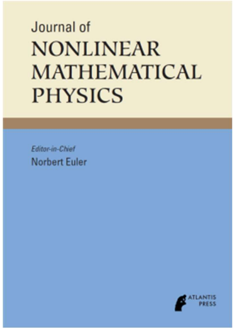

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On Peakon and Kink-peakon Solutions to a $(2+1)$ Dimensional Generalized Camassa-Holm Equation

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In this paper, we study a $(2+1)$-dimensional generalized Camassa-Holm (2dgCH) equation with both quadratic and cubic nonlinearity. We derive a peaked soliton (peakon) solution, double-peakon solutions, and kink-peakon solutions. In particular, weak kink - peakon solution is the first time to address in the $2+1$-dimensional integrable system.

Keywords: Camassa Holm (CH) equation, 2dgCH equation, peakon, kink-peakon.

2000 Mathematics Subject Classification: 35C07, 35C08, 37K40

1. Introduction

In recent years, the Camassa-Holm (CH) equation [4]

$$m_t - \alpha u_x + 2mu_x + m_x u = 0, \quad m = u - u_{xx}, \tag{1.1}$$

has attracted much attention in the theory of integrable systems and solitons. Since the work of Camassa and Holm [4], various studies on this equation have remarkably been developed [1, 2, 6, 7, 11, 13, 15, 16]. One of the most interesting features of the CH equation (1.1) is that it admits peaked soliton (peakon) solutions in the case of $\alpha = 0$ [4, 5]. A peakon is a weak solution in a suitable Sobolev space with a corner at its crest. The stability and dynamical interaction of peakons were discussed in several references [1, 2, 6, 7, 13]. In addition to the CH equation being an integrable model with peakon solutions, other integrable peakon models have been found, including the Degasperis-Procesi (DP) equation [8], the cubic nonlinear peakon equations [14, 17],and a

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generalized CH equation (gCH) with both quadratic and cubic nonlinearity [18]

$$m_t = k_1[m(u^2 - u_x^2)]_x + \frac{1}{2}k_2(2mu_x + m_x u), \quad m = u - u_{xx}, \tag{1.2}$$

where k_1 and k_2 are two arbitrary constants.

It is an interesting task to study the $(2+1)$-dimensional generalizations of the peakon equations. For example, in [9, 10] the authors provided a $(2+1)$-dimensional extension of the CH hierarchy, and they further studied peakon solutions for their $(2+1)$-dimensional CH equation. In this paper, we study a generalized $(2+1)$-dimensional Camassa-Holm equation (2dgCH) proposed by Xia and Qiao in [19]

$$\begin{cases} m_t &= bu_x + k_1[m\partial_x^{-1} mu_x]_x + \frac{1}{2}k_2(2mu_x + m_x u), \\ m_y &= u_x - u_{xxx}, \end{cases} \tag{1.3}$$

where b, k_1, k_2 are arbitrary constants. Moreover, we discuss the single-peakon solution found in [19] and derive two-peakon solutions for (1.3). For the double-peakon case, the peakon dynamical system is explicitly presented and their collisions are discussed in details. We also show that equation (1.3) with $k_2 = 0$ allows the weak kink solution in the case $k_1 \neq 0, b \neq 0$. Different from the double-peakon solutions in the form of linear superposition of the single-peakon, equation (1.3) with $k_2 = 0$ and $b \neq 0$ does not allow the multi-kink solution in the form of superposition of single-kink. However, we find that equation (1.3) with $k_2 = 0$ and $k_1 \neq 0, b \neq 0$ allows the solutions in the form of superposition of kink and peakon. In particular, the weak kink and kink-peakon interactional solutions are analyzed and plotted. Within our knowledge, this is the first time discussing double-peakon, weak kink and kink-peakon interactional solutions in $(2+1)$ dimensions.

2. Single-peakon solutions

In [19], the authors showed integrability and found single-peakon solutions for (1.3) with $b = 0$. Let us briefly discuss their results for a single-peakon solution and add some additional comments. Assume the single-peakon solution of the $(2+1)$-dimensional general Camassa-Holm equation (1.3) is given in the form

$$u = p(y,t)e^{-|x-q(y,t)|}, \quad m = 2r(y,t)\delta(x - q(y,t)), \tag{2.1}$$

where $p(y,t)$, $q(y,t)$ and $r(y,t)$ are to be determined. Differentiating in a distributional sense, we can find derivatives of (2.1)

$$\begin{aligned} u_x &= -pe^{-|x-q(y,t)|}sgn(x - q(y,t)), \\ u_{xx} &= pe^{-|x-q(y,t)|} - 2p\delta(x - q(y,t)), \\ m_x &= 2r\delta'(x - q(y,t)), \\ m_t &= 2r_t\delta(x - q(y,t)) - 2rq_t\delta'(x - q(y,t)). \end{aligned}$$

After substituting the derivatives into (1.3) and integrating against a test function, we arrive at

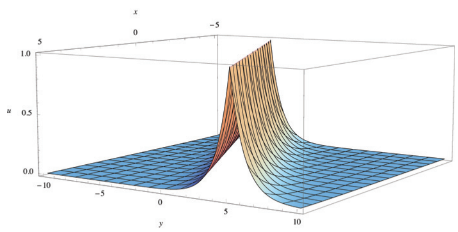

Fig. 1: Single-peakon solution $u(x,y,t)$ in (2.8) with $c=-1$ and $t=0$.

$$
\begin{aligned}
r_y &= 0,\\
r_t &= 0,\\
p &= -rq_y,\\
q_t &= -\frac{1}{3}k_1 rp - \frac{1}{2}k_2 p,
\end{aligned}
$$

which yields

$$r = c, \tag{2.2}$$

$$q = F\big(y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t\big), \tag{2.3}$$

$$p = -cq_y, \tag{2.4}$$

where c is an arbitrary constant, and F is an arbitrary smooth function. Thus, the single-peakon solution of (1.3) is given by

$$u(x,y,t) = -cF_y(y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t) \times e^{-|x - F(y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t)|}, \tag{2.5}$$

$$m(x,y,t) = 2c\delta(x - F(y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t)). \tag{2.6}$$

As $k_1 = 0$, $k_2 = 2$ we recover the single peakon solution of the $(2+1)$-dimensional CH equation proposed in [10]. In particular, if we take $F(y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t) = y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t$, then the single-peakon solution is given by

$$u(x,y,t) = -ce^{-|x-y-(\frac{1}{3}k_1c^2+\frac{1}{2}k_2c)t|}, \tag{2.7}$$

$$m(x,y,t) = 2c\delta(x - y - (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t). \tag{2.8}$$

See Figure 1 for the graph of the single-peakon solution $u(x,y,t)$ at $t=0$.

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An interesting case to note is the special case $c = i$ and $k_2 = 0$, then we see the nonlinear cubic equation

$$\begin{cases} m_t &= k_1[m\partial_x^{-1} m u_x]_x \\ m_y &= u_x - u_{xxx}, \end{cases} \tag{2.9}$$

has the complex peakon solution

$$u(x,y,t) = -ie^{-|x-y+\frac{1}{3}k_1 t|}. \tag{2.10}$$

3. Double-peakon solutions and their dynamics

We are interested in a double-peakon solution to equation (1.3). Assume

$$u = p_1(y,t)e^{-|x-q_1(y,t)|} + p_2(y,t)e^{-|x-q_2(y,t)|}, \tag{3.1}$$

$$m = 2r_1(y,t)\delta(x-q_1(y,t)) + 2r_2(y,t)\delta(x-q_2(y,t)), \tag{3.2}$$

where p_i, q_i, r_i are to be determined. The expression of u has two peaks at positions $x = q_1$ and $x = q_2$. Similarly as in the previous section, we differentiate (3.1) and (3.2) in a distributional sense. Then assuming

$$\frac{p_1}{r_1} = \frac{p_2}{r_2} = \bar{k}, \tag{3.3}$$

and substituting the derivatives of (3.1) and (3.2) into (1.3) and integrating against a test function, we have the following double-peakon dynamical system:

$$r_{1,y} = 0, \tag{3.4}$$

$$r_{2,y} = 0, \tag{3.5}$$

$$p_1 = -r_1 q_{1,y}, \tag{3.6}$$

$$p_2 = -r_2 q_{2,y}, \tag{3.7}$$

$$r_{1,t} = -\frac{1}{2}k_2 r_1 p_2 e^{-|q_1-q_2|}\operatorname{sgn}(q_1-q_2), \tag{3.8}$$

$$r_{2,t} = -\frac{1}{2}k_2 r_2 p_1 e^{-|q_2-q_1|}\operatorname{sgn}(q_2-q_1), \tag{3.9}$$

$$\begin{aligned} q_{1,t} = &-\frac{1}{3}k_1 p_1 r_1 - \frac{1}{2}k_2 p_1 - \frac{1}{2}k_2 p_2 e^{-|q_1-q_2|} \\ &-\frac{1}{2}k_1 r_1 p_2 e^{-|q_1-q_2|} - \frac{1}{2}k_1 r_2 p_1 e^{-|q_2-q_1|}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} q_{2,t} = &-\frac{1}{3}k_1 p_2 r_2 - \frac{1}{2}k_2 p_2 - \frac{1}{2}k_2 p_1 e^{-|q_2-q_1|} \\ &-\frac{1}{2}k_1 r_1 p_2 e^{-|q_1-q_2|} - \frac{1}{2}k_1 r_2 p_1 e^{-|q_2-q_1|}. \end{aligned} \tag{3.11}$$

Substituting (3.6) and (3.7) into (3.8)-(3.11), and using relationship (3.3) along with the transformations

$$\begin{aligned} r(t) &= r_1(t)+r_2(t),\\ R(t) &= r_1(t)-r_2(t),\\ q(y,t) &= q_1(y,t)+q_2(y,t),\\ Q(y,t) &= q_1(y,t)-q_2(y,t), \end{aligned}$$

the double-peakon dynamical system (3.4)-(3.11) becomes

$$r'(t) = 0, \tag{3.12}$$

$$R'(t) = -\frac{1}{4}\bar{k}k_2(r^2-R^2)e^{-|Q|}\mathrm{sgn}(Q), \tag{3.13}$$

$$q_t(y,t) = -\frac{1}{6}\bar{k}k_1(r^2+R^2)-\frac{1}{2}\bar{k}k_2 r(1+e^{-|Q|})-\frac{1}{2}\bar{k}k_1(r^2-R^2)e^{-|Q|}, \tag{3.14}$$

$$Q_t(t) = \frac{1}{2}\bar{k}k_2R(e^{-|Q|}-1)-\frac{1}{3}\bar{k}k_1 rR. \tag{3.15}$$

From (3.12), we see that $r(t)=A_1$, where A_1 is an arbitrary constant, which suggests we will have a peakon-antipeakon interaction. Letting $\Gamma = 1+\frac{2}{3}\frac{k1}{k2}A_1$, we have the following results.

Case 1. If $0<\Gamma\le 1$, then (3.12)-(3.15) admit the following solutions

$$\begin{aligned} R(t) &= \pm a_2\frac{1+\tilde{A}_3e^{Bt}}{1-\tilde{A}_3e^{Bt}}, \quad Q(t)=\pm\ln\frac{4\Gamma a_2^2\tilde{A}_3e^{Bt}}{a_2^2(1+\tilde{A}_3e^{Bt})^2-A_1^2(1-\tilde{A}_3e^{Bt})^2},\\ q(y,t) &= -2\bar{k}y-\ln\frac{|\tilde{A}_3e^{Bt}-\frac{A_1+a_2}{A_1-a_2}|}{|\tilde{A}_3e^{Bt}-\frac{A_1-a_2}{A_1+a_2}|}-\frac{2\bar{k}k_1a_2^2(3\Gamma-1)}{3B(\tilde{A}_3e^{Bt}-1)}-\frac{1}{2}\bar{k}[k_2A_1+\frac{1}{3}k_1(A_1^2+a_2^2)]t+D, \end{aligned}$$

where $a_2>|A_1|, \tilde{A}_3>0, B=-\frac{1}{2}a_2\bar{k}k_2\Gamma$ and D is an arbitrary constant.

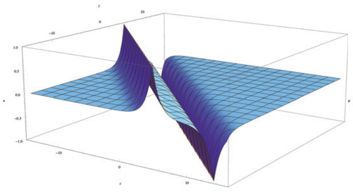
Case 2. If $\Gamma>1$, then we have the following solutions

$$\begin{aligned} R(t) &= \pm a_2\frac{1-\tilde{A}_3e^{Bt}}{1+\tilde{A}_3e^{Bt}}, \quad Q(t)=\pm\ln\frac{-4\Gamma a_2^2\tilde{A}_3e^{Bt}}{a_2^2(1-\tilde{A}_3e^{Bt})^2-A_1^2(1+\tilde{A}_3e^{Bt})^2},\\ q(y,t) &= -2\bar{k}y-\ln\frac{|\tilde{A}_3e^{Bt}+\frac{A_1+a_2}{A_1-a_2}|}{|\tilde{A}_3e^{Bt}+\frac{A_1-a_2}{A_1+a_2}|}-\frac{2\bar{k}k_1a_2^2(3\Gamma-1)}{3B(\tilde{A}_3e^{Bt}+1)}-\frac{1}{2}\bar{k}[k_2A_1+\frac{1}{3}k_1(A_1^2+a_2^2)]t+D, \end{aligned}$$

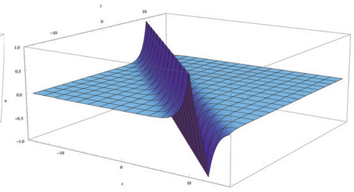
where $0<a_2<\frac{|A_1|}{\sqrt{\Gamma}}, \tilde{A}_3>0, B=-\frac{1}{2}a_2\bar{k}k_2\Gamma$, and D is an arbitrary constant.

For **Case 1**, let $A_1=0, a_2=2, \tilde{A}_3=1, k_1=k_2=-2, \bar{k}=1$ and $D=0$, then we get

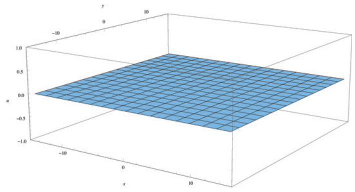
$$r(t)=0, \quad R(t)=2\coth(t), \quad q(y,t)=-2y+\frac{16}{3(e^{2t}-1)}+\frac{4}{3}t, \quad Q(t)=-\ln\frac{4e^{2t}}{(1+e^{2t})^2}. \tag{3.16}$$



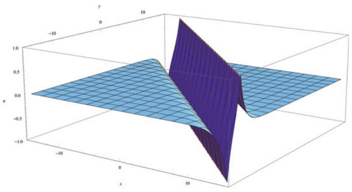
(a) Double-peakon solution at $t = -5$



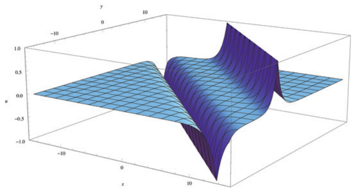
(b) Double-peakon solution at $t = -2$



(c) Collision of double-peakon solution at $t = 0$



(d) Double-peakon solution at $t = 2$



(e) Double-peakon solution at $t = 7$

Fig. 2: Double-peakon solution (3.18) of the 2dgCH equation (1.3) at different time steps.

These yield

$$\begin{aligned} r_1(t) &= \coth(t), \quad q_1(y,t) = -y + \frac{8}{3(e^{2t}-1)} + \ln(e^{2t}+1) - \frac{1}{3}t - \ln 2, \\ r_2(t) &= -\coth(t), \quad q_2(y,t) = -y + \frac{8}{3(e^{2t}-1)} - \ln(e^{2t}+1) + \frac{5}{3}t + \ln 2. \end{aligned} \tag{3.17}$$

Therefore, we obtain the following $(2+1)$ peakon-antipeakon solution

$$u(x,y,t) = \coth(t)\left(e^{-|x-q_1(y,t)|} - e^{-|x-q_2(y,t)|}\right), \tag{3.18}$$

where q_1 and q_2 are defined above. Profiles of the solution (3.18) for different time steps can be see in Figure 2.

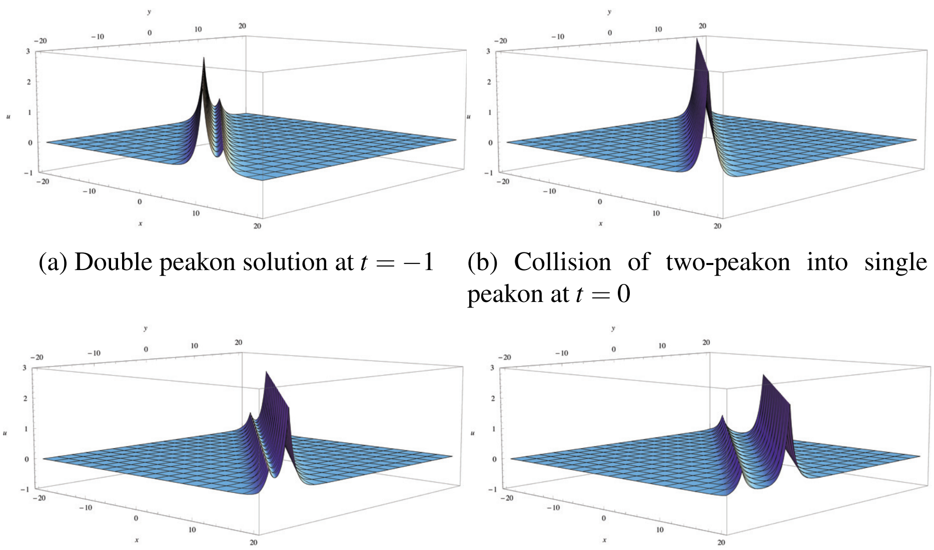
From (3.16), one can easily see the collision of the peakon-antipeakon interaction happens at the moment $t = 0$ since $Q(0) = 0$. From (3.17), we may compute [18]

$$\lim_{t\to 0} r_1(t) = -\lim_{t\to 0} r_2(t) = \infty, \quad \lim_{t\to 0} q_1(y,t) = \lim_{t\to 0} q_2(y,t) = \infty. \tag{3.19}$$

But from (3.18), we may infer that

$$\lim_{t\to 0} u(x,y,t) = 0, \quad \text{for every } x \in \mathbb{R}, \tag{3.20}$$

which indicates that the peakon and antipeakon vanish when they overlap. Using our results, we can discuss the dynamics of the peakon-antipeakon solution (3.18). When $t < 0$, we see that the peak is at $q_2(y,t)$ and the trough is at $q_1(y,t)$. The peak and trough are approaching each other as t goes to 0, with the peak traveling faster than the trough. At the moment of $t = 0$, the peakon and antipeakon collide and then vanish. After the collision ($t > 0$), they depart and redevelop with the trough at $q_2(y,t)$ and the peak at $q_1(y,t)$.



(a) Double peakon solution at $t=-1$ (b) Collision of two-peakon into single peakon at $t=0$

(c) Double peakon solution at $t=1$ (d) Double peakon solution at $t=2$

Fig. 3: Double peakon solution with (3.22) of the 2dgCH equation (1.3) at different time steps.

For **Case 2**, let $A_1=3$, $a_2=\sqrt{3}$, $\Gamma=3$, $\tilde{A}_3=1$, $k_1=k_2=-2$, $\bar{k}=1$, $B=3\sqrt{3}$ and $D=0$. Then the first two equations of Case 2 yield

$$R(t)=\sqrt{3}\,\mathrm{sgn}(t)\frac{1-e^{3\sqrt{3}t}}{1+e^{3\sqrt{3}t}},\quad Q(t)=\mathrm{sgn}(t)\ln\frac{6e^{3\sqrt{3}t}}{e^{6\sqrt{3}t}+4e^{3\sqrt{3}t}+1}. \tag{3.21}$$

Then using the equation for $q(y,t)$ we see that

$$\begin{aligned}
r_1(t)&=\frac{3}{2}-\frac{\sqrt{3}}{2}\mathrm{sgn}(t)\tanh\left(\frac{3\sqrt{3}}{2}t\right),\quad r_2(t)=\frac{3}{2}+\frac{\sqrt{3}}{2}\mathrm{sgn}(t)\tanh\left(\frac{3\sqrt{3}}{2}t\right),\\
q_1(y,t)&=-y+\frac{1}{2}\mathrm{sgn}(t)\ln\frac{6e^{3\sqrt{3}t}}{e^{6\sqrt{3}t}+4e^{3\sqrt{3}t}+1}-\frac{1}{2}\ln\frac{e^{3\sqrt{3}t}+2+\sqrt{3}}{e^{3\sqrt{3}t}+2-\sqrt{3}}-\frac{16\sqrt{3}}{9(e^{3\sqrt{3}t}+1)}+\frac{7}{2}t,\\
q_2(y,t)&=-y-\frac{1}{2}\mathrm{sgn}(t)\ln\frac{6e^{3\sqrt{3}t}}{e^{6\sqrt{3}t}+4e^{3\sqrt{3}t}+1}-\frac{1}{2}\ln\frac{e^{3\sqrt{3}t}+2+\sqrt{3}}{e^{3\sqrt{3}t}+2-\sqrt{3}}-\frac{16\sqrt{3}}{9(e^{3\sqrt{3}t}+1)}+\frac{7}{2}t.
\end{aligned} \tag{3.22}$$

Notice that (3.21) shows that the collision of the two-peakon happens at the moment $t=0$ since $Q(0)=0$. We can see from (3.22) that the double-peakon collides and overlaps into the peakon

$$u(x,y,0)=3e^{-|x+y+\frac{1}{2}\ln(2+\sqrt{3})+\frac{8\sqrt{3}}{9}|}, \tag{3.23}$$

at the moment $t=0$. After the collision, the two-peakon departs and redevelops. See Figure 3 for the profile of the two-peakon dynamics at different time steps.

For $1+1$ dimensional integrable peakon systems, an analysis approach for peakon-antipeakon collisions was developed for the CH-equation in the papers [3, 12]. However, it is open for the $(2+1)$-dimensional 2dgCH system (1.3).

4. Single Weak Kink Solution

We have already shown that the 2dgCH equation (1.3) admits peakon and double-peakon solutions in the case of $b = 0$. We are also interested in what kind of solutions arise in the case $b \neq 0$. We will see that equation (1.3) with $k_2 = 0$ and $b \neq 0$ (i.e. only keeping the cubic term) possesses weak kink solutions and kink-peakon interacted solutions. We seek weak kink solutions in the form of

$$\begin{cases} u(x,y,t) & = p(y,t)\mathrm{sgn}(x - q(y,t))\left(e^{-|x-q(y,t)|} - 1\right), \\ m(x,y,t) & = -r(y,t)\mathrm{sgn}(x - q(y,t)), \end{cases} \tag{4.1}$$

where p, q, r are to be determined. Substituting the distributional derivatives of (4.1) into (1.3) with $k_2 = 0$ and $b \neq 0$, we have

$$\begin{aligned} r_y &= 0, \\ r_t &= 0, \\ r &= \pm\sqrt{\frac{-b}{k_1}}, \\ p &= -rq_y, \\ q_t &= k_1 pr, \end{aligned}$$

which yield the following results

$$r = \pm\sqrt{\frac{-b}{k_1}}, \tag{4.2}$$

$$q = F(bt + y), \tag{4.3}$$

$$p = -rq_y, \tag{4.4}$$

where F is an arbitrary smooth function. Then we obtain the following weak kink solution

$$u(x,y,t) = \mp\sqrt{\frac{-b}{k_1}}F_y(bt + y)\mathrm{sgn}(x - F(bt + y))\left(e^{-|x-F(bt+y)|} - 1\right). \tag{4.5}$$

Although F may be arbitrary, one must check that (4.5) still satisfies the definition of a kink. We can choose q to be the identity function $F(bt + y) = bt + y$. Thus, we have

$$u(x,y,t) = \mp\sqrt{\frac{-b}{k_1}}\mathrm{sgn}(x - y - bt)\left(e^{-|x-y-bt|} - 1\right), \tag{4.6}$$

$$\tag{4.7}$$

where

$$\lim_{t\to\infty} u = -\lim_{t\to-\infty} u = \mp\sqrt{\frac{-b}{k_1}}.$$

In the special case of $b = 1, k_1 = -1$ and forcing $r = -1$ yields

$$u(x,y,t) = \mathrm{sgn}(x - y - t)\left(e^{-|x-y-t|} - 1\right).. \tag{4.8}$$

See Figure 4 and Figure 5 for the profile of this weak kink wave solution at different time steps. Note that we see a white line in the middle of the kink's profile which shows that this is indeed a

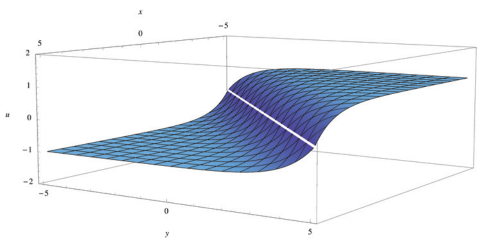


Fig. 4: Weak kink solution of (1.3) with $b \neq 0$ and $k_2 = 0$ at $t = 0$.

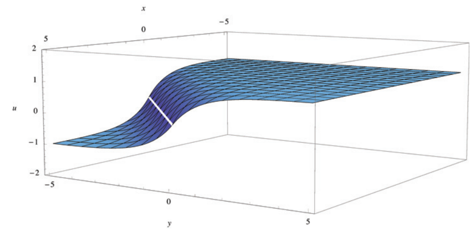


Fig. 5: Weak kink solution of (1.3) with $b \neq 0$ and $k_2 = 0$ at $t = 5$.

kink solution, but in a weak sense (namely, the first order derivative exists on the white line, but the 2nd order does not).

We also want to point out that equation (1.3) with $k_2 = 0$ and $b \neq 0$ does not allow a double-kink solution in the form of the superposition of two single-kink solutions:

$$u = p_1(y,t)\mathrm{sgn}(x-q_1(y,t))\left(e^{-|x-q_1(y,t)|}-1\right) + p_2(y,t)\mathrm{sgn}(x-q_2(y,t))\left(e^{-|x-q_2(y,t)|}-1\right). \tag{4.9}$$

5. Kink-Peakon Solutions

We now look for a new phenomena of kink-peakon interacted dynamics in soliton theory for $2+1$-dimensional systems. This phenomena was first proposed in 2012 by Xia, Qiao, and Li in [18]. To our knowledge, this will be the first time this phenomena is presented in $(2+1)$-dimensions. Let us consider the solution

$$\begin{cases} u(x,y,t) &= p_1(y,t)\mathrm{sgn}(x-q_1(y,t))\left(e^{-|x-q_1(y,t)|}-1\right) + p_2(y,t)e^{-|x-q_2(y,t)|}, \\ m(x,y,t) &= -r_1(y,t)\mathrm{sgn}(x-q_1(y,t)) + 2r_2(y,t)\delta(x-q_2(y,t)), \end{cases} \tag{5.1}$$

where p_1, q_1 are the amplitude and the position of the kink, respectively, and p_2, q_2 are the amplitude and position of the peakon. Assuming the following relationship

$$\frac{p_1}{r_1} = \frac{p_2}{r_2} = \tilde{k}, \tag{5.2}$$

and substituting the distributional derivatives of (5.1) into (1.3) with $k_2 = 0, b \neq 0$ and integrating against a test function, we arrive at the following kink-peakon dynamical system:

$$r_{1,t} = 0, \tag{5.3}$$
$$r_{1,y} = 0, \tag{5.4}$$
$$r_{2,y} = 0, \tag{5.5}$$
$$p_1 = -r_1 q_{1,y,}, \tag{5.6}$$
$$p_2 = -r_2 q_{2,y}, \tag{5.7}$$
$$r_1 = \pm\sqrt{\frac{-b}{k_1}}, \tag{5.8}$$
$$r_{2,t} = k_1 r_1^2 p_2 e^{-|q_2-q_1|}\mathrm{sgn}(q_2-q_1), \tag{5.9}$$
$$q_{1,t} = \frac{1}{2}p_1 r_1 - k_1 r_1 p_2 e^{-|q_2-q_1|}\mathrm{sgn}(q_2-q_1), \tag{5.10}$$
$$q_{2,t} = -\frac{1}{3}k_1 p_2 r_2 - \frac{1}{2}k_1 p_1 r_1 + k_1(p_1 r_1 - r_1 p_2 \mathrm{sgn}(q_2-q_1))e^{-|q_2-q_1|} + k_1 r_1 r_2 p_2 \mathrm{sgn}(q_2-q_1). \tag{5.11}$$

Then using (5.2) and substituting (5.6) and (5.7) into (5.8)-(5.11) lead to

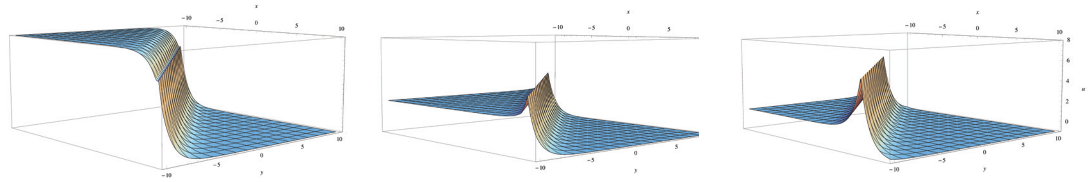
$$r_1 = \pm\sqrt{\frac{-b}{k_1}}, \tag{5.12}$$
$$r_{2,t} = \tilde{k}k_1 r_1^2 r_2 e^{-|q_2-q_1|}\mathrm{sgn}(q_2-q_1), \tag{5.13}$$
$$q_{1,t} = -\frac{1}{2}\tilde{k}b - \tilde{k}k_1 r_1 r_2 e^{-|q_2-q_1|}\mathrm{sgn}(q_2-q_1), \tag{5.14}$$
$$q_{2,t} = -\frac{1}{3}\tilde{k}k_1 r_2^2 - \frac{1}{2}\tilde{k}k_1 r_1^2 + \tilde{k}k_1(r_1^2 - r_1 r_2 \mathrm{sgn}(q_2-q_1))e^{-|q_2-q_1|} + \tilde{k}k_1 r_1 r_2 \mathrm{sgn}(q_2-q_1). \tag{5.15}$$

Choosing $k_1 = -b = 2$, then $r_1 = \pm 1$. Without loss of generality, taking $r_1 = 1$ forces (5.12)-(5.15) to become

$$r_1 = 1, \tag{5.16}$$
$$r_{2,t} = 2\tilde{k}r_2 e^{-|q_2-q_1|}\mathrm{sgn}(q_2-q_1), \tag{5.17}$$
$$q_{1,t} = \tilde{k} - 2\tilde{k}r_2 e^{-|q_2-q_1|}\mathrm{sgn}(q_2-q_1), \tag{5.18}$$
$$q_{2,t} = -\frac{2}{3}\tilde{k}r_2^2 - \tilde{k} + 2\tilde{k}(1 - r_2\mathrm{sgn}(q_2-q_1))e^{-|q_2-q_1|} + 2\tilde{k}r_2\mathrm{sgn}(q_2-q_1). \tag{5.19}$$

To solve the system above, we may assume that $q_1 < q_2$. Then integrating, we have

$$q_1 = -\tilde{k}y + \tilde{k}t - r_2 + A_1, \tag{5.20}$$
$$q_2 = -\tilde{k}y + \tilde{k}t - r_2 - \ln\left(\frac{1}{9}\tilde{k}r_2^2 - \frac{1}{2}\tilde{k}r_2 + \tilde{k} + \frac{\tilde{k}A_2}{2r_2}\right) + A_1, \tag{5.21}$$
$$r_{2,t} = \frac{2}{9}\tilde{k}r_2^3 - \tilde{k}r_2^2 + 2\tilde{k}r_2 + \tilde{k}A_2, \tag{5.22}$$



(a) Kink-peakon interaction at $t = 0$. (b) Kink-peakon interaction at $t = 1$. (c) Kink-peakon interaction at $t = 2$.

Fig. 6: Kink-peakon interactional solution (5.24) of the 2dgCH equation (1.3) with $k_2 = 0$ at different time steps.

where A_1 and A_2 are arbitrary constants. Letting $A_2 = 0$, we may solve equation (5.22) for r_2, which has the following implicit form:

$$\ln|r_2| - \frac{1}{2}\ln|r_2^2 - \frac{9}{2}r_2 + 9| + \frac{3\sqrt{7}}{7}\tan^{-1}(\frac{4r_2-9}{3\sqrt{7}}) = 2\tilde{k}t + A_3. \tag{5.23}$$

Then with $A_1 = A_2 = A_3 = 0$ the kink-peakon solution (5.1) can be rewritten as

$$u(x,y,t) = \tilde{k}\operatorname{sgn}(x+\tilde{k}y-\tilde{k}t+r_2)\left(e^{-|x+\tilde{k}y-\tilde{k}t+r_2|}-1\right) + \tilde{k}r_2 e^{-|x+\tilde{k}y-\tilde{k}t+r_2+\ln(\frac{1}{9}\tilde{k}r_2^2-\frac{1}{2}\tilde{k}r_2+\tilde{k})|} \tag{5.24}$$

Because of the implicit form of (5.23), we do not have explicit results about the collision of the kink and peakon. See Figure 6 for a profile of the kink-peakon interaction with $\tilde{k} = 1, A_1 = A_2 = A_3 = 0$.

6. Conclusion

We have shown single-peakon solutions and double-peakon solutions for the $(2+1)$ dimensional generalized Camassa-Holm equation (1.3) with $b = 0, k_1, k_2 \neq 0$ and discussed the double-peakon dynamical system in detail. We also showed that (1.3) admits weak-kink and kink-peakon interaction solutions with $b, k_1 \neq 0, k_2 = 0$.

For further study, we would like to investigate the kink-peakon solutions in more detail and provide more information about the interaction of the kink and peakon. We may also study topics such as all possible traveling wave solutions, peakon stability, kink-peakon stability, etc.

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References

[1] R. Beals, D. H. Sattinger, and J. Szmigielski. Acoustic scattering and the extended Korteweg–de Vries hierarchy. *Advances in Mathematics*, 140(2):190–206, 1998.

[2] R. Beals, D. H. Sattinger, and J. Szmigielski. Multipeakons and the classical moment problem. *Advances in Mathematics*, 154(2):229–257, 2000.

[3] A. Bressan and A. Constantin. Global conservative solutions of the Camassa–Holm equation. *Archive for Rational Mechanics and Analysis*, 183(2):215–239, 2007.

[4] R. Camassa and D. D. Holm. An integrable shallow water equation with peaked solitons. *Physical Review Letters*, 71(11):1661, 1993.

[5] R. Camassa, D. D. Holm, and J. M. Hyman. A new integrable shallow water equation. *Advances in Applied Mechanics*, 31(31):1–33, 1994.

[6] A. Constantin and W. A. Strauss. Stability of peakons. *Communications on Pure and Applied Mathematics*, 53(5):603–610, 2000.

[7] A. Constantin and W. A. Strauss. Stability of the Camassa-Holm solitons. *Journal of Nonlinear Science*, 12(4):415–422, 2002.

[8] A. Degasperis and M. Procesi. Asymptotic integrability. *Symmetry and perturbation theory*, 1(1):23–37, 1999.

[9] P. G. Estévez and J. Prada. Hodograph transformations for a Camassa–Holm hierarchy in 2+ 1 dimensions. *Journal of Physics A: Mathematical and General*, 38(6):1287, 2005.

[10] P. G. Estévez and J. Prada. Solutions of a Camassa–Holm hierarchy in 2+ 1 dimensions. *Theoretical and mathematical physics*, 144(2):1132–1137, 2005.

[11] F. Gesztesy and H. Holden. Algebro-geometric solutions of the Camassa–Holm hierarchy. *Revista Matematica Iberoamericana*, 19(1):73–142, 2003.

[12] H. Holden and X. Raynaud. Global conservative solutions of the Camassa–Holm equation-a lagrangian point of view. *Communications in Partial Differential Equations*, 32(10):1511–1549, 2007.

[13] R. S. Johnson. On solutions of the Camassa–Holm equation. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 459, pages 1687–1708. The Royal Society, 2003.

[14] V. Novikov. Generalisations of the Camassa-Holm equation. *J. Phys. A: Math. Theor.*, 42(34):342002, 2009.

[15] P. J. Olver and P. Rosenau. Tri-hamiltonian duality between solitons and solitary-wave solutions having compact support. *Physical Review E*, 53(2):1900, 1996.

[16] Z. Qiao. The Camassa–Holm hierarchy, n-dimensional integrable systems, and algebro-geometric solution on a symplectic submanifold. *Communications in Mathematical Physics*, 239(1-2):309–341, 2003.

[17] Z. Qiao. A new integrable equation with cuspons and W/M-shape-peaks solitons. *Journal of mathematical physics*, 47(11):112701–112900, 2006.

[18] Z. Qiao, B. Xia, and J. Li. Integrable system with peakon, weak kink, and kink-peakon interactional solutions. *arXiv:1205.2028*, 2012.

[19] B. Xia and Z. Qiao. An integrable (2+1)-dimensional Camassa–Holm hierarchy with peakon solutions. *Physica Scripta*, 89(10):105209, 2014.