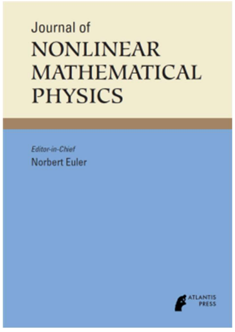

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LETTER TO THE EDITOR

Multipotentializations and nonlocal symmetries: Kupershmidt, Kaup-Kupershmidt and Sawada-Kotera equations

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In this letter we report a new invariant for the Sawada-Kotera equation that is obtained by a systematic potentialization of the Kupershmidt equation. We show that this result can be derived from nonlocal symmetries and that, conversely, a previously known invariant of the Kaup-Kupershmidt equation can be recovered using potentializations.

Dedicated in memory to Wilhelm I Fushchich (1936-1997)

1. Introduction

We report a new invariant for the Sawada-Kotera equation by a systematic potentialization of the Kupershmidt equation. We deduce an invariant of the Kaup-Kupershmidt equation, obtained in [6] with the help of nonlocal symmetries. We furthermore show that the same invariant of the Sawada-Kotera equation can be derived by considering nonlocal symmetries.

For the benefit of clarity we first review some of the results that have been reported in [2] (see also [1] for more details on recursion operators and multipotentialization of semilinear fifth-order evolution equations).

The Kupershmidt equation[a]

$$K_t = K_{xxxxx} + \lambda\left(K_x K_{xxx} + K_{xx}^2\right) - \frac{\lambda^2}{5}\left(K^2 K_{xxx} + 4KK_xK_{xx} + K_x^3\right) + \frac{\lambda^4}{125}K^4K_x \tag{1.1}$$

(λ is an arbitrary non-zero constant) potentializes in the so-called first potential Kupershmidt equation,

$$U_t = U_{xxxxx} + \lambda U_{xx}U_{xxx} - \frac{\lambda^2}{5}\left(U_x^2 U_{xxx} + U_x U_{xx}^2\right) + \left(\frac{\lambda}{5}\right)^4 U_x^5, \tag{1.2}$$

[a]A typing error appeared in this equation in [2] (see eq. (2.7) in [2]), which did however not affect the results reported in [2]

by the potentialization

$$U_x = K. \tag{1.3}$$

Moreover, (1.1) is, by the second potentialization

$$u_x = -\frac{5}{2\lambda} e^{-2\lambda U/5}, \tag{1.4}$$

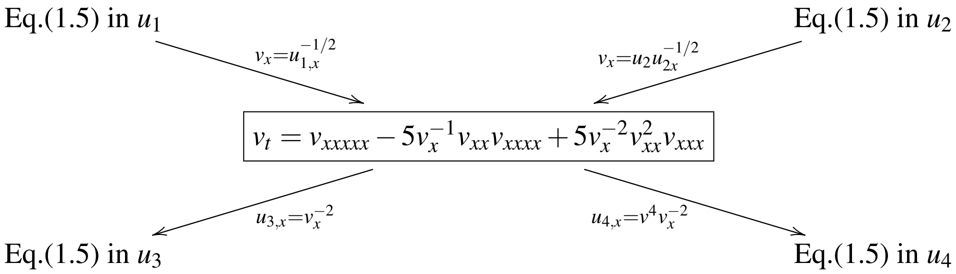
connected to the equation [b]

$$u_t = u_{xxxxx} - \frac{5u_{xx}u_{xxxx}}{u_x} - \frac{15u_{xxx}^2}{4u_x} + \frac{65u_{xx}^2 u_{xxx}}{4u_x^2} - \frac{135u_{xx}^4}{16u_x^3}, \tag{1.5}$$

called the second-potential Kupershmidt equation. Combining the potentializations (1.3) and (1.4), we find that (1.5) and (1.1) are related by the differential substitution

$$K(x,t) = -\frac{5}{2\lambda}\frac{u_{xx}}{u_x}. \tag{1.6}$$

Diagram 1:



Equation (1.5) admits the following △-auto-Bäcklund transformations that are obtained by combining potentializations as shown in Diagram 1 (see [2] for details):

$$u_{j+1,xx} = u_{j+1,x}\left[\frac{u_{j,xx}}{u_{j,x}} - 2\frac{u_{j,x}}{u_j}\right] + 4u_{j+1,x}^{3/4}\left[\frac{u_j^{1/2}}{u_{j,x}^{1/4}}\right] \tag{1.7a}$$

$$u_{j+1,xx} = u_{j+1,x}\left[\frac{u_{j,xx}}{u_{j,x}}\right] + 4u_{j+1,x}^{3/4}\left[\frac{1}{u_{j,x}^{1/4}}\right], \tag{1.7b}$$

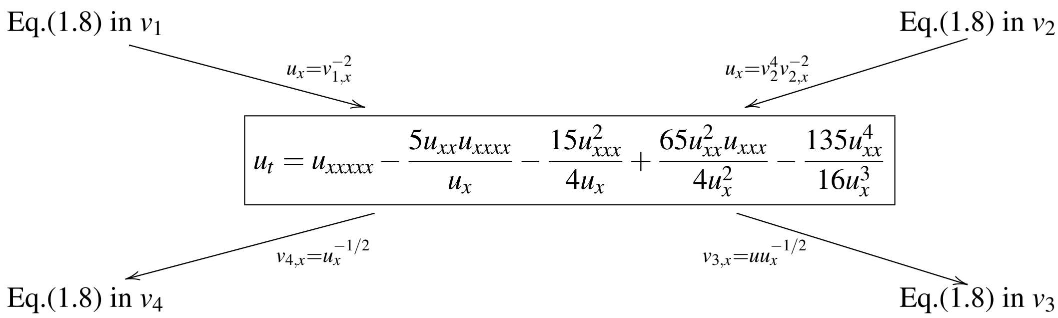
where u_j and u_{j+1} satisfy (1.5) for all natural numbers j. On the other hand, (1.5) is also related to

$$v_t = v_{xxxxx} - 5\frac{v_{xx}v_{xxxx}}{v_x} + 5\frac{v_{xx}^2 v_{xxx}}{v_x^2}, \tag{1.8}$$

as shown in Diagram 2.

[b]A typing error appeared in this equation in [2] (see eq. (2.11) in [2]), which did however not affect the rerults reported in [2]

Diagram 2:



This leads to the $\triangle$-auto-Bäcklund transformations for (1.8) (see Diagram 2 and [2] for details)

$$v_{j+1,xx} = v_{j+1,x}\left[\frac{v_{j,xx}}{v_{j,x}} - 2\frac{v_{j,x}}{v_j}\right] + \frac{v_j^2}{v_{j,x}} \tag{1.9a}$$

$$v_{j+1,xx} = v_{j+1,x}\left[\frac{v_{j,xx}}{v_{j,x}}\right] + \frac{1}{v_{j,x}} \tag{1.9b}$$

$$v_{j+1,xx} = v_{j+1,x}\left[\frac{v_{j,xx}}{v_{j,x}}\right] - \frac{1}{v_{j,x}} \tag{1.9c}$$

where v_j and v_{j+1} satisfy (1.8) for all natural numbers j.

We remark that relation (1.9c) follows by applying the discrete symmetry $v \mapsto -v$ that is admitted by (1.8) to v_j in (1.9b).

2. Connections to the Sawada-Kotera equation

We find that the Sawada-Kotera equation [9]

$$S_t = S_{xxxxx} + \nu S S_{xxx} + \nu S_x S_{xx} + \frac{1}{5}\nu^2 S^2 S_x \tag{2.1}$$

(ν is an arbitrary non-zero constant) is related to the first potential Kupershmidt equation (1.2) by the differential substitution

$$S = -\frac{\lambda}{\nu} U_{xx} - \frac{1}{5}\frac{\lambda^2}{\nu} U_x^2. \tag{2.2}$$

(Note that a similar differential substitution to (2.2) was given in [4]). On the other hand, (1.2) potentializes in (1.5) by

$$u_x = -\frac{5}{2}\frac{1}{\lambda}\exp\left(-\frac{2}{5}\lambda U\right). \tag{2.3}$$

Combining these transformations, we obtain the following

Proposition 1: *The Sawada-Kotera equation (2.1) admits the solutions*

$$S(x,t) = \frac{5}{2\nu}\{u, x\}, \tag{2.4}$$

where $u(x,t)$ is any non-constant solution of the second potential Kupershmidt equation (1.5) and $\{u,\ x\}$ is the Schwarzian derivative

$$\{u, x\} := \left(\frac{u_{xx}}{u_x}\right)_x - \frac{1}{2}\left(\frac{u_{xx}}{u_x}\right)^2. \tag{2.5}$$

Moreover, relation (1.6) implies that solutions of the Sawada-Kotera equation (2.1) are

$$S(x,t) = -\frac{\lambda}{\nu}\left(K_x + \frac{\lambda}{5}K^2\right), \tag{2.6}$$

where $K(x,t)$ is any solution of the Kupershmidt equation (1.1).

The $\triangle$-auto-Bäcklund transformations (1.7a)–(1.7b) can now be applied to generate solutions for (1.5), and hence for (2.1).

We remark that relation (2.6) was previously obtained by Fordy and Gibbons [3] by factorizing a third-order linear operator.

As an example, we apply the $\triangle$-auto-Bäcklund transformation (1.7a), viz.

$$u_{j+1,xx} = u_{j+1,x}\left[\frac{u_{j,xx}}{u_{j,x}} - 2\frac{u_{j,x}}{u_j}\right] + 4\,u_{j+1,x}^{3/4}\left[\frac{u_j^{1/2}}{u_{j,x}^{1/4}}\right].$$

Since (1.7a) is in the form of a Bernoulli equation in the variable $u_{j+1,x}$, we can easily integrate this equation to obtain

$$u_{j+1,x} = \frac{u_{j,x}}{u_j^2}\left[\int\left(\frac{u_j}{u_{j,x}^{1/2}}\right)dx + c_j(t)\right]^4, \tag{2.7}$$

where $c_j(t)$ is an arbitrary function of t that appears as a constant of integration. Inserting (2.7) with $u = u_{j+1,x}$ into (2.4), we find that $c_j(t)$ is an arbitrary constant.

As an explicit example, we use the following seed solution for (1.5):

$$u_1(x,t) = x^5 - 180t.$$

Applying now relation (2.7), with $j = 1$, we obtain

$$u_{2,x}(x,t) = (x^5 - 180t)^{-2}\left[5^{3/4}\left(\frac{1}{20}x^5 + 36t\right) + c_1 x\right]^4. \tag{2.8}$$

Using $u = u_{2,x}$ given by (2.8) in relation (2.4), an explicit solution for (2.1) takes the form

$$S(x,t) = -\frac{30(5x^8 - 14400tx^3 - 40x^4 + 16)}{\nu(x^5 + 720t + 4x)^2},$$

where we have chosen $c_1 = 5^{-1/4}$ for simplicity.

Now we obtain an invariant for the Sawada-Kotera equation (2.1). Applying the two potentializations

$$u_x = v_x^{-2} \tag{2.9a}$$

$$u_x = v^4 v_x^{-2} \tag{2.9b}$$

of (1.8), with the connection to the Sawada-Kotera equation given by (2.4), we obtain the following (see Diagram 3)

Corollary: *The Sawada-Kotera equation (2.1) is invariant under the transformation $S(x,t) \mapsto \bar{S}(x,t)$, in which*

$$\bar{S}(x,t) = S(x,t) + \frac{30}{v}(\ln v)_{xx}, \tag{2.10}$$

where the variables $S(x,t)$ and $v(x,t)$ are related by

$$S(x,t) = -\frac{5}{v}\frac{v_{xxx}}{v_x} \tag{2.11}$$

and $v(x,t)$ is a solution to (1.8), viz.

$$v_t = v_{xxxxx} - 5\frac{v_{xx}v_{xxxx}}{v_x} + 5\frac{v_{xx}^2 v_{xxx}}{v_x^2}.$$

This gives a linearization of (1.8) in terms of the Sawada-Kotera equation, namely

$$v_t = v_{xxxxx} + v\, v_{xx} S_x.$$

Using any of the $\triangle$-auto-Bäcklund transformations (1.9a), (1.9b) or (1.9c), we can construct solutions of the Sawada-Kotera equation (2.1) with the above Corollary.

3. Regarding the Kaup-Kupershmidt equation

In this and the following section we connect the above results with nonlocal symmetries. In the paper [6], Reyes obtained an invariance transformation for the Kaup-Kupershmidt equation

$$V_t = V_{xxxxx} + 5VV_{xxx} + \frac{25}{2}V_xV_{xx} + 5V^2V_x. \tag{3.1}$$

In particular he reported the following

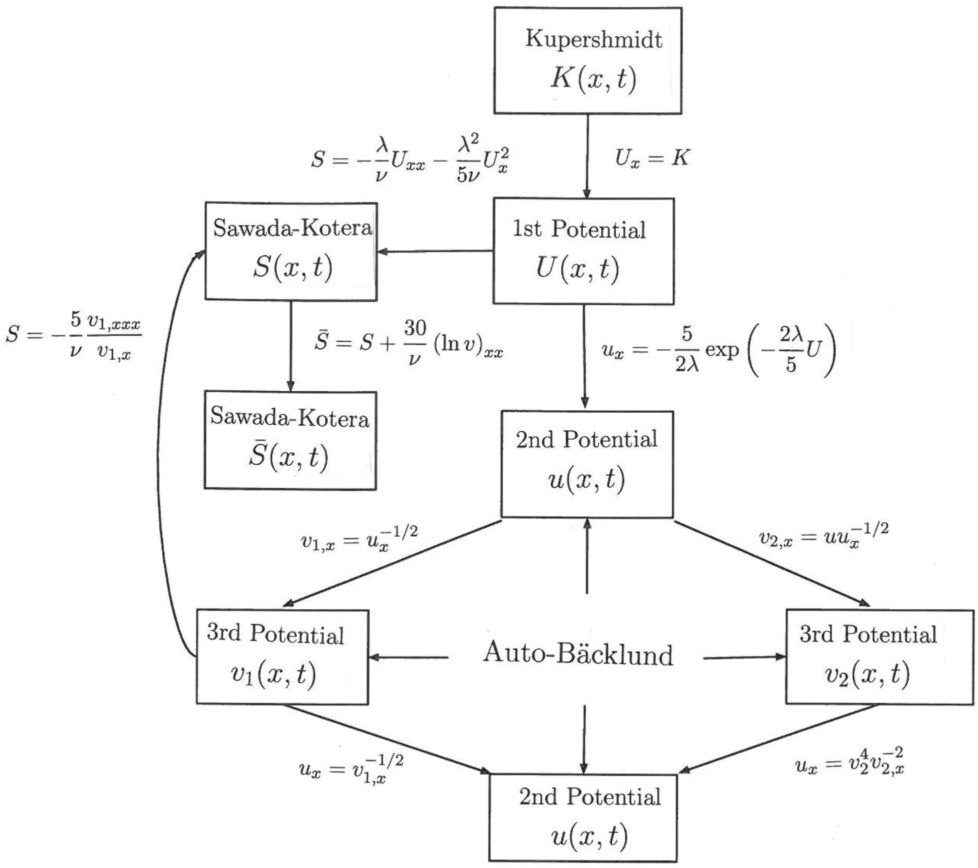


Diagram 3: An invariance transformation for the Sawada-Kotera equation (2.1)

Proposition 2: [6] *The Kaup-Kupershmidt equation (3.1) is invariant under the transformation* $V \mapsto \bar{V}$*, in which*

$$\bar{V} = V + 3(\ln u)_{xx}, \tag{3.2}$$

where the variables V *and* u *are related by*

$$V(x,t) = -\frac{u_{xxx}}{u_x} + \frac{3}{4}\left(\frac{u_{xx}}{u_x}\right)^2 \tag{3.3}$$

and $u(x,t)$ *is a solution to (1.5) viz.*

$$u_t = u_{xxxxx} - \frac{5u_{xx}u_{xxxx}}{u_x} - \frac{15u_{xxx}^2}{4u_x} + \frac{65u_{xx}^2 u_{xxx}}{4u_x^2} - \frac{135u_{xx}^4}{16u_x^3}.$$

This invariance was obtained with the help of nonlocal symmetries: Equation (3.1) admits a nonlocal symmetry whose flow can be explicitly computed; consideration of this flow yields (3.2) and (3.3). We present a related computation in the next section. Now, the result given in Proposition 2 can be obtained by our multipotentialization method, namely in a similar way as was done in the

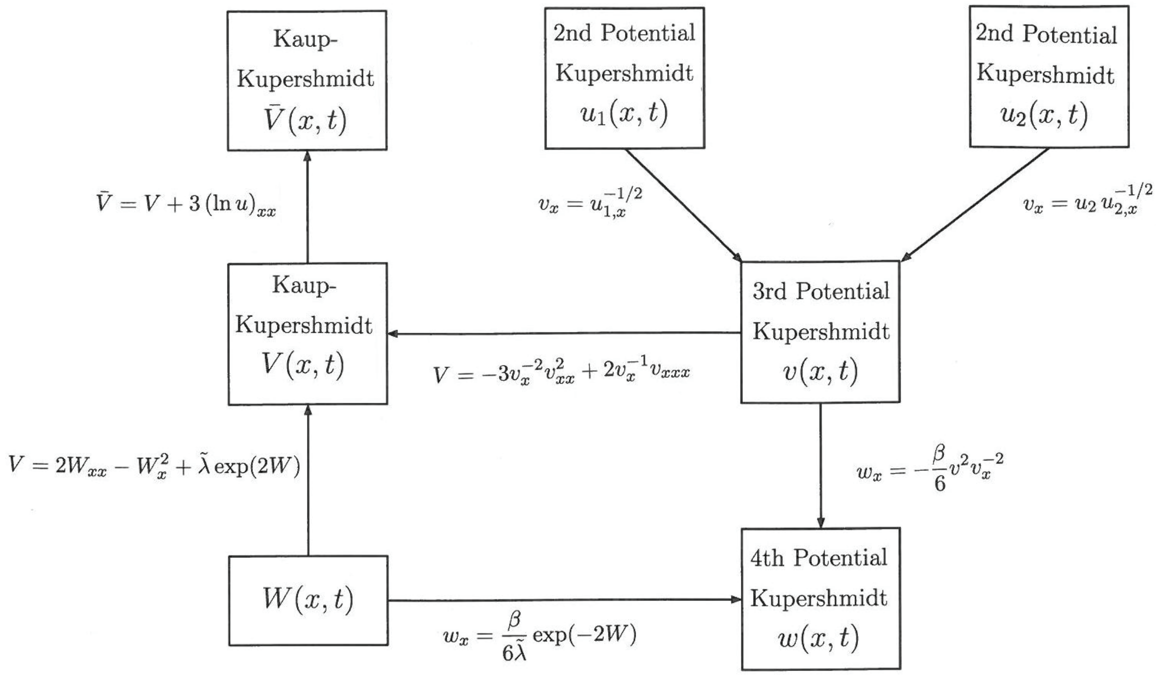


Diagram 4: An invariance transformation for the Kaup-Kupershmidt equation (3.1)

previous section for the Sawada-Kotera equation. Diagram 4 shows the connections between the equations. Besides the second potentialization of the Kupershmidt equation (1.5) (in the variables $u_1(x,t)$ and $u_2(x,t)$) and third potentialization of the Kupershmidt equation (1.8) (in the variable $v(x,t)$), Diagram 4 also includes a fourth potentialization of the Kupershmidt equation, namely

$$w_t = w_{xxxxx} - 5w_x^{-1}w_{xx}w_{xxxx} - \frac{15}{4}w_x^{-1}w_{xxx}^2 + \frac{65}{4}w_x^{-2}w_{xx}^2 w_{xxx} - \frac{135}{16}w_x^{-3}w_{xx}^4$$
$$+\frac{5\beta}{6}\left(w_x^{-1}w_{xxx} - \frac{7}{4}w_x^{-2}w_{xx}^2\right) - \frac{5}{36}\beta^2 w_x^{-1}. \tag{3.4}$$

Furthermore we have the equation

$$W_t = W_{xxxxx} + 5\left(W_{xx} - W_x^2 + \tilde{\lambda}e^{2W}\right)W_{xxx} - 5W_xW_{xx}^2 + 15\tilde{\lambda}e^{2W}W_xW_{xx}$$
$$+W_x^5 + 5\tilde{\lambda}^2 e^{4W}W_x \tag{3.5}$$

which is related to the fourth potential Kupershmidt equation (3.4) by a potentialization of (3.5), namely

$$w_x = \frac{\beta}{6\tilde{\lambda}}\exp(-2W), \tag{3.6}$$

and to the Kaup-Kupershmidt equation (3.1) by the differential substitution (given in [4])

$$V = 2W_{xx} - W_x^2 + \tilde{\lambda}\exp(-2W). \tag{3.7}$$

Combining these change of variables (see Diagram 4), we obtain the differential substitution between the third potential Kupershmidt equation (1.8) and the Kaup-Kupershmidt equation (3.1), namely

$$V(x,t) = -3v_x^{-2}v_{xx}^2 + 2v_x^{-1}v_{xxx}. \tag{3.8}$$

The invariance transformation given in Proposition 2 then follows (see Diagram 4).

4. The Sawada-Kotera invariance via nonlocal symmetries

We show that the invariance transformation (2.10) for Sawada-Kotera can be recovered with the help of nonlocal symmetries.

We replace S for $(5/\nu)S$ in (2.1) and we obtain the Sawada-Kotera equation in the standard form

$$S_t = S_{xxxxx} + 5SS_{xxx} + 5S_xS_{xx} + 5S^2S_x\,. \tag{4.1}$$

This equation is a member of a one-parameter family of equations admitting zero curvature representations. Indeed, we recall from [7, Section 6]:

Proposition 3: [7] *The family of equations*

$$S_t = S_{xxxxx} - \left(4y + \frac{1}{y}\right) S S_{xxx} + 5S^2S_x - \left(2y + \frac{3}{y}\right) S_xS_{xx}\,, \tag{4.2}$$

in which y is a non-zero real parameter, is the integrability condition of the $sl(2,\mathbb{R})$-valued linear problem $X\psi = \psi_x$, $T\psi = \psi_t$ where

$$X = \begin{bmatrix} 0 & -y/\eta^2 \\ -\eta^2 S & 0 \end{bmatrix} \tag{4.3}$$

and

$$T = \begin{bmatrix} yS_{xxx} - SS_x & 2y^2S_{xx}/\eta^2 - yS^2/\eta^2 \\ \eta^2(-S_{xxxx} + (2y+1/y)SS_{xx} + S_x^2/y - S^3) & -yS_{xxx} + SS_x \end{bmatrix}. \tag{4.4}$$

The real number η appearing in (4.3) and (4.4) is not essential, since this "spectral parameter" can be eliminated via a simple gauge transformation. However, this linear problem does encode non-trivial information on Equation (4.2), as we will see below.

The Kaup-Kupershmidt equation corresponds to (4.2) with $y = -1/4$, while the Sawada-Kotera equation (4.1) is (4.2) with $y = -1$. This family contains the fifth order Korteweg-de Vries equation as well (it is enough to take $y = -1/\sqrt{6}$) but we will not use this observation here. Proposition 3 allows us to find a quadratic pseudo-potential for Equation (4.2):

Lemma: *Equation* (4.2) *admits the quadratic pseudo-potential*

$$\alpha_x = -\eta^2 S + \frac{y}{\eta^2}\,\alpha^2 \tag{4.5a}$$

$$\alpha_t = -\eta^2 S_{xxxx} + \left(2\eta^2 y + \frac{\eta^2}{y}\right) S S_{xx} + \frac{\eta^2}{y} S_x^2 - \eta^2 S^3 + (2 S S_x - 2y S_{xxx})\,\alpha + \left(\frac{y}{\eta^2} S^2 - \frac{2y^2}{\eta^2} S_{xx}\right)\alpha^2\,, \tag{4.5b}$$

that is, the system (4.5*a*) *and* (4.5*b*) *is completely integrable for* $\alpha(x,t)$ *whenever* $S(x,t)$ *is a solution to Equation* (4.2).

This result generalizes some interesting computations carried out by Nucci in [5]. We write Equation (4.5b) as a conservation law, and define a corresponding potential δ. We find that δ is determined by the following two compatible equations:

$$\delta_x = \frac{2y}{\eta^2}\,\alpha \tag{4.6a}$$

$$\delta_t = -2y S_{xxx} + 2 S S_x + \left(-\frac{4y^2}{\eta^2} S_{xx} + \frac{2y}{\eta^2} S^2\right)\alpha\,. \tag{4.6b}$$

We would like to find a shadow of a nonlocal symmetry for (4.2), that is, a solution to the formal linearization of (4.2) depending on α and δ. For that, the following theorem, given in [7], where the reader can also find further references on nonlocal symmetries, is essential:

Theorem 1: [7] *Consider the function*

$$G = \alpha \exp(-L(y)\,\delta), \tag{4.7}$$

in which α *and* δ *satisfy Equations* (4.5*a*)-(4.6*b*). *G is the shadow of a nonlocal symmetry for Equation (4.2) if and only if*

$$L(y) = \frac{4y^2+1}{10y^2}\,, \tag{4.8}$$

and the parameter y satisfies the equation

$$-125y^2(96y^6 - 118y^4 - 1 + 23y^2) = 0\,. \tag{4.9}$$

Since y cannot be equal to zero, Equation (4.9) gives exactly six values for which (4.7) is the shadow of a nonlocal symmetry, namely

$$y = 1,\, -1\,;\, \frac{1}{4},\, -\frac{1}{4}\,;\, \frac{1}{\sqrt{6}},\, -\frac{1}{\sqrt{6}}\,.$$

The corresponding values of L are $L(1) = L(-1) = 1/2$, $L(1/4) = L(-1/4) = 2$, and $L(1/\sqrt{6}) = L(-1/\sqrt{6}) = 1$. The equations obtained by replacing these values of y into (4.2) are, respectively, the Sawada-Kotera, Kaup-Kupershmidt, and fifth-order KdV equations! Thus, the shadow (4.7) recognizes precisely the only 2-homogeneous polynomial evolution equations which possess an infinite number of symmetries, from a whole family of equations which are the integrability condition of overdetermined $sl(2,\mathbb{R})$-valued linear problems, and which admit quadratic pseudo-potentials and conservation laws (the notion of λ-homogeneous equations, $\lambda \in \mathbf{R}$, and the classification cited above, appears in the paper [8] by Sanders and Wang).

We complete the shadow G to a bona-fide nonlocal symmetry of the Sawada-Kotera equation following [7]:

Theorem 2: *The system of equations formed by the Sawada-Kotera equation (4.1), (4.5a)–(4.6b) with $y = -1$, and the equations*

$$\beta_x = \frac{2\eta^2}{3} \exp(-\frac{1}{2}\,\delta) \tag{4.10}$$

and

$$\beta_t = \frac{2\,\exp\left(-\dfrac{\delta}{2}\right)}{3\eta^6} \left[3\eta^6 S_x\, \alpha + \eta^8 (S^2 - S_{xx})\right] , \tag{4.11}$$

admits the classical symmetry

$$W = \alpha\,\exp(-\frac{1}{2}\,\delta)\,\frac{\partial}{\partial S} - \frac{\eta^4}{3}\,\exp(-\frac{1}{2}\,\delta)\,\frac{\partial}{\partial \alpha} + \beta\,\frac{\partial}{\partial \delta} - \frac{1}{4}\,\beta^2\,\frac{\partial}{\partial \beta} , \tag{4.12}$$

and therefore this vector field is a nonlocal symmetry of the Sawada-Kotera equation (4.1).

The flow of the vector field (4.12) is found by solving the system of equations

$$\frac{\partial S}{\partial \tau} = \alpha\,\exp(-\frac{1}{2}\,\delta) ; \quad \frac{\partial \alpha}{\partial \tau} = \frac{-\eta^4}{3}\,\exp(-\frac{1}{2}\,\delta) ; \tag{4.13a}$$

$$\frac{\partial \delta}{\partial \tau} = \beta ; \quad \frac{\partial \beta}{\partial \tau} = -\frac{1}{4}\,\beta^2 \tag{4.13b}$$

with initial conditions

$$S(x,t,0) = S_0 ; \quad \alpha(x,t,0) = \alpha_0 ; \quad \delta(x,t,0) = \delta_0 ; \quad \beta(x,t,0) = \beta_0 , \tag{4.14}$$

in which S_0, α_0, δ_0 and β_0 are arbitrary particular solutions to the compatible system of equations given in Theorem 2. The solution to this initial value problem is

$$\alpha(\tau) = -\frac{4\eta^4\tau}{3\beta_0\tau + 12}\exp\left(-\frac{1}{2}\delta\right) + \alpha_0 \tag{4.15a}$$

$$\delta(\tau) = 4\ln\left|\frac{\beta_0\tau + 4}{4}\right| + \delta_0 \tag{4.15b}$$

$$\beta(\tau) = \frac{4\beta_0}{\beta_0\tau + 4} . \tag{4.15c}$$

The corresponding formula for $S(x,t,\tau)$ is obtained from the first equation in (4.13a) by using (4.15a), (4.15b) and the initial conditions above. We find the family of solutions

$$S(x,t,\tau) = -\frac{8\eta^4 e^{-\delta_0}\tau^2}{3\,(\beta_0\,\tau + 4)^2} + \frac{4\tau}{(\beta_0\,\tau + 4)}\,\alpha_0\,e^{-(1/2)\,\delta_0} + S_0 . \tag{4.16}$$

Now we remark that the foregoing analysis allows us to recover transformation (2.10). Indeed, we start from (4.16) and eliminate δ_0 using (4.10). We obtain

$$S = \frac{-6\beta_{0,x}^2\tau^2}{(\beta_0\tau+4)^2} + \frac{6\tau\beta_{0,x}\alpha_0}{\eta^2(\beta_0\tau+4)} + S_0 \,.$$

Now we eliminate α_0 using (4.6a) [with $y=-1$]. We find

$$S = \frac{-6\beta_{0,x}^2\tau^2}{(\beta_0\tau+4)^2} - \frac{3\tau\beta_{0,x}\delta_{0,x}}{\beta_0\tau+4} + S_0 \,.$$

We re-write the second summand of this expression by using the equation

$$\beta_{0,xx} = (-1/2)\beta_{0,x}\delta_{0,x} \,, \tag{4.17}$$

which is obtained by differentiating (4.10) with respect to x and simplifying the result using again (4.10). We obtain

$$S = \frac{-6\beta_{0,x}^2\tau^2}{(\beta_0\tau+4)^2} + \frac{6\tau\beta_{0,xx}}{\beta_0\tau+4} + S_0 \,,$$

or, equivalently,

$$S = 6\,\frac{\partial^2}{\partial x^2}\ln(B) + S_0 \,, \tag{4.18}$$

in which $B=\beta_0\tau+4$. This is exactly transformation (2.10).

We also recover Equation (2.11). Replacing (4.6a) [with $y=-1$] into Equation (4.17) we obtain

$$\beta_{0,xx} = \frac{1}{\eta^2}\,\alpha_0\,\beta_{0,x} \,. \tag{4.19}$$

Differentiating (4.19) with respect to x and using (4.5a) [with $y=-1$] we find

$$\beta_{0,xxx} = \frac{\beta_{0,xx}}{\eta^2}\,\alpha_0 - \beta_{0,x}S_0 - \frac{\beta_{0,x}}{\eta^4}\,\alpha_0^2 \,. \tag{4.20}$$

Now we eliminate α_0 from (4.20) by means of (4.19) and then we simplify the resulting expression. We obtain $S_0 = -\beta_{0,xxx}/\beta_{0,x}$, which is equivalent to Equation (2.11) for B.

Finally, we consider Equation (4.11) for β_0. Using (4.10) we can write (4.11) as

$$\beta_{0,t} = \frac{3\beta_{0,x}S_{0,x}\alpha_0}{\eta^2} + \beta_{0,x}S_0^2 - \beta_{0,x}S_{0,xx} \,.$$

We eliminate α_0 using (4.19), and we eliminate S_0 and its derivatives using the relation $S_0 = -\beta_{0,xxx}/\beta_{0,x}$ and its differential consequences. A straightforward calculation then yields

$$\beta_{0,t} = \frac{-5\beta_{0,xxxx}\beta_{0,xx}}{\beta_{0,x}} + \frac{5\beta_{0,xxx}\beta_{0,xx}^2}{\beta_{0,x}^2} + \beta_{0,xxxxx} \,.$$

This equation is equivalent to Equation (1.8) for B.

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