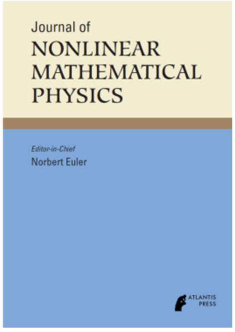

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Riemann–Hilbert method and N–soliton for two–component Gerdjikov-Ivanov equation

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We consider the Riemann–Hilbert method for initial problem of the vector Gerdjikov–Ivanov equation, and obtain the formula for its N-soliton solution, which is expressed as a ratio of $(N+1)\times(N+1)$ determinant and $N\times N$ determinant. Furthermore, by applying asymptotic analysis, the simple elastic interactions of N-soliton are confirmed, and the shifts of phase and position are also explicitly displayed.

Keywords: Riemann–Hilbert problem; vector Gerdjikov-Ivanov equation; asymptotic analysis; soliton.

2000 Mathematics Subject Classification: 35Q55,35C08,35C11,37K40

1. Introduction

The nonlinear Schrödinger equation (NLS) is an important integrable equation, which governs weakly nonlinear and dispersive wave packets in one-dimensional physical systems. It was first derived by Zakharov [26] in his study of modulational stability of deep water waves. Then Hasegawa and Tappert [15] display that the NLS equation governs light pulse propagation in optical fibers. Besides, the same equation will be extended to a vector case [4, 19], if more than one packets of different carrier frequencies appear simultaneously.

In order to study the effect of higher order perturbations, a derivative–type nonlinear equation

$$\mathrm{i}q_t+q_{xx}-\mathrm{i}q^2q_x^*+\frac{1}{2}q^3q^{*2}=0, \tag{1.1}$$

is derived by Gerdjikov-Ivanov [12], which is called the GI equation or DNLSIII equation. Since its discovery, its Darboux transformation [6] , Hamiltonian structures [7], algebra–geometric solutions

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[5], Wronskian type solution [16], soliton [6,25], breather and rogue wave solutions [14,22] have been obtained, even though some of them may be also obtained in theory by taking into account some gauge transformations [18,21]. Actually, the above results give us the results directly and help us to avoid calculating complicated integrals involved in gauge transformations, which may not be evaluated at all.

The GI equation, similar to the NLS equation, also admits a vector case

$$\mathbf{q}_t = \mathrm{i}\mathbf{q}_{xx} + \mathbf{q}_x^\dagger \mathbf{q}^2 + \frac{\mathrm{i}}{2}(\mathbf{q}^\dagger \mathbf{q})^2 \mathbf{q}, \tag{1.2}$$

where $\mathbf{q} = (q_1, q_2, \cdots, q_n)^T$, and the superscript "†" denotes Hermitian conjugate of a matrix or vector. When $n = 2$, it yields

$$q_{1t} = \mathrm{i}q_{1xx} + q_1(q_1 q_{1x}^* + q_2 q_{2x}^*) + \frac{\mathrm{i}}{2} q_1(|q_1|^4 + |q_2|^4) + \mathrm{i}|q_1 q_2|^2 q_1, \tag{1.3a}$$

$$q_{2t} = \mathrm{i}q_{2xx} + q_2(q_1 q_{1x}^* + q_2 q_{2x}^*) + \frac{\mathrm{i}}{2} q_2(|q_1|^4 + |q_2|^4) + \mathrm{i}|q_1 q_2|^2 q_2. \tag{1.3b}$$

In this paper, we will consider the Riemann–Hilbert method for the vector GI (vGI) equation (1.3), and display the determinant expression of its N–soliton solution, which is easy to be used for considering asymptotic behavior.

The Riemann–Hilbert method, derived by Novikov et al. [20], streamlines and simplifies the original inverse scattering transformation [1, 3, 9, 10] based on the Gel'fand–Levitan–Marchenko integral equations. Recently, the Riemann–Hilbert method has been adopted to solve the vector nonlinear equation, 3×3 spectral problems, squared eigenfunctions and so on [2,8,11,13,17,23,24]. In the next section, we will construct the Riemann–Hilbert problem based on the Jost solutions to the Lax pair of vGI (1.3) equation and scattering data $S(\lambda)$ (2.6). In § 3, we discuss solutions to the regular and non–regular Riemann–Hilbert problems by applying Plemelj formula. In § 4, the determinant expressions of N–soliton solutions to the vGI equation are obtained, as well as asymptotic behaviors. Also, the expression of one–soliton is displayed explicitly. The conclusions are given in the final section.

2. The construction of Riemann–Hilbert problem

The vector GI equation (1.3) is the compatibility condition of the following two linear equations:

$$\Phi_x = U\Phi = (-\mathrm{i}\lambda^2\sigma - \lambda Q\sigma + \frac{\mathrm{i}}{2}Q^2\sigma)\Phi, \tag{2.1a}$$

$$\Phi_t = V\Phi = (-2\mathrm{i}\lambda^4\sigma - 2\lambda^3 Q\sigma + \mathrm{i}\lambda^2\sigma Q^2 + \mathrm{i}\lambda Q_x - \frac{1}{2}[Q_x, Q] + \frac{\mathrm{i}}{4}Q^4\sigma)\Phi, \tag{2.1b}$$

with

$$Q = \begin{pmatrix} 0 & q_1 & q_2 \\ q_1^* & 0 & 0 \\ q_2^* & 0 & 0 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Here λ is a spectral parameter, $\Phi(x,t,\lambda)$ is a vector or matrix function, the superscript "*" represents complex conjugation, and $[A, B]$ denotes $AB - BA$. In our analysis, we assume that

$$q_1(x,0) \to 0, \quad q_2(x,0) \to 0, \qquad \text{as} \qquad x \to \infty, \tag{2.2}$$

which belong to Schwartz space. Obviously, $\hat{E} = \mathrm{e}^{-\mathrm{i}(\lambda^2\sigma x+2\lambda^4\sigma t)}$ is a solution for the linear equations (2.1) at this time. Let $\Phi = J\hat{E}$, then the spectral problems about $J(x,t,\lambda)$ are defined as

$$J_x + \mathrm{i}\lambda^2[\sigma, J] = \hat{U}J = (-\lambda Q\sigma + \frac{\mathrm{i}}{2}Q^2\sigma)J, \tag{2.3a}$$

$$J_t + 2\mathrm{i}\lambda^4[\sigma, J] = \hat{V}J = (-2\lambda^3 Q\sigma + \mathrm{i}\lambda^2\sigma Q^2 + \mathrm{i}\lambda Q_x - \frac{1}{2}[Q_x, Q] + \frac{\mathrm{i}}{4}Q^4\sigma)J, \tag{2.3b}$$

Under boundary conditions

$$J_\pm \to I, \qquad x \to \pm\infty, \tag{2.4}$$

the Jost solutions for the spectral problem (2.3a) can be solved as

$$J_\pm(x,\lambda) = I + \int_{\pm\infty}^{x} \mathrm{e}^{-\mathrm{i}\lambda^2\sigma(x-y)}\hat{U}(y)J_\pm(y)\mathrm{e}^{\mathrm{i}\lambda^2\sigma(x-y)}\mathrm{d}y. \tag{2.5}$$

Let $[J_\pm]_k$ denote the k-th column vector of $J_\pm$, then it leads that, after simple analysis, $[J_-]_1$, $[J_+]_2$ and $[J_+]_3$ are analytic for $\lambda \in C_+$ and continuous for $\lambda \in C_+ \cup \mathbb{R} \cup \mathrm{i}\mathbb{R}$, and $[J_+]_1$, $[J_-]_2$ and $[J_-]_3$ are analytic for $\lambda \in C_-$ and continuous for $\lambda \in C_- \cup \mathbb{R} \cup \mathrm{i}\mathbb{R}$, where

$$C_+ = \left\{\lambda \,|\, \arg\lambda \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})\right\}, \qquad C_- = \left\{\lambda \,|\, \arg\lambda \in (\frac{\pi}{2}, \pi) \cup (\frac{3\pi}{2}, 2\pi)\right\},$$

which are displayed explicitly in Fig. 1.

Defining $E = \mathrm{e}^{-\mathrm{i}\lambda^2\sigma x}$, then J_-E and J_+E are different solutions for the linear equation (2.1a), so they are not independent and are linearly related by a scattering matrix $S(\lambda)$:

$$J_-E = J_+ES(\lambda), \qquad \lambda \in \mathbb{R} \cup \mathrm{i}\mathbb{R}, \tag{2.6}$$

or

$$J_- = J_+ES(\lambda)E^{-1}, \qquad \lambda \in \mathbb{R} \cup \mathrm{i}\mathbb{R}. \tag{2.7}$$

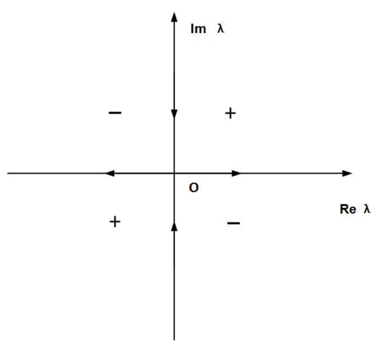

Fig. 1. The jump contour in the complex λ-plane. The positive (negative) side lies on the left (right) as one traverses the contour.

Owing to the Abel's identity and $\mathrm{tr}(Q) = 0$, the determinants of $J_\pm$ are constants for all x. Then using the boundary conditions (2.4), we have

$$\det J_\pm = 1.$$

According to (2.7), $\det S(\lambda) = 1$ is obtained. Furthermore, let x go to $+\infty$, $S(\lambda)$ is given as

$$S(\lambda) = (s_{ij})_{3\times 3} = \lim_{x\to+\infty} E^{-1}J_-E = I + \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\lambda^2\sigma y}\hat{U}J_-\mathrm{e}^{-\mathrm{i}\lambda^2\sigma y}\mathrm{d}y, \qquad \lambda \in \mathbb{R}\cup\mathrm{i}\mathbb{R}.$$

Based on the analytic property of J_-, s_{11} is analytic extension to C_+, while s_{22}, s_{23}, s_{32} and s_{33} are analytic extension to C_-.

In order to obtain behavior of Jost solution for very large λ, we consider the following expansion

$$J = J_0 + \frac{J_1}{\lambda} + \frac{J_2}{\lambda^2} + \frac{J_3}{\lambda^3} + O(\frac{1}{\lambda^4}), \tag{2.8}$$

and substitute it into the spectral problem (2.3a). By comparing the coefficients of λ, it leads to

$$[\sigma, J_0] = 0, \qquad \mathrm{i}[\sigma, J_1] + Q\sigma J_0 = 0, \qquad J_{0x} = \frac{\mathrm{i}}{2}Q^2\sigma J_0 - Q\sigma J_1 - \mathrm{i}[\sigma, J_2], \tag{2.9}$$

which imply

$$\mathrm{i}[\sigma, J_1] = -Q\sigma J_0, \qquad J_{0x} = 0. \tag{2.10}$$

If there exists a solution Q of the vector GI, and set $J_0 = I$ without loss of generality, then it is given by

$$Q = \mathrm{i}\sigma[\sigma, J_1] = \mathrm{i}(J_1 - \sigma J_1 \sigma). \tag{2.11}$$

To construct the Riemann–Hilbert problem, we define a new Jost solution for (2.3a) as

$$P_+ = ([J_-]_1, [J_+]_2, [J_+]_3) = J_+ES_+E^{-1} = J_+E\begin{pmatrix} s_{11} & 0 & 0 \\ s_{21} & 1 & 0 \\ s_{31} & 0 & 1 \end{pmatrix}E^{-1}, \tag{2.12}$$

which is analytic for $\lambda \in C_+$ and possesses asymptotic behavior for very large λ as

$$P_+ \to I, \qquad \lambda \in C_+ \to +\infty. \tag{2.13}$$

Indeed, the analytic counterpart of P_+ in C_-, denoted by P_-, can be derived from the adjoint scattering equation of (2.3a):

$$K_x + \mathrm{i}\lambda^2[\sigma, K] = -K\hat{U}. \tag{2.14}$$

It is easy to see that $J_\pm^{-1}$ solve the above adjoint equation (2.14) and satisfy the boundary conditions $J_\pm^{-1} \to I$ as $x \to \pm\infty$ respectively. Taking the similar procedure as above and denoting the k-th row vector of $J_{\pm}^{-1}$ as $[J_{\pm}^{-1}]^k$ for convenience, the desired P_- is expressed as follows:

$$P_- = \begin{pmatrix} [J_-^{-1}]^1 \\ [J_+^{-1}]^2 \\ [J_+^{-1}]^3 \end{pmatrix}. \tag{2.15}$$

That is, P_- is analytic in $\lambda \in C_-$, and goes to I as $\lambda \to -\infty$. Also, assuming $R(\lambda) = S^{-1}(\lambda)$, then

$$J_-^{-1} = ERE^{-1}J_+^{-1}, \tag{2.16}$$

and

$$P_- = ER_+E^{-1}J_+^{-1} = E \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} E^{-1}J_+^{-1}. \tag{2.17}$$

Hence, we have constructed two matrix functions $P_{\pm}(x,\lambda)$ which are analytic for $\lambda \in C_{\pm}$ respectively. Furthermore, these two functions can construct a Riemann–Hilbert problem as follows:

$$P_-P_+ = G(x,\lambda) = ER_+S_+E^{-1} = E \begin{pmatrix} 1 & r_{12} & r_{13} \\ s_{21} & 1 & 0 \\ s_{31} & 0 & 1 \end{pmatrix} E^{-1}, \qquad \lambda \in \mathbb{R} \cup \mathrm{i}\mathbb{R}. \tag{2.18}$$

Here we have adopted the identity $s_{11}r_{11} + s_{12}r_{21} + s_{13}r_{31} = 1$, and the jump contour is given in Fig. 1.

At the end of this section, we consider the time evolution of the scattering matrices $S(\lambda)$ and $R(\lambda)$. Due to J_- solving scattering problem (2.3b), inserting $J_- = J_+ESE^{-1}$ into (2.3b), taking the limit $x \to +\infty$, and taking into account the boundary condition of J_+ as well as the fact that $V \to 0$ as $x \to +\infty$, we obtain

$$S_t = -2\mathrm{i}\lambda^4[\sigma, S]. \tag{2.19}$$

And then the time evolution of $R(\lambda)$ can be gotten immediately

$$R_t = -2\mathrm{i}\lambda^4[\sigma, R]. \tag{2.20}$$

These two equations show that s_{11}, r_{11} are time independent, and

$$r_{12}(t,\lambda) = r_{12}(0,\lambda)\mathrm{e}^{-4\mathrm{i}\lambda^4 t}, \quad s_{21}(t,\lambda) = s_{21}(0,\lambda)\mathrm{e}^{4\mathrm{i}\lambda^4 t}, \tag{2.21}$$

$$r_{13}(t,\lambda) = r_{13}(0,\lambda)\mathrm{e}^{-4\mathrm{i}\lambda^4 t}, \quad s_{31}(t,\lambda) = s_{31}(0,\lambda)\mathrm{e}^{4\mathrm{i}\lambda^4 t}. \tag{2.22}$$

We have obtained the time evolution of scattering matrices. Later on, the non–regular Riemann–Hilbert problem will be solved by using these scattering data. Furthermore, the inverse problem will be also considered, and the solution Q will be constructed from the solution of non–regular Riemann–Hilbert problem.

3. Solution for Riemann–Hilbert problem

The Riemann–Hilbert problem (2.18) constructed in above section is regular when $\det(P_+) = s_{11} \neq 0$ and $\det(P_-) = r_{11} \neq 0$ for all λ, and is non–regular when $\det(P_+)$ and $\det(P_-)$ can be zero at certain discrete locations of λ. In fact, a non–regular Riemann–Hilbert problem can be transformed into a regular one, thus we consider the regular case at first.

3.1. *Solution for a regular Riemann–Hilbert problem*

From (2.18), we have

$$P_{+}^{-1}-P_{-}=\hat{G}P_{+}^{-1}=(I-G)P_{+}^{-1}, \qquad \lambda\in\mathbb{R}\cup\mathrm{i}\mathbb{R}, \tag{3.1}$$

with the canonical normalization condition

$$P_{\pm}\to I, \qquad |\lambda|\to+\infty. \tag{3.2}$$

Taking into account the Plemelj formula [2], the solution for above Riemann–Hilbert problem is solved as

$$P_{+}^{-1}(\lambda)=I+\frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\frac{\hat{G}(\xi)P_{+}^{-1}(\xi)}{\xi-\lambda}\mathrm{d}\xi, \tag{3.3}$$

with $\Gamma=[0,+\infty)\cup[0,-\infty)\cup(+\mathrm{i}\infty,0]\cup(-\mathrm{i}\infty,0]$.

In what follows, we shall show that the solution to this regular Riemann–Hilbert problem is unique. It can be proved as follows. Set $P_{\pm}$ and $\tilde{P}_{\pm}$ are two set solutions we desired. Then

$$P_{-}(\lambda)P_{+}(\lambda)=\tilde{P}_{-}(\lambda)\tilde{P}_{+}(\lambda), \tag{3.4}$$

which yields

$$\tilde{P}_{-}^{-1}(\lambda)P_{-}(\lambda)=\tilde{P}_{+}(\lambda)P_{+}^{-1}(\lambda), \qquad \lambda\in\mathbb{R}\cup\mathrm{i}\mathbb{R}. \tag{3.5}$$

Since $\tilde{P}_{-}^{-1}(\lambda)P_{-}(\lambda)$ and $\tilde{P}_{+}(\lambda)P_{+}^{-1}(\lambda)$ are analytic in C_{-} and C_{+} respectively, and they are equal to each other on $\mathbb{R}\cup\mathrm{i}\mathbb{R}$, they together define a matrix function which is analytic in the whole plane of λ. Due to the boundary condition (3.2), we have

$$\tilde{P}_{-}^{-1}(\lambda)P_{-}(\lambda)=\tilde{P}_{+}(\lambda)P_{+}^{-1}(\lambda)=I \tag{3.6}$$

for all λ by applying the Liouville's theorem. That is, $\tilde{P}_{\pm}=P_{\pm}$, which implies the uniqueness of solution to the above Riemann–Hilbert problem (2.18).

3.2. *Solution for a non–regular Riemann–Hilbert problem*

In order to study a non–regular Riemann–Hilbert problem, we shall consider symmetric property of these zero points, which are determined by $s_{11}(\lambda)=0$.

Note that s_{11} and r_{11} are time independent, so the roots of $s_{11}=0$ and $r_{11}=0$ are also time independent. Furthermore, owing to $\sigma Q\sigma=Q$ and $Q^{\dagger}=Q$, it is easy to see that

$$J(x,t,-\lambda)=\sigma J(x,t,\lambda)\sigma, \qquad J^{-1}(x,t,\lambda)=J^{\dagger}(x,t,\lambda^{*}). \tag{3.7}$$

Applying these two reduction conditions to (2.7), then

$$S(-\lambda)=\sigma S(\lambda)\sigma, \qquad S^{\dagger}(\lambda^{*})=R(\lambda). \tag{3.8}$$

It yields that $s_{11}(\lambda)$ is an odd function, so each zero λ_k of $s_{11}(\lambda)$ is accompanied by another zero at $-\lambda_k$. Assuming $s_{11}(\pm\lambda_k)=0\,(k=1,2,\cdots,N)$, then $r_{11}(\pm\lambda_k^{*})=0\,(k=1,2,\cdots,N)$. For simplicity, assuming that all zeroes are simple, the kernels of $P_+(\lambda_k)$ and $P_-(\lambda_k)$ contain only a single vector $|v_k\rangle$ and $\langle v_k|$, respectively,

$$P_+(\lambda_k)|v_k\rangle = 0, \qquad \langle v_k|P_-(\lambda_k^*) = 0, \qquad k = 1,2,\cdots,N. \tag{3.9}$$

Taking the x–derivative and t–derivative to the first equation of (3.9), and recalling the Lax pair equation (2.3), it infers

$$P_+(\lambda_k)\left(|v_k\rangle_x + \mathrm{i}\lambda_k^2\sigma|v_k\rangle\right) = 0, \qquad P_+(\lambda_k)\left(|v_k\rangle_t + 2\mathrm{i}\lambda_k^4\sigma|v_k\rangle\right) = 0. \tag{3.10}$$

Thus,

$$|v_k\rangle = \mathrm{e}^{-\mathrm{i}(\lambda_k^2 x + 2\lambda_k^4 t)\sigma} v_{k0} \mathrm{e}^{\int_{x_0}^x \alpha_k(y)\mathrm{d}y + \int_{t_0}^t \beta_k(\tau)\mathrm{d}\tau}. \tag{3.11}$$

where $\alpha_k(x)$ and $\beta_k(t)$ are two scalar functions.

Based on above results, we have the following theorem for the solution to the non–regular Riemann–Hilbert problem with canonical normalization condition (3.2).

Theorem 1. *The solution to a non–regular Riemann–Hilbert problem* (2.18) *with simple zeroes under the canonical normalization condition* (3.2) *is*

$$P_+(\lambda) = P^+(\lambda)T(\lambda), \qquad P_-(\lambda) = T^{-1}(\lambda)P^-(\lambda), \tag{3.12a}$$

where

$$T(\lambda) = \prod_{k=1}^{k=N} T_j(\lambda_j) = \prod_{k=1}^{k=N}\left(I + \frac{A_j}{\lambda - \lambda_j^*} - \frac{\sigma A_j \sigma}{\lambda + \lambda_j^*}\right),$$
$$T^{-1}(\lambda) = \prod_{k=1}^{k=N} T_j^{-1}(\lambda_j) = \prod_{k=1}^{k=N}\left(I + \frac{A_j^\dagger}{\lambda - \lambda_j} - \frac{\sigma A_j^\dagger \sigma}{\lambda + \lambda_j}\right),$$

$$A_j = \frac{\lambda_j^{*2} - \lambda_j^2}{2}\begin{pmatrix}\alpha_j & 0 & 0\\ 0 & \alpha_j^* & 0\\ 0 & 0 & \alpha_j^*\end{pmatrix}|w_j\rangle\langle w_j|, \qquad \alpha_j^{-1} = \langle w_j|\begin{pmatrix}\lambda_j^* & 0 & 0\\ 0 & \lambda_j & 0\\ 0 & 0 & \lambda_j\end{pmatrix}|w_j\rangle,$$

$|w_j\rangle$ is a column vector and defined by $|w_j\rangle = T_{j-1}(\lambda_j)\cdots T_1(\lambda_j)|v_j\rangle$, and $P^\pm$ is the unique solution to the following regular Riemann–Hilbert problem:

$$P^-(\lambda)P^+(\lambda) = T(\lambda)G(\lambda)T^{-1}(\lambda), \qquad \lambda \in \mathbb{R}\cup\mathrm{i}\mathbb{R}, \tag{3.12b}$$

where $P^\pm(\lambda)$ are analytic in $C_\pm$ respectively, and $P^\pm \to I$ as $\lambda \to \infty$.

Proof. We will use a constructional method to prove the theorem. First, construct a matrix function

$$T_1(\lambda) = I + \frac{A_1}{\lambda - \lambda_1^*} - \frac{\sigma A_1 \sigma}{\lambda + \lambda_1^*},$$

which is meromorphic with two simple pole singularities at $\lambda = \pm\lambda_1^* \in C_-$. After a simple calculation, it is obtained immediately that

$$T_1^{-1}(\lambda) = I + \frac{A_1^\dagger}{\lambda - \lambda_1} - \frac{\sigma A_1^\dagger \sigma}{\lambda + \lambda_1}, \qquad \det T_1(\lambda) = \frac{\lambda^2 - \lambda_1^2}{\lambda^2 - \lambda_1^{*2}}.$$

Moreover, we have $\det\left(P_+(\lambda)T_1^{-1}(\lambda)\right)\neq 0$ when $\lambda=\pm\lambda_1$, and $\det\left(T_1(\lambda)P_-(\lambda)\right)\neq 0$ when $\lambda=\pm\lambda_1^*$. Hence, define

$$T(\lambda)=T_N(\lambda)T_{N-1}(\lambda)\cdots T_1(\lambda),\qquad T^{-1}(\lambda)=T_1^{-1}(\lambda)T_2^{-1}(\lambda)\cdots T_N^{-1}(\lambda),$$

and

$$P^+(\lambda)=P_+(\lambda)T^{-1}(\lambda),\qquad P^-(\lambda)=T(\lambda)P_-(\lambda),\tag{3.13a}$$

with

$$T_k(\lambda)=I+\frac{A_k}{\lambda-\lambda_k^*}-\frac{\sigma A_k\sigma}{\lambda+\lambda_k^*},\qquad T_k^{-1}(\lambda)=I+\frac{A_k^\dagger}{\lambda-\lambda_k}-\frac{\sigma A_k^\dagger\sigma}{\lambda+\lambda_k},\qquad k=2,\cdots,N,$$

a regular Riemann–Hilbert problem is obtained as follows:

$$P^-(\lambda)P^+(\lambda)=T(\lambda)G(\lambda)T^{-1}(\lambda),\qquad \lambda\in\mathbb{R}\cup\mathrm{i}\mathbb{R}.\tag{3.13b}$$

Furthermore, $P^\pm$ have the canonical normalization condition $P^\pm\to I$ as $\lambda\to\infty$. It is just the Riemann–Hilbert problem (3.12b) we need.

To finish the proof, we shall accomplish the mission of solving $T_k(\lambda)$. Due to $T(\lambda)T^{-1}(\lambda)=I$ for all λ, we have

$$\mathrm{Res}_{\lambda=\lambda_k}T(\lambda)T^{-1}(\lambda)=\mathrm{Res}_{\lambda=\lambda_k}T^{-1}(\lambda)T(\lambda)=T(\lambda_k)A_k^\dagger=0,\tag{3.13c}$$

which yields that $A_k^\dagger$ must be one dimensional. On the other hand, based on (3.9) and analytic property of $P^\pm$, we have $T(\lambda_k)|v_k\rangle=0$ and $\langle v_k|T^{-1}(\lambda_k)=0$. This implies that A_k is linear related to $\langle v_k|$. Actually, through direct calculation, A_k can be solved as

$$A_k=\frac{\lambda_k^{*2}-\lambda_k^2}{2}\begin{pmatrix}\alpha_k&0&0\\0&\alpha_k^*&0\\0&0&\alpha_k^*\end{pmatrix}|w_k\rangle\langle w_k|,\qquad \alpha_k^{-1}=\langle w_k|\begin{pmatrix}\lambda_k^*&0&0\\0&\lambda_k&0\\0&0&\lambda_k\end{pmatrix}|w_k\rangle,$$

with $|w_k\rangle=T_{k-1}(\lambda_k)\cdots T_1(\lambda_k)|v_k\rangle$. Hence $T(\lambda)$ and $T^{-1}(\lambda)$ are obtained. This completes the proof. □

4. The inverse problem

Based on (2.11), the potential can be obtained from the asymptotic expansion of Jost solutions as $\lambda\to\infty$. To this end, it is necessary to simplify the expression of $T(\lambda)$ at first. Due to $T_k(\lambda)=\sigma T_k(-\lambda)\sigma$, $T(\lambda)$ and $T^{-1}(\lambda)$ can be expressed by

$$T(\lambda)=I+\sum_{j=1}^N\left(\frac{B_j}{\lambda-\lambda_j^*}-\frac{\sigma B_j\sigma}{\lambda+\lambda_j^*}\right),\qquad T^{-1}(\lambda)=I+\sum_{j=1}^N\left(\frac{B_j^\dagger}{\lambda-\lambda_j}-\frac{\sigma B_j^\dagger\sigma}{\lambda+\lambda_j}\right)\tag{4.1}$$

with $B_j=|z_j\rangle\langle y_j|$. Taking into account the same residue condition (3.13c), it yields that

$$\left[I+\sum_{k=1}^N\left(\frac{|z_k\rangle\langle y_k|}{\lambda_j-\lambda_k^*}-\frac{\sigma|z_k\rangle\langle y_k|\sigma}{\lambda_j+\lambda_k^*}\right)\right]|y_j\rangle=0,\qquad j=1,2,\cdots,N,\tag{4.2}$$

or

$$|y_j\rangle = \sum_{k=1}^{N} \left(\frac{\sigma|z_k\rangle\langle y_k|\sigma|y_j\rangle}{\lambda_j+\lambda_k^*} - \frac{|z_k\rangle\langle y_k|y_j\rangle}{\lambda_j-\lambda_k^*} \right), \qquad j=1,2,\cdots,N. \tag{4.3}$$

Solving the above linear equations, then

$$\begin{pmatrix} |z_1\rangle_1 \\ |z_2\rangle_1 \\ \vdots \\ |z_N\rangle_1 \end{pmatrix} = M^{-1} \begin{pmatrix} |y_1\rangle_1 \\ |y_2\rangle_1 \\ \vdots \\ |y_N\rangle_1 \end{pmatrix}, \quad \begin{pmatrix} |z_1\rangle_2 \\ |z_2\rangle_2 \\ \vdots \\ |z_N\rangle_2 \end{pmatrix} = \hat{M}^{-1} \begin{pmatrix} |y_1\rangle_2 \\ |y_2\rangle_2 \\ \vdots \\ |y_N\rangle_2 \end{pmatrix}, \quad \begin{pmatrix} |z_1\rangle_3 \\ |z_2\rangle_3 \\ \vdots \\ |z_N\rangle_3 \end{pmatrix} = \hat{M}^{-1} \begin{pmatrix} |y_1\rangle_3 \\ |y_2\rangle_3 \\ \vdots \\ |y_N\rangle_3 \end{pmatrix},$$

where $|z_j\rangle_k$ denotes the k-th element of $|z_j\rangle$, $|y_j\rangle$ is equal to $|v_j\rangle$, and entries of $N\times N$ matrices M and $\hat{M}$ are defined as

$$M_{jk} = \frac{\langle y_k|\sigma|y_j\rangle}{\lambda_j+\lambda_k^*} - \frac{\langle y_k|y_j\rangle}{\lambda_j-\lambda_k^*}, \quad \hat{M}_{jk} = -\frac{\langle y_k|\sigma|y_j\rangle}{\lambda_j+\lambda_k^*} - \frac{\langle y_k|y_j\rangle}{\lambda_j-\lambda_k^*}, \quad j,k=1,2,\cdots,N. \tag{4.4}$$

We are in a position to calculate the potential. According to the Plemelj formula [2], the solution of (3.12b) can be expressed as

$$\left(P^+(\lambda)\right)^{-1} = I + \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{T(\xi)\hat{G}(\xi)T^{-1}(\xi)\left(P^+(\xi)\right)^{-1}}{\xi-\lambda} \mathrm{d}\xi, \quad \lambda \in C_+. \tag{4.5}$$

As $\lambda \to \infty$,

$$\left(P^+(\lambda)\right)^{-1} \to I - \frac{1}{2\pi \mathrm{i}\lambda} \int_\Gamma T(\xi)\hat{G}(\xi)T^{-1}(\xi)\left(P^+(\xi)\right)^{-1}\mathrm{d}\xi,$$

and thus

$$P^+(\lambda) \to I + \frac{1}{2\pi \mathrm{i}\lambda} \int_\Gamma T(\xi)\hat{G}(\xi)T^{-1}(\xi)\left(P^+(\xi)\right)^{-1}\mathrm{d}\xi.$$

From (4.1), we see that as $\lambda \to \infty$,

$$T(\lambda) \to I + \frac{1}{\lambda} \sum_{k=1}^{N} \left(|z_k\rangle\langle y_k| - \sigma|z_k\rangle\langle y_k|\sigma\right).$$

So J_1 displayed in (2.11) can be expressed as

$$J_1 = \sum_{k=1}^{N} \left(|z_k\rangle\langle y_k| - \sigma|z_k\rangle\langle y_k|\sigma\right) + \frac{1}{2\pi \mathrm{i}} \int_\Gamma T(\xi)\hat{G}(\xi)T^{-1}(\xi)\left(P^+(\xi)\right)^{-1}\mathrm{d}\xi. \tag{4.6}$$

In order to obtain N-soliton solutions to the vector GI (1.3) equation, set $G = I$, i.e. $r_{12} = r_{13} = s_{21} = s_{31} = 0$, which is called reflection–less. In this case $\hat{G} = 0$, from (2.11) and (4.6), the formula for N-soliton is

$$Q = \mathrm{i}\sigma[\sigma, \sum_{k=1}^{N} \left(|z_k\rangle\langle y_k| - \sigma|z_k\rangle\langle y_k|\sigma\right)], \tag{4.7}$$

which implies

$$q_1 = 4\mathrm{i}\sum_{k=1}^{N} |z_k\rangle_1\langle y_k|_2 = -4\mathrm{i}\frac{\det M_1}{\det M}, \qquad q_2 = 4\mathrm{i}\sum_{k=1}^{N} |z_k\rangle_1\langle y_k|_3 = -4\mathrm{i}\frac{\det M_2}{\det M}. \tag{4.8}$$

Here

$$M_1=\begin{pmatrix} M_{11} & \cdots & M_{1N} & |y_1\rangle_1 \\ \vdots & \cdots & \vdots & \vdots \\ M_{N1} & \cdots & M_{NN} & |y_N\rangle_1 \\ \langle y_1|_2 & \cdots & \langle y_N|_2 & 0 \end{pmatrix}, \qquad M_2=\begin{pmatrix} M_{11} & \cdots & M_{1N} & |y_1\rangle_1 \\ \vdots & \cdots & \vdots & \vdots \\ M_{N1} & \cdots & M_{NN} & |y_N\rangle_1 \\ \langle y_1|_3 & \cdots & \langle y_N|_3 & 0 \end{pmatrix}. \tag{4.9}$$

Based on the dressing method [20], it is straightforward to verify that (4.8) satisfies the vGI equation.

Let $N=1$, $\theta_k=-\mathrm{i}\lambda_k^2x-2\mathrm{i}\lambda_k^4t$ and $v_{k0}=(1,a_k,b_k)^T$, according to (4.8), then it yields

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}=-\frac{2\mathrm{i}(\lambda_1^2-\lambda_1^{*2})\exp(\theta_1-\theta_1^*)}{\lambda_1(|a_1|^2+|b_1|^2)\exp(-\theta_1-\theta_1^*)+\lambda_1^*\exp(\theta_1+\theta_1^*)}\begin{pmatrix} a_1^* \\ b_1^* \end{pmatrix} \tag{4.10}$$

which is the first order soliton solution of the vGI (1.3). Let $a_k=b_k=1$, then

$$q_1=q_2=\frac{-2\mathrm{i}(\lambda_1^2-\lambda_1^{*2})}{2\lambda_1\exp(-2\theta_{1,R})+\lambda_1^*\exp(2\theta_{1,R})}\exp(2\mathrm{i}\theta_{1,I}), \tag{4.11}$$

where the subscript "R" and "I" denote the real part and the imaginary part, respectively. For θ_1, its real and imaginary parts are displayed as

$$\begin{aligned} \theta_{1,R}&=2\lambda_{1,R}\lambda_{1,I}\left[x+4(\lambda_{1,R}^2-\lambda_{1,I}^2)t\right], \\ \theta_{1,I}&=-(\lambda_{1,R}^2-\lambda_{1,I}^2)x-(2\lambda_{1,R}^4-12\lambda_{1,R}^2\lambda_{1,I}^2+2\lambda_{1,I}^4)t. \end{aligned}$$

When $x+4(\lambda_{1,R}^2-\lambda_{1,I}^2)t=0$, one–soliton reaches to its amplitude $|\frac{8\lambda_{1,R}\lambda_{1,I}}{3\lambda_{1,R}+\mathrm{i}\lambda_{1,I}}|$. Besides, it is found that $\alpha(x)$ and $\beta(t)$ are eliminated automatically interior the calculation, so set $\alpha(x)=\beta(t)=0$ below without loss of generality. Inspired by the elastic collisions of multi–soliton, we have the following theorem for the interactions of N solitons.

Theorem 2. *Set* $\mathrm{Im}\lambda_k^2>0\,(k=1,2,\cdots,N)$*, and* $\mathrm{Re}\lambda_N^2<\cdots<\mathrm{Re}\lambda_2^2<\mathrm{Re}\lambda_1^2$*, then N-soliton* (4.8) *has a following simple asymptotic behavior as*

$$q_1\sim\sum_{k=1}^{N}\frac{-2\mathrm{i}(\lambda_k^2-\lambda_k^{*2})\exp(2\mathrm{i}\theta_{k,I}\pm\mathrm{i}\varphi_{k,I}\pm\mathrm{i}\psi_{k,I})}{2\lambda_k\exp(-2\theta_{k,R}\mp\varphi_{k,R}\pm\psi_{k,R})+\lambda_k^*\exp(2\theta_{k,R}\pm\varphi_{k,R}\mp\psi_{k,R})} \tag{4.12}$$

as $t\to\pm\infty$*, where*

$$\varphi_k=\sum_{l=1}^{k-1}\ln\left|\frac{\lambda_k^2-\lambda_l^2}{\lambda_k^2-\lambda_l^{*2}}\right|+\mathrm{i}\sum_{l=1}^{k-1}\arg\frac{\lambda_k^2-\lambda_l^2}{\lambda_k^2-\lambda_l^{*2}},\quad \psi_k=\sum_{j=k+1}^{N}\ln\left|\frac{\lambda_j^{*2}-\lambda_k^{*2}}{\lambda_j^2-\lambda_k^{*2}}\right|+\mathrm{i}\sum_{j=k+1}^{N}\arg\frac{\lambda_j^{*2}-\lambda_k^{*2}}{\lambda_j^2-\lambda_k^{*2}}.$$

Proof. Since $\mathrm{Im}\lambda_k^2>0\,(k=1,2,\cdots,N)$, the asymptotic behavior of $\exp(-\theta_k)$ is decided by $x+4(\mathrm{Re}\,\lambda_k^2)t$. Denoting the vicinity of $x=-4(\mathrm{Re}\,\lambda_k^2)t$ as Ω_k, in the limit of $t\to+\infty$, these vicinities are separated from each other. In the vicinity of Ω_k,

$$\begin{aligned} x+4(\mathrm{Re}\,\lambda_j^2)t&=-4(\mathrm{Re}\,\lambda_k^2-\mathrm{Re}\,\lambda_j^2)t\to+\infty \qquad j<k, \\ x+4(\mathrm{Re}\,\lambda_j^2)t&=-4(\mathrm{Re}\,\lambda_k^2-\mathrm{Re}\,\lambda_j^2)t\to-\infty \qquad j>k. \end{aligned}$$

That is,

$$\exp(-\theta_j)\to0 \qquad j<k; \qquad \exp(\theta_j)\to0 \qquad j>k.$$

Thus, in the vicinity of Ω_k,

$$
\begin{aligned}
\det M &= \begin{vmatrix}
\frac{-2\lambda_1^*}{\lambda_1^2-\lambda_1^{*2}} & \cdots & \frac{-2\lambda_{k-1}^*}{\lambda_1^2-\lambda_{k-1}^{*2}} & \frac{-2\lambda_k^*}{\lambda_1^2-\lambda_k^{*2}} & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{-2\lambda_1^*}{\lambda_{k-1}^2-\lambda_1^{*2}} & \cdots & \frac{-2\lambda_{k-1}^*}{\lambda_{k-1}^2-\lambda_{k-1}^{*2}} & \frac{-2\lambda_k^*}{\lambda_{k-1}^2-\lambda_k^{*2}} & 0 & \cdots & 0 \\
\frac{-2\lambda_1^*}{\lambda_k^2-\lambda_1^{*2}} & \cdots & \frac{-2\lambda_{k-1}^*}{\lambda_k^2-\lambda_{k-1}^{*2}} & \frac{-2\lambda_k^*-4\lambda_k \mathrm{e}^{-2\theta_k-2\theta_k^*}}{\lambda_k^2-\lambda_k^{*2}} & \frac{-4\lambda_k \mathrm{e}^{-2\theta_k}}{\lambda_k^2-\lambda_{k+1}^{*2}} & \cdots & \frac{-4\lambda_k \mathrm{e}^{-2\theta_k}}{\lambda_k^2-\lambda_N^{*2}} \\
0 & \cdots & 0 & \frac{-4\lambda_{k+1} \mathrm{e}^{-2\theta_k^*}}{\lambda_{k+1}^2-\lambda_k^{*2}} & \frac{-4\lambda_{k+1}}{\lambda_{k+1}^2-\lambda_{k+1}^{*2}} & \cdots & \frac{-4\lambda_{k+1}}{\lambda_{k+1}^2-\lambda_N^{*2}} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & \frac{-4\lambda_N \mathrm{e}^{-2\theta_k^*}}{\lambda_N^2-\lambda_k^{*2}} & \frac{-4\lambda_N}{\lambda_N^2-\lambda_{k+1}^{*2}} & \cdots & \frac{-4\lambda_N}{\lambda_N^2-\lambda_N^{*2}}
\end{vmatrix} \\
&\quad \times \exp\left[\sum_{l=1}^{k}(\theta_l+\theta_l^*) - \sum_{j=k+1}^{N}(\theta_j+\theta_j^*)\right] \\
&= \left[\begin{vmatrix}
\frac{-2\lambda_1^*}{\lambda_1^2-\lambda_1^{*2}} & \cdots & \frac{-2\lambda_{k-1}^*}{\lambda_1^2-\lambda_{k-1}^{*2}} \\
\vdots & & \vdots \\
\frac{-2\lambda_1^*}{\lambda_{k-1}^2-\lambda_1^{*2}} & \cdots & \frac{-2\lambda_{k-1}^2}{\lambda_{k-1}^2-\lambda_{k-1}^{*2}}
\end{vmatrix}
\begin{vmatrix}
\frac{-4\lambda_k \mathrm{e}^{-2\theta_k-2\theta_k^*}}{\lambda_k^2-\lambda_k^{*2}} & \frac{-4\lambda_k \mathrm{e}^{-2\theta_k}}{\lambda_k^2-\lambda_{k+1}^{*2}} & \cdots & \frac{-4\lambda_k \mathrm{e}^{-2\theta_k}}{\lambda_k^2-\lambda_N^{*2}} \\
\frac{-4\lambda_{k+1} \mathrm{e}^{-2\theta_k^*}}{\lambda_{k+1}^2-\lambda_k^{*2}} & \frac{-4\lambda_{k+1}}{\lambda_{k+1}^2-\lambda_{k+1}^{*2}} & \cdots & \frac{-4\lambda_{k+1}}{\lambda_{k+1}^2-\lambda_N^{*2}} \\
\vdots & \vdots & & \vdots \\
\frac{-4\lambda_N \mathrm{e}^{-2\theta_k^*}}{\lambda_N^2-\lambda_k^{*2}} & \frac{-4\lambda_N}{\lambda_N^2-\lambda_{k+1}^{*2}} & \cdots & \frac{-4\lambda_N}{\lambda_N^2-\lambda_N^{*2}}
\end{vmatrix} \right. \\
&\quad + \left. \begin{vmatrix}
\frac{-2\lambda_1^*}{\lambda_1^2-\lambda_1^{*2}} & \cdots & \frac{-2\lambda_k^*}{\lambda_1^2-\lambda_k^{*2}} \\
\vdots & & \vdots \\
\frac{-2\lambda_1^*}{\lambda_k^2-\lambda_1^{*2}} & \cdots & \frac{-2\lambda_k^*}{\lambda_k^2-\lambda_k^{*2}}
\end{vmatrix}
\begin{vmatrix}
\frac{-4\lambda_{k+1}}{\lambda_{k+1}^2-\lambda_{k+1}^{*2}} & \cdots & \frac{-4\lambda_{k+1}}{\lambda_{k+1}^2-\lambda_N^{*2}} \\
\vdots & & \vdots \\
\frac{-4\lambda_N}{\lambda_N^2-\lambda_{k+1}^{*2}} & \cdots & \frac{-4\lambda_N}{\lambda_N^2-\lambda_N^{*2}}
\end{vmatrix} \right] \times \exp\left[\sum_{l=1}^{k}(\theta_l+\theta_l^*) - \sum_{j=k+1}^{N}(\theta_j+\theta_j^*)\right] \\
&= \left[\prod_{j=1}^{k}(-2\lambda_j^*)\prod_{l=k+1}^{N}(-4\lambda_l)C(\lambda_1^2,\cdots,\lambda_k^2)C(\lambda_{k+1}^2,\cdots,\lambda_N^2) + \prod_{j=1}^{k-1}(-2\lambda_j^*)\prod_{l=k}^{N}(-4\lambda_l)\right. \\
&\quad \left. C(\lambda_1^2,\cdots,\lambda_{k-1}^2)C(\lambda_k^2,\cdots,\lambda_N^2)\mathrm{e}^{-2\theta_k-2\theta_k^*}\right] \times \exp\left[\sum_{l=1}^{k}(\theta_l+\theta_l^*) - \sum_{j=k+1}^{N}(\theta_j+\theta_j^*)\right],
\end{aligned}
$$

and

$$
\det M_1 = \begin{vmatrix}
\frac{-2\lambda_1^*}{\lambda_1^2-\lambda_1^{*2}} & \cdots & \frac{-2\lambda_{k-1}^*}{\lambda_1^2-\lambda_{k-1}^{*2}} & \frac{-2\lambda_k^*}{\lambda_1^2-\lambda_k^{*2}} & 0 & \cdots & 0 & 1 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\frac{-2\lambda_1^*}{\lambda_{k-1}^2-\lambda_1^{*2}} & \cdots & \frac{-2\lambda_{k-1}^*}{\lambda_{k-1}^2-\lambda_{k-1}^{*2}} & \frac{-2\lambda_k^*}{\lambda_{k-1}^2-\lambda_k^{*2}} & 0 & \cdots & 0 & 1 \\
\frac{-2\lambda_1^*}{\lambda_k^2-\lambda_1^{*2}} & \cdots & \frac{-2\lambda_{k-1}^*}{\lambda_k^2-\lambda_{k-1}^{*2}} & \frac{-2\lambda_k^*-4\lambda_k \mathrm{e}^{-2\theta_k-2\theta_k^*}}{\lambda_k^2-\lambda_k^{*2}} & \frac{-4\lambda_k \mathrm{e}^{-2\theta_k}}{\lambda_k^2-\lambda_{k+1}^{*2}} & \cdots & \frac{-4\lambda_k \mathrm{e}^{-2\theta_k}}{\lambda_k^2-\lambda_N^{*2}} & 1 \\
0 & \cdots & 0 & \frac{-4\lambda_{k+1} \mathrm{e}^{-2\theta_k^*}}{\lambda_{k+1}^2-\lambda_k^{*2}} & \frac{-4\lambda_{k+1}}{\lambda_{k+1}^2-\lambda_{k+1}^{*2}} & \cdots & \frac{-4\lambda_{k+1}}{\lambda_{k+1}^2-\lambda_N^{*2}} & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & \frac{-4\lambda_N \mathrm{e}^{-2\theta_k^*}}{\lambda_N^2-\lambda_k^{*2}} & \frac{-4\lambda_N}{\lambda_N^2-\lambda_{k+1}^{*2}} & \cdots & \frac{-4\lambda_N}{\lambda_N^2-\lambda_N^{*2}} & 0 \\
0 & \cdots & 0 & \mathrm{e}^{-2\theta_k^*} & 1 & \cdots & 1 & 0
\end{vmatrix}
$$

$$
\begin{aligned}
&\times \exp\left[\sum_{l=1}^{k}(\theta_l+\theta_l^*)-\sum_{j=k+1}^{N}(\theta_j+\theta_j^*)\right]\\
=&(-1)^{N-k+1}\begin{vmatrix} \frac{-2\lambda_1^*}{\lambda_1^2-\lambda_1^{*2}} & \cdots & \frac{-2\lambda_{k-1}^*}{\lambda_1^2-\lambda_{k-1}^{*2}} & 1\\ \vdots & & \vdots & \vdots\\ \frac{-2\lambda_1^*}{\lambda_k^2-\lambda_1^{*2}} & \cdots & \frac{-2\lambda_{k-1}^*}{\lambda_k^2-\lambda_{k-1}^{*2}} & 1 \end{vmatrix}\begin{vmatrix} \frac{-4\lambda_{k+1}e^{-2\theta_k^*}}{\lambda_{k+1}^2-\lambda_k^{*2}} & \frac{-4\lambda_{k+1}}{\lambda_{k+1}^2-\lambda_{k+1}^{*2}} & \cdots & \frac{-4\lambda_{k+1}}{\lambda_{k+1}^2-\lambda_N^{*2}}\\ \vdots & \vdots & & \vdots\\ \frac{-4\lambda_N e^{-2\theta_k^*}}{\lambda_N^2-\lambda_k^{*2}} & \frac{-4\lambda_N}{\lambda_N^2-\lambda_{k+1}^{*2}} & \cdots & \frac{-4\lambda_N}{\lambda_N^2-\lambda_N^{*2}}\\ e^{-2\theta_k^*} & 1 & \cdots & 1 \end{vmatrix}\\
&\times \exp\left[\sum_{l=1}^{k}(\theta_l+\theta_l^*)-\sum_{j=k+1}^{N}(\theta_j+\theta_j^*)\right]\\
=&-\prod_{j=1}^{k-1}\frac{-2\lambda_j^*(\lambda_k^2-\lambda_l^2)}{\lambda_k^2-\lambda_l^{*2}}\prod_{l=k+1}^{N}\frac{-4\lambda_l(\lambda_j^{*2}-\lambda_k^{*2})}{\lambda_j^2-\lambda_k^{*2}}C(\lambda_1^2,\cdots,\lambda_{k-1}^2)C(\lambda_{k+1}^2,\cdots\lambda_N^2)e^{-2\theta_k^*}\\
&\times \exp\left[\sum_{l=1}^{k}(\theta_l+\theta_l^*)-\sum_{j=k+1}^{N}(\theta_j+\theta_j^*)\right],
\end{aligned}
$$

where $C(\lambda_1,\cdots,\lambda_k)$ denotes the determinant of Cauchy matrix $(\frac{1}{\lambda_j^2-\lambda_l^{*2}})_{k\times k}$, $j,l=1,2,\cdots,k$. Let

$$
\varphi_k=\sum_{l=1}^{k-1}\ln\left|\frac{\lambda_k^2-\lambda_l^2}{\lambda_k^2-\lambda_l^{*2}}\right|+\mathrm{i}\sum_{l=1}^{k-1}\arg\frac{\lambda_k^2-\lambda_l^2}{\lambda_k^2-\lambda_l^{*2}},\quad \psi_k=\sum_{j=k+1}^{N}\ln\left|\frac{\lambda_j^{*2}-\lambda_k^{*2}}{\lambda_j^2-\lambda_k^{*2}}\right|+\mathrm{i}\sum_{j=k+1}^{N}\arg\frac{\lambda_j^{*2}-\lambda_k^{*2}}{\lambda_j^2-\lambda_k^{*2}}.
$$

By a simple analysis,

$$
q_1=-4\mathrm{i}\frac{\det M_1}{\det M}\sim\frac{-2\mathrm{i}(\lambda_k^2-\lambda_k^{*2})\exp(2\mathrm{i}\theta_{k,I}+\mathrm{i}\varphi_{k,I}+\mathrm{i}\psi_{k,I})}{2\lambda_k\exp(-2\theta_{k,R}-\varphi_{k,R}+\psi_{k,R})+\lambda_k^*\exp(2\theta_{k,R}+\varphi_{k,R}-\psi_{k,R})}.
$$

When $t\to-\infty$, taking the similar procedure as above, we have

$$
q_1=-4\mathrm{i}\frac{\det M_1}{\det M}\sim\frac{-2\mathrm{i}(\lambda_k^2-\lambda_k^{*2})\exp(2\mathrm{i}\theta_{k,I}-\mathrm{i}\varphi_{k,I}-\mathrm{i}\psi_{k,I})}{2\lambda_k\exp(-2\theta_{k,R}+\varphi_{k,R}-\psi_{k,R})+\lambda_k^*\exp(2\theta_{k,R}-\varphi_{k,R}+\psi_{k,R})}.
$$

Thus, on the whole plane, q_1 has the asymptotic behavior as (4.12). □

The asymptotic behavior (4.12) displays the elastic collisions of multi–soliton. Comparing with the single soliton, the additional phase shifts and displacements, Θ_k and Δ_k, are easily obtained

$$
\Theta_k=\sum_{l=1}^{k-1}\arg\frac{\lambda_k^2-\lambda_l^2}{\lambda_k^2-\lambda_l^{*2}}-\sum_{l=k+1}^{N}\arg\frac{\lambda_k^2-\lambda_l^2}{\lambda_k^2-\lambda_l^{*2}}, \tag{4.13}
$$

$$
\Delta_k=\frac{1}{2\lambda_{kR}\lambda_{kI}}\left[\sum_{l=1}^{k-1}\ln\left|\frac{\lambda_k^2-\lambda_l^2}{\lambda_k^2-\lambda_l^{*2}}\right|-\sum_{l=k+1}^{N}\ln\left|\frac{\lambda_k^2-\lambda_l^2}{\lambda_k^2-\lambda_l^{*2}}\right|\right]. \tag{4.14}
$$

5. Conclusion

Basing on the Jost solutions to the Lax pair of the vGI equation and the scattering matrix $S(\lambda)$, we formulated the corresponding Riemann–Hilbert problem, which admitted simple zero points generated by the roots of $\det s_{11}(\lambda)$. By taking spectral analysis, we found that the zero points were paired, since $\det s_{11}(\lambda)$ was an odd function. In view of the symmetry relations of zero points, we constructed a transformation, which eliminated the zero points and made the Riemann–Hilbert problem be regular. Applying the Plemelj formulae, N–soliton solutions to the vGI equation were obtained from the solutions of Riemann–Hilbert problem with vanishing scattering coefficients, which was just the reflection-less case. Moreover,the asymptotic behavior of N-soliton was also provided, and the simple elastic interactions of multi–soliton were observed directly from it.

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