives the following restriction on the integer μ:

$$\sum_{k=0}^{n-1} \cos(\frac{2k\pi}{n}\mu) = 0, \tag{9.9}$$

i.e.,

$$\mu \in \mathbb{Z} \setminus n\mathbb{Z}. \tag{9.10}$$

Now consider the case B), i.e., $\varkappa = -1$, $\theta = 1$, $\mu = z + \frac{1}{2}$, where $z \in \mathbb{Z}$. Namely, consider the following two equations of the system (9.2):

$$g_k^{str}(t_{k,0}) = \frac{2}{\mu}(\cos(\frac{2\pi kz}{n} + \frac{\pi k}{n} + \pi z + \frac{\pi}{2}) - 1)sp(L_0) - \frac{(1+\lambda^k)^2}{\lambda^k} str(S_k) = 0,$$
$$g_k^{str}(t_{k,1}) = \frac{2}{\mu}(\cos(\frac{2\pi kz}{n} + \frac{\pi k}{n} + \pi z + \frac{\pi}{2} + \pi) - 1)sp(L_0) - \frac{(1+\lambda^k)^2}{\lambda^k} str(S_k) = 0,$$

which give

$$\cos(\frac{2\pi kz}{n} + \frac{\pi k}{n} + \pi z + \frac{\pi}{2}) = 0 \tag{9.11}$$

or

$$2z + 1 = nr \text{ for some odd } r, \text{ or } \mu = \frac{nr}{2}. \tag{9.12}$$

One easily checks that for every μ found, the system (9.2) does not depend on l_k and so has a nonzero solution.

Thus, we have proved the following theorem:

Theorem 9.1. *Let $m \in \mathbb{Z}$, $m \geqslant 1$ and $n = 2m+1$. Then*

1) The associative algebra $H_{1,\nu}(I_2(n))$ has a 1-parametric set of nonzero traces tr_z such that the symmetric invariant bilinear form $B_{tr_z}(x,y) = tr(xy)$ is degenerate if and only if $\nu = \frac{z}{n}$, where $z \in \mathbb{Z} \setminus n\mathbb{Z}$. These traces are completely defined by their values on S_k for $k = 1, \ldots, m$:

$$tr_z(S_k) = \frac{\tau}{n \sin^2(\frac{\pi k}{n})}(1 - \cos(\frac{2\pi kz}{n})), \text{ where } \tau \in \mathbb{C},\ \tau \neq 0. \tag{9.13}$$

2) The associative superalgebra $H_{1,\nu}(I_2(n))$ has a 1-parametric set of nonzero supertraces str_z such that the supersymmetric invariant bilinear form $B_{str_z}(x,y) = str(xy)$ is degenerate if $\nu = \frac{z}{n}$, where $z \in \mathbb{Z} \setminus n\mathbb{Z}$. These supertraces are completely defined by their values on S_k for $k = 0, \ldots, m$:

$$str_z(S_k) = \frac{\tau}{n \cos^2(\frac{\pi k}{n})}(1 - (-1)^z \cos(\frac{2\pi kz}{n})), \text{ where } \tau \in \mathbb{C},\ \tau \neq 0. \tag{9.14}$$

3) The associative superalgebra $H_{1,\nu}(I_2(n))$ has a 1-parametric set of nonzero supertraces $str_{1/2}$ such that the supersymmetric invariant bilinear form $B_{str_{1/2}}(x,y) = str_{1/2}(xy)$ is degenerate if $\nu =$

$z+\frac{1}{2}$, where $z\in\mathbb{Z}$. These supertraces are completely defined by their values on S_k for $k=0,\dots,m$:

$$str_{1/2}(S_k)=\frac{\tau}{n\cos^2(\frac{\pi k}{n})},\ \text{where}\ \tau\in\mathbb{C},\ \tau\neq 0. \tag{9.15}$$

4) For all other values of ν, all nonzero traces and supertraces are nondegenerate.

Remark 9.1. Theorem 9.1 implies that if $z\in\mathbb{Z}\setminus n\mathbb{Z}$, then the trace (9.13) generates the ideal $\mathscr{I}_{tr_z}$ consisting of null-vectors of the degenerate form $B_{tr_z}(x,y)=tr_z(xy)$, and simultaneously the supertrace (9.14) generates the ideal $\mathscr{I}_{str_z}$ consisting of null-vectors of the degenerate form $B_{str_z}(x,y)=str_z(xy)$. A question arises: is it true that $\mathscr{I}_{tr_z}=\mathscr{I}_{str_z}$?

Conjecture 9.1. $\mathscr{I}_{tr_z}=\mathscr{I}_{str_z}$.

Our observation, that the set of coefficients $\omega_{j,p}$ in Eq. (6.1) for $F_p^{tr_z}$ is the same as for $F_p^{str_z}$, is an argument in favor of this conjecture.

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Appendix A. The case $H_{1,\nu_1,\nu_2}(I_2(n))$ with n even

Here we, following [6], briefly describe the degenerate traces generating the ideals in the Symplectic Reflection Algebra $H_{1,\nu_1,\nu_2}(I_2(2m))$.

This algebra has two complex parameters; for every value of these parameters the algebra has an m-dimensional space of traces and, due to presence of the Klein operator, the isomorphic space of supertraces.

A.1. *The group* $I_2(2m)$

Definition A.1. The group $I_2(2m)$ is a finite subgroup of $O(2,\mathbb{R})$, generated by the root system $I_2(2m)$. It consists of $2m$ reflections R_k, acting on $z\in\mathbb{C}$ as follows

$$R_k z=-z^*v_k^2R_k,\quad k=0,...,2m-1 \tag{A.1}$$

and $2m$ rotations $S_k:=R_kR_0$, where S_0 is the unit in $I_2(2m)$ and S_m is the Klein operator. As we see from (A.1), these elements satisfy the relations

$$R_kR_l=S_{k-l},\qquad S_kS_l=S_{k+l},\qquad R_kS_l=R_{k-l},\qquad S_kR_l=R_{k+l}.$$

Evidently, the R_{2k} belong to one conjugacy class and the R_{2k+1} belong to another class. The rotations S_k and S_l constitute a conjugacy class if $k+l=2m$.

Definition A.2.

$$L_p := \frac{1}{n}\sum_{k=0}^{2m-1}\lambda^{kp}R_k, \qquad Q_p := \frac{1}{n}\sum_{k=0}^{2m-1}\lambda^{-kp}S_k, \tag{A.2}$$

where $\lambda = \exp\left(\frac{\pi i}{m}\right)$.

A.2. *Symplectic reflection algebra* $H_{1,\nu_0,\nu_1}(I_2(2m))$

Definition A.3. The *symplectic reflection algebra* $\mathscr{H} := H_{1,\nu_0,\nu_1}(I_2(2m))$ is an associative algebra of polynomials in a^α, b^α, where $\alpha = 0,1$, with coefficients in $\mathbb{C}[I_2(2m)]$, satisfying the relations

$$\begin{aligned} R_k a^\alpha &= -\lambda^k b^\alpha R_k, \qquad & R_k b^\alpha &= -\lambda^{-k} a^\alpha R_k, \\ S_k a^\alpha &= \lambda^{-k} a^\alpha S_k, \qquad & S_k b^\alpha &= \lambda^k b^\alpha S_k, \end{aligned}$$

$$\begin{aligned} L_p a^\alpha &= -b^\alpha L_{p+1}, \qquad & L_p b^\alpha &= -a^\alpha L_{p-1}, \\ Q_p a^\alpha &= a^\alpha Q_{p+1}, \qquad & Q_p b^\alpha &= b^\alpha Q_{p-1}, \\ L_k L_l &= \delta_{k+l} Q_l, \qquad & L_k Q_l &= \delta_{k-l} L_l, \\ Q_k L_l &= \delta_{k+l} L_l, \qquad & Q_k Q_l &= \delta_{k-l} Q_l, \ \text{ where } \delta_k := \delta_{k0}, \end{aligned} \tag{A.3}$$

$$\begin{aligned} \left[a^\alpha, b^\beta\right] &= \varepsilon^{\alpha\beta}\left(1 + \mu_0 L_0 + \mu_1 L_m\right), \\ \left[a^\alpha, a^\beta\right] &= \varepsilon^{\alpha\beta}\left(\mu_0 L_1 + \mu_1 L_{m+1}\right), \\ \left[b^\alpha, b^\beta\right] &= \varepsilon^{\alpha\beta}\left(\mu_0 L_{-1} + \mu_1 L_{m-1}\right), \end{aligned}$$

where $\varepsilon^{\alpha\beta}$ is the skew-symmetric tensor with $\varepsilon^{01} = 1$ and

$$\mu_0 := m(\nu_0 + \nu_1), \quad \mu_1 := m(\nu_0 - \nu_1). \tag{A.4}$$

The basis elements of Lie algebra sl_2 of inner derivations $T^{\alpha\beta} := \frac{1}{2}(\{a^\alpha, b^\beta\} + \{b^\alpha, a^\beta\})$ act on $\mathscr{H}$ as follows

$$f \mapsto \left[f, T^{\alpha\beta}\right] \ \text{ for each } f \in \mathscr{H}.$$

Let the skew-symmetric tensor $\varepsilon_{\alpha\beta}$ be such that $\varepsilon_{01} = 1$. Set

$$\mathfrak{s} := \sum_{\alpha,\beta=0,1} \frac{1}{4i}(\{a^\alpha, b^\beta\} - \{b^\alpha, a^\beta\})\varepsilon_{\alpha\beta}.$$

Then

$$\begin{aligned} [\mathfrak{s}, Q_p] = [\mathfrak{s}, S_k] &= [T^{\alpha\beta}, \mathfrak{s}] = 0, \\ \mathfrak{s}L_p = -L_p\mathfrak{s}, &\qquad \mathfrak{s}R_k = -R_k\mathfrak{s}, \\ (\mathfrak{s} - i(\mu_0 L_0 + \mu_1 L_m))a^\alpha &= a^\alpha(\mathfrak{s} + i + i(\mu_0 L_0 + \mu_1 L_m)). \end{aligned}$$

A.3. *The values of the trace on* $\mathbb{C}[I_2(2m)]$

The group $I_2(2m)$ has m conjugacy classes without the eigenvalue +1 in their spectra:

$$\{S_p, S_{n-p}\}, \text{ where } p = 1, ..., m-1, \text{ and also } \{S_m\}.$$

Due to Theorem 2.3 in [9], the values of the trace on these conjugacy classes

$$s_k := tr(S_k), \quad \text{where } s_{2m-k} = s_k, \;\; k = 1, ..., m, \tag{A.5}$$

completely define the trace on $\mathscr{H}$, and therefore the dimension of the space of traces is equal to m.

The group $I_2(2m)$ has two conjugacy classes each having one eigenvalue +1 in its spectrum:

$$\{R_{2l} \mid l = 0, ..., m-1\}, \; \{R_{2l+1} \mid l = 0, ..., m-1\},$$

and one conjugacy class with two eigenvalues +1 in its spectrum: $\{S_0\}$.

The traces on these conjugacy classes are calculated via Ground Level Conditions [9]:

$$tr([c_k^0, c_k^1]R_k) = 0, \text{ where } c_k^\alpha := a^\alpha - \lambda^k b^\alpha \text{ are eigenvectors of } R_k, \quad R_k c_k^\alpha = c_k^\alpha R_k,$$
$$tr([a^0, b^1]S_0) = 0$$

and are equal to

$$\begin{aligned} tr(R_{2l}) &= -2\nu_2 X_1 - 2\nu_1 X_2, \\ tr(R_{2l+1}) &= -2\nu_1 X_1 - 2\nu_2 X_2, \\ & \quad l = 0, 1, ..., m-1, \end{aligned} \tag{A.6}$$

$$tr(S_0) = 2(\nu_1^2 + \nu_2^2) m X_1 + 4\nu_1 \nu_2 m X_2, \tag{A.7}$$

where

$$X_1 := \sum_{l=1}^{m-1} s_{2l} \sin^2\left(\frac{\pi l}{m}\right),$$
$$X_2 := \sum_{l=0}^{m-1} s_{2l+1} \sin^2\left(\frac{\pi(2l+1)}{2m}\right).$$

We note also that

$$tr(L_0) = -\frac{\mu_0}{m}(X_1 + X_2), \quad tr(L_m) = -\frac{\mu_1}{m}(X_1 - X_2), \quad tr(L_p) = 0 \text{ for } p \neq 0, m,$$
$$tr(S_0) = -\mu_0 tr(L_0) - \mu_1 tr(L_m).$$

A.4. *Generating functions of the trace*

Set $\mathscr{L} := \mu_0 L_0 + \mu_1 L_m$.

For each trace tr, we define the following set of generating functions on $\mathscr{H}$:

$$F_p(t) := tr(\exp(t(\mathfrak{s} - i\mathscr{L}))Q_p), \qquad \text{(A.8)}$$
$$\Psi_p(t) := tr(\exp(t\mathfrak{s})L_p),$$

where $p = 0, ..., 2m-1$. From $\mathfrak{s}L_p = -L_p\mathfrak{s}$ and definition of the trace it follows that

$$\Psi_p(t) = \Psi_p(0).$$

We also consider the functions $\Phi_p(t) := tr(\exp(t(\mathfrak{s} + i\mathscr{L}))Q_p)$ related with the functions F_p by the formula

$$\Phi_p(t) = F_p(t) + 2i\Delta_p(t), \text{where } \Delta_p(t) = \delta_p \sin(\mu_0 t)tr(L_0) + \delta_{m-p}\sin(\mu_1 t)tr(L_m).$$

Analogously to our previous consideration, one can get the following system of equations

$$\frac{d}{dt}F_p - e^{it}\frac{d}{dt}F_{p+1} = iF_p + ie^{it}F_{p+1} + 2i\frac{d}{dt}\left(e^{it}\Delta_{p+1}\right). \qquad \text{(A.9)}$$

Next, we consider the Fourier transform of (A.9), namely, we consider

$$G_k := \sum_{p=0}^{2m-1} \lambda^{kp}F_p, \text{ where } k = 0, ..., 2m-1,$$
$$\widetilde{\Delta}_k := \sum_{p=0}^{2m-1} \lambda^{kp}\Delta_{p+1} = \lambda^{-k}\left(\sin(\mu_0 t)tr(L_0) + \lambda^{km}\sin(\mu_1 t)tr(L_m)\right),$$
$$\text{where } k = 0, ..., 2m-1 \qquad \text{and } \lambda := e^{i\pi/m},$$

and obtain the system of equation

$$\frac{d}{dt}G_k = i\frac{\lambda^k + e^{it}}{\lambda^k - e^{it}}G_k + \frac{2i}{\lambda^k - e^{it}}\frac{d}{dt}\left(e^{it}\widetilde{\Delta}_k\right)$$

with initial conditions

$$G_k(0) = s_k, \text{ where } k = 0, ..., 2m-1, \qquad \text{(A.10)}$$

and where the s_k are defined by Eq. (A.5) for $k = 1, ..., 2m-1$ and $s_0 := tr(S_0)$ is defined by Eq. (A.7). The value s_0 depends linearly on s_k, where $k = 1, ..., m$ (see Eq. (A.7) and take into account the relations $s_k = s_{2m-k}$).

The solution of the equations for G_k has the form:

$$G_k(t) = \frac{e^{it}f_k(t)}{(e^{it} - \lambda^k)^2}, \qquad \text{(A.11)}$$

where

$$f_k(t) = \frac{2\lambda^k}{m}X_+[1 - \cos(t\mu_0)] + (-1)^k\frac{2\lambda^k}{m}X_-[1 - \cos(t\mu_1)] + (1 - \lambda^k)^2 s_k +$$
$$+ \frac{2i}{m}(e^{it} - \lambda^k)[\mu_0 X_+ \sin(t\mu_0) + (-1)^k\mu_1 X_- \sin(t\mu_1)], \qquad \text{(A.12)}$$

and where $X_\pm := X_1 \pm X_2$.

The following proposition is analogous to Proposition 6.1 but its proof is slightly more difficult:

Proposition A.1. *The trace on the algebra $\mathscr{H}$ is degenerate if and only if the generating functions F_p^{tr} defined by formula (A.8) have the following form*

$$F_p^{tr}(t) = \sum_{j=1}^{j_p} \exp(t\omega_{j,p})\varphi_{j,p}(t), \tag{A.13}$$

where $\omega_{j,p} \in \mathbb{C}$ and $\varphi_{j,p} \in \mathbb{C}[t]$.

A.5. *The degeneracy conditions for the trace*

We now find the values of the parameters μ_0 and μ_1 for which there exists a nonzero trace tr, (i.e., the values s_k (A.5), not all zero) such that the generating functions (A.11) are of the form (6.1). Obviously, it is necessary that the numerator of Eq. (A.11) contains all zeros of the denominator of this expression. The denominator of G_k vanishes at the points

$$t_{k,l} = \frac{\pi}{m}k + 2\pi l, \text{ where } \quad l = 0, \pm 1, \pm 2, ...$$

It so happens that it is sufficient to consider only the points $t_{k,0}$.
Set

$$s'_k := s_k \sin^2\left(\frac{\pi k}{2m}\right), \quad k = 1, ..., 2m-1, \qquad s'_0 = 0.$$

Then the system of linear equations for s'_k has the form

$$\left(1 - \cos\left(\frac{\pi}{m}k\mu_0\right)\right) X_+ + (-1)^k \left(1 - \cos\left(\frac{\pi}{m}k\mu_1\right)\right) X_- = 2ms'_k, \quad k = 1, ..., 2m-1, \tag{A.14}$$

$$s'_{2m-r} = s'_r, \quad r = 1, ..., m, \tag{A.15}$$

$$X_\pm = X_1 \pm X_2, \tag{A.16}$$

$$X_1 = \sum_{1 \le l \le m-1} s'_{2l}, \tag{A.17}$$

$$X_2 = \sum_{0 \le l \le m-1} s'_{2l+1}, \tag{A.18}$$

and the parameters μ_0 and μ_1 are defined from the condition that this system has a nonzero solution.

Eqs. (A.14) – (A.18) imply that the dimension of the space of solutions is $\leqslant 2$ and we can take the values X_1 and X_2 as parameters determining the solutions.

Theorem A.1. *Let $m \geqslant 2$. Then the system of equations (A.14)-(A.18) has nonzero solutions at the following values of the parameters μ_0 and μ_1 only:*

$$\mu_0 \in \mathbb{Z} \setminus m\mathbb{Z}, \qquad \mu_1 \in \mathbb{Z} \setminus m\mathbb{Z}, \tag{A.19}$$

$$\mu_0 \in \mathbb{Z} \setminus m\mathbb{Z}, \qquad \text{any } \mu_1, \tag{A.20}$$

$$\mu_1 \in \mathbb{Z} \setminus m\mathbb{Z}, \qquad \text{any } \mu_0, \tag{A.21}$$

$$\mu_0 = \pm\mu_1 + m(2l+1), \qquad l = 0, \pm 1, \pm 2, ... \tag{A.22}$$

Here,

1. In case (A.19), the system of equations (A.14)-(A.18) has a 2*-parametric family of solutions;*

2. In case (A.20), if $\mu_1 \notin \mathbb{Z} \setminus m\mathbb{Z}$*, then the system of equations (A.14)-(A.18) has a* 1*-parametric family of solutions with* $X_- = 0$*,*

3. In case (A.21), if $\mu_0 \notin \mathbb{Z} \setminus m\mathbb{Z}$*, then the system of equations (A.14)-(A.18) has a* 1*-parametric family of solutions with* $X_+ = 0$*,*

4. In case (A.22), if $\mu_0, \mu_1 \notin \mathbb{Z} \setminus m\mathbb{Z}$*, then the system of equations (A.14)-(A.18) has a* 1*-parametric family of solutions with* $X_1 = 0$*.*

Remark A.1. Theorem A.5 is proved for $m \geqslant 2$, nevertheless it describes also the case $m = 1$ correctly.

If $m = 1$, then the cases (A.19) – (A.21) disappear, and the case (A.22) shows that

$$\text{at least one of } \nu_1 \text{ and } \nu_2 \text{ is half-integer.} \tag{A.23}$$

Because $H_{1,\nu_1,\nu_2}(I_2(2)) \cong H_{1,\nu_1}(A_1) \otimes H_{1,\nu_2}(A_1)$, the statement (A.23) follows also from [7], where the singular values of ν and ideals in $H_{1,\nu}(A_1)$ were found.

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