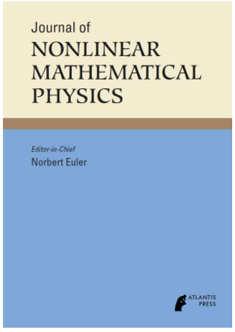

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Additional symmetries and string equations of the noncommutative B and C type KP hierarchies

Qiufang Liu and Chuanzhong Li*
Department of Mathematics, Ningbo University
Ningbo, 315211, Zhejiang, P. R. China

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In this paper, we construct the noncommutative B and C type KP hierarchies using pseudo-differential operators and reducing conditions. Further a series of additional flows of the noncommutative B and C type KP hierarchies will be defined and the additional symmetries constitute the B and C type infinite dimensional Lie algebra $W_{1+\infty}$. In addition, the generating function of the additional symmetries can also be proved to have a nice form in terms of wave functions. Further, the string equations of the noncommutative B and C type KP hierarchies are derived.

Keywords: noncommutative B and C type KP hierarchies, additional symmetry, $W_{1+\infty}$ Lie algebra, String equation.

2000 Mathematics Subject Classification: 37K05, 37K10, 35Q53

1. Introduction

The Kakomtsev-Petviashvili(KP) hierarchy([4], [6]) is one of the most important integrable hierarchy and it arises in many different fields of mathematics and physics such as enumerative algebraic geometry, topological field and string theory. It has perfect structures such as the Virasoro type additional symmetry which has been extensively studied in literature([6]- [19]). From the point of the B type and C type reductions on Lie algebras, the KP hierarchy which corresponds to a A type Lie algebra has two kinds of sub-hierarchies including the B type KP (BKP) hierarchy [2,4,5] and the C type KP (CKP) hierarchy [3].

The noncommutative field theory is a fruitful subject in both mathematics and physics particularly in noncommutative integrable systems [18, 27]. The noncommutative theory gives rise to various new physical objects in quantum mechanics such as the canonical commutation relation $[q, p] = i\hbar$. As in [11], the noncommutative parameter is closely related to the existence of a background flux. Also in the effective theory of D-branes, in the presence of background magnetic fields the noncommutative gauge theories are found to be equivalent to ordinary gauge theories and noncommutative solitons play important roles in the study of D-brane dynamics. These noncommutative theories are known to emerge from limits of M theory and string theory [10].

Similar as the KP hierarchy, the noncommutative KP hierarchy [11, 12, 27] has also two kinds of sub-hierarchies: the noncommutative BKP hierarchy and noncommutative CKP hierarchy [8]. In this paper, the additional symmetries of the noncommutative BKP and CKP hierarchies will be analyzed by using two different ways. In the first one, similar to the additional symmetry flows of

*Corresponding author. E-mail: lichuanzhong@nbu.edu.cn.

the KP hierarchy which was given by Orlov and Shulman([24]), the explicit form of the additional symmetry flows that act on the wave function, or equivalently on the Lax operator L through the Orlov and Shulman operator M, which can form a centerless algebra $W_{1+\infty}$. In the second way, motivated by the results on the KP hierarchy([1], [7]), the BKP and CKP hierarchies([30], [13]), we will derive a nice form of generating functions of the additional symmetries of the noncommutative B and C type KP hierarchies in terms of wave functions.

As we know, the string equation formally as $[P,Q]=1$ which connects the Lax operator and Orlov-Shulman operator is very useful in application on the partition function of the string theory [20]. The possible physical interest in a non-commutative generalization of Orlov's work [25, 26] might be an exciting and interesting subject and this becomes one important motivation for us to consider additional symmetries of noncommutative integrable systems.

For the BKP system, lots of work on the additional symmetries have been done, such as, the additional symmetry of the BKP hierarchy([30]), the additional symmetry of the supersymmetric BKP hierarchy([17]), and the additional symmetry of the constrained BKP hierarchy([29]). About the CKP hierarchy, many studies on the additional symmetries can also be found, for example, the additional symmetry of the CKP hierarchy([13]) and the additional symmetry of the constrained CKP hierarchy([29]). But there is no study on the additional symmetry of noncommutative integrable KP type systems. This is one motivation for us to study the symmetries of the noncommutative KP type hierarchies and their application in noncommutative fields theory.

The organization of this paper is as follows. We firstly review the Lax equation of the noncommutative B and C type KP hierarchies in Section 2. In Section 3, under the basic Sato theory, we construct the additional symmetry of the noncommutative B and C type KP hierarchies. The String equations of the noncommutative B and C type KP hierarchies will be studied in Section 4.

2. The noncommutative B and C type KP hierarchies

As we all know, the noncommutative KP hierarchy is one of the most important topics in the area of classical integrable systems. In the noncommutative system, $\star$ is defined by

$$f\star g=\exp\left(\frac{i}{2}\theta^{uv}\partial_{a^u}\partial_{b^v}\right)f(a)g(b)\mid_{a=b=x}=f(x)g(x)+\frac{i}{2}\theta^{uv}\partial_{x^u}f(x)\partial_{x^v}g(x)+\vartheta(\theta^2)$$

where $\vartheta(\theta^2)$ means the higher order terms. We can get that $[x^u,x^v]_\star=x^u\star x^v-x^v\star x^u=i\theta^{uv}$, and when $\theta^{uv}\to 0$, the noncommutative system can be reduced to the commutative ones. The noncommutative KP hierarchy is constructed by the pseudo-differential operator $L=\partial+u_2\partial^{-1}+u_3\partial^{-2}+\ldots$. like this:

$$L_{t_n}=[B_n,L]_\star:=B_n\star L-L\star B_n,$$

where $B_n=(L^n)_+$ and "+" means the nonnegative projection on powers of ∂. In order to define the noncommutative B and C type KP hierarchies, we need a formal adjoint operation $*$ for an arbitrary pseudo-differential operator $P=\sum_i p_i\star\partial^i$, with $P^*=\sum_i(-1)^i\partial^i\star p_i$. Meanwhile, we have $\partial^*=-\partial$, $(\partial^{-1})^*=-\partial^{-1}$, and $(A\star B)^*=B^*\star A^*$ for two operators. The noncommutative B and C type KP hierarchies are reduced from the noncommutative KP hierarchy by the constraint

$$L^*=-\partial^h\star L\star\partial^{-h},\ \ h=0,1, \tag{2.1}$$

which freezes all even flows of the noncommutative KP hierarchy. The noncommutative B and C type KP hierarchies have only odd flows and the Lax equation has the form

$$\frac{\partial L}{\partial t_{2n+1}} = [B_{2n+1}, L]_\star, \ \ n = 0,1,2,..., \tag{2.2}$$

where $u_i = u_i(t_1, t_3, t_5, ...)$.

When $h = 1$, this hierarchy contains the $(2+1)$ dimensional noncommutative BKP equation:

$$9v_{x,t_5} - 5v_{t_3,t_3} + (v_{xxxxx} + 15v_x \star v_{xxx} + 15v_x^3 - 15v_x \star v_{t_3})_x + 15[v_x, \int [v_{t_3}, v_x]_\star x]_\star = 0, \tag{2.3}$$

where $v = \int u_2$. Let $v_{t_3} = 0$, eq.(2.3) becomes a well-known equation called noncommutative Sawada-Kotera equation ([8, 21, 23, 28])

$$9u_{t_5} + (u_{xxxx} + 15u \star u_{xx} + 15u^3)_x = 0, \tag{2.4}$$

where $u = u_2$.

When $h = 0$, this hierarchy contains the $(2+1)$ dimensional noncommutative CKP equation:

$$9v_{x,t_5} + (v_{xxxxx} + \frac{15}{2}v_x \star v_{xxx} + \frac{15}{2}v_{xxx} \star v_x + 15v_x^3 - 5v_{xx,t_3} - \frac{15}{2}v_x \star v_{t_3} - \frac{15}{2}v_{t_3} \star v_x + \frac{45}{4}v_{xx}^2)_x$$

$$-5v_{t_3 t_3} + 15[v_x, \int [v_{t_3}, v_x]_\star x]_\star = 0, \tag{2.5}$$

where $v = \int u_2$. Let $v_{t_3} = 0$, eq.(2.5) becomes a well-known equation called noncommutative Kaup-Kupershmidt equation ([8, 15, 16, 21, 23])

$$9u_{t_5} + (u_{xxxx} + \frac{15}{2}u \star u_{xx} + \frac{15}{2}u_{xx} \star u + 15u^3 + \frac{45}{4}u_x^2)_x = 0, \tag{2.6}$$

where $u = u_2$.

Now, we can give the noncommutative B and C type KP hierarchies by the consistency conditions of the following set of linear partial differential equations

$$L \star \omega(t,\lambda) = \lambda \star \omega(t,\lambda), \ \ \frac{\partial \omega(t,\lambda)}{\partial t_{2n+1}} = B_{2n+1} \star \omega(t,\lambda), \ \ t = (t_1, t_3, t_5, ...). \tag{2.7}$$

Here, $\omega(t,\lambda)$ is defined as a wave function. Define $\phi = 1 + \sum_{i=1}^{\infty} \omega_i \star \partial^{-i}$ to be the wave operator of the noncommutative B and C type KP hierarchies. The Lax operator and the wave function can be represented as

$$L = \phi \star \partial \star \phi^{-1}, \ \ \omega(t,\lambda) = \phi(t) \star e^{\xi(t,\lambda)}, \tag{2.8}$$

in which $\xi(t,\lambda) = \lambda \star t_1 + \lambda^3 \star t_3 + ... + \lambda^{2n+1} \star t_{2n+1} + ...$.

The Lax equation is equivalent to the following Sato equation

$$\frac{\partial \phi}{\partial t_{2n+1}} = -L_-^{2n+1} \star \phi, \tag{2.9}$$

and the constraints on L in eq.(2.1) can be generalized to the constraints on the dressing operator

$$\phi^* = \partial^h \star \phi^{-1} \star \partial^{-h}, \ \ h = 0, 1. \tag{2.10}$$

Eq.(2.10) is a crucial condition to construct the additional symmetry of the noncommutative B and C type KP hierarchies.

3. Additional symmetries of the noncommutative B and C type KP hierarchies

Firstly, we define the operator Γ and the Orlov-Shulman's operator M as

$$M=\phi\star\Gamma\star\phi^{-1},\ \ \Gamma=\sum_{i=0}^{\infty}(2i+1)t_{2i+1}\star\partial^{2i}. \tag{3.1}$$

Meanwhile, the Orlov-Shulman's operator M satisfy the following identities

$$[L,M]_\star=1,\ \ \partial_{t_{2n+1}}M=[B_{2n+1},M]_\star,\ \ M\star\omega(t,z)=\partial_z\omega(t,z). \tag{3.2}$$

Further, we can get

$$\frac{\partial M^m}{\partial t_{2n+1}}=[B_{2n+1},M^m]_\star,\ \ \frac{\partial M^mL^l}{\partial t_{2n+1}}=[B_{2n+1},M^mL^l]_\star. \tag{3.3}$$

Moreover, by acting on the wave function $\omega(t,z)$, (L,M) is anti-isomorphic to (z,∂_z) with $[z,\partial_z]_\star=-1$ as

$$M^m\star L^l\star\omega(t,z)=z^l\star(\partial_z^m\omega(t,z)),L^l\star M^m\star\omega(t,z)=\partial_z^m(z^l\star\omega(t,z)),m,l\in\mathbb{Z}_+. \tag{3.4}$$

Next, we should consider the adjoint wave function ω^* and the adjoint Orlov-Shulman's operator M^* that are useful for constructing the additional symmetry of the noncommutative B and C type KP hierarchies and we have

$$\omega^*(t,z)=(\phi^*)^{-1}\star e^{-\xi(t,z)}=-z^{-h}\partial_x^h\omega(t,-z),\ \ h=0,1, \tag{3.5}$$

and

$$M^*=(\phi\star\Gamma\star\phi^{-1})^*=(L^*)^{-h}\partial^hM\partial^{-h}(L^*)^h=\partial^hL^{-h}\star M\star L^h\partial^{-h},\ \ h=0,1. \tag{3.6}$$

With the eq.(2.10), we have $\Gamma^*=\Gamma$. However, L^* and M^* satisfy $[L^*,M^*]_\star=[M,L]_\star^*=-1$. Furthermore, we have

$$L^*\star\omega^*=z\star\omega^*,M^*\star\omega^*=-\partial_z\omega^*,\partial_{t_{2n+1}}\omega^*=-B_{2n+1}^*\star\omega^*. \tag{3.7}$$

Next, we give the additional symmetries of the noncommutative B and C type KP hierarchies. Firstly, we introduce additional independent variables $t_{m,l}^*$ and define the action of the additional flows on the wave operator ϕ as

$$\frac{\partial\phi}{\partial t_{m,l}^*}=-(A_{m,l})_-\star\phi, \tag{3.8}$$

in which $A_{m,l}=A_{m,l}(L,M)$ are monomials in L and M and their explicit forms are undetermined. In addition, we give some useful identities in the following proposition.

Proposition 3.1. *The operators L and M acted by the additional flows are defined as*

$$\frac{\partial L}{\partial t_{m,l}^*}=-[(A_{m,l})_-,L]_\star,\frac{\partial M}{\partial t_{m,l}^*}=-[(A_{m,l})_-,M]_\star. \tag{3.9}$$

Proof. Using eq.(2.8) and eq.(3.8), we get

$$\begin{aligned}\partial_{t^*_{m,l}} L &= \partial_{t^*_{m,l}}(\phi \star \partial \star \phi^{-1}) \\ &= (\partial_{t^*_{m,l}}\phi) \star \partial \star \phi^{-1} + \phi \star \partial \star (\partial_{t^*_{m,l}}\phi^{-1}) \\ &= -(A_{m,l})_- \star \phi \star \partial \star \phi^{-1} - \phi \star \partial \star \phi^{-1} \star (\partial_{t^*_{m,l}}\phi) \star \phi^{-1} \\ &= -(A_{m,l})_- \star L + L \star (A_{m,l})_- \\ &= -[(A_{m,l})_-, L]_\star.\end{aligned}$$

With above preparation and $M = \phi \star \Gamma \star \phi^{-1}$, we have

$$\begin{aligned}\partial_{t^*_{m,l}} M &= \partial_{t^*_{m,l}}(\phi \star \Gamma \star \phi^{-1}) \\ &= (\partial_{t^*_{m,l}}\phi) \star \Gamma \star \phi^{-1} + \phi \star \Gamma \star (\partial_{t^*_{m,l}}\phi^{-1}) \\ &= -(A_{m,l})_- \star \phi \star \Gamma \star \phi^{-1} - \phi \star \Gamma \star \phi^{-1} \star (\partial_{t^*_{m,l}}\phi) \star \phi^{-1} \\ &= -(A_{m,l})_- \star M + M \star (A_{m,l})_- \\ &= -[(A_{m,l})_-, M]_\star.\end{aligned}$$

□

Similar to the Proposition 3.1, we will show some identities as in the following corollary.

Corollary 3.1. *The following identities hold true*

$$\frac{\partial L^n}{\partial t^*_{m,l}} = -[(A_{m,l})_-, L^n]_\star, \frac{\partial M^m}{\partial t^*_{m,l}} = -[(A_{m,l})_-, M^m]_\star. \tag{3.10}$$

Proof. Here, we only give the proof of the first equation, the others can be proved by the same way. So we have

$$\begin{aligned}\frac{\partial L^n}{\partial t^*_{m,l}} &= \frac{\partial L}{\partial t^*_{m,l}} \star L^{n-1} + L \star \frac{\partial L}{\partial t^*_{m,l}} \star L^{n-2} + \ldots + L^{n-2} \star \frac{\partial L}{\partial t^*_{m,l}} \star L + L^{n-1} \star \frac{\partial L}{\partial t^*_{m,l}} \\ &= \sum_{k=1}^{n} L^{k-1} \star \frac{\partial L}{\partial t^*_{m,l}} \star L^{n-k}.\end{aligned}$$

Using the formula $\frac{\partial L}{\partial t^*_{m,l}} = -[(A_{m,l})_-, L]_\star$, we can get

$$\frac{\partial L^n}{\partial t^*_{m,l}} = -\sum_{k=1}^{n} L^{k-1} \star [(A_{m,l})_-, L]_\star \star L^{n-k} = -[(A_{m,l})_-, L^n]_\star.$$

□

Now we consider the form of the monomial $A_{m,l}$ of the noncommutative B and C type KP hierarchies in the following proposition.

Proposition 3.2. *By setting*

$$A_{m,l} = M^m \star L^l - (-1)^l L^{l-h} \star M^m \star L^h, \quad h = 0, 1; \tag{3.11}$$

the monomial $A_{m,l}$ of the noncommutative B and C type KP hierarchies satisfy the following condition

$$A^*_{m,l} = -\partial^h \star A_{m,l} \star \partial^{-h}, \quad h = 0, 1. \tag{3.12}$$

Proof. . We can get the action of the additional flows $\partial_{t^*_{m,l}}$ on the adjoint wave operator ϕ^* by two different ways. The first way is that we do a formal adjoint operation on eq.(3.8), so we have

$$\partial_{t^*_{m,l}} \phi^* = -\phi^* \star (A_{m,l})^*_-. \tag{3.13}$$

The second way is that we take a derivative with respect to $t^*_{m,l}$ on ϕ^* with the constraints relation $\phi^* = \partial^h \star \phi^{-1} \star \partial^{-h}$,

$$\begin{aligned}
\partial_{t^*_{m,l}} \phi^* &= \partial^h (\partial_{t^*_{m,l}} \phi^{-1}) \partial^{-h} \\
&= -\partial^h \phi^{-1} \star (\partial_{t^*_{m,l}} \phi) \star \phi^{-1} \partial^{-h} \\
&= \partial^h \phi^{-1} \star (A_{m,l})_- \star \phi \star \phi^{-1} \partial^{-h} \\
&= \partial^h \phi^{-1} \star (A_{m,l})_- \partial^{-h} \\
&= \partial^h \phi^{-1} \partial^{-1} \partial (A_{m,l})_- \partial^{-h} \\
&= \phi^* \partial^h (A_{m,l})_- \partial^{-h}.
\end{aligned}$$

So we get $-\phi^* \star (A_{m,l})^*_- = \phi^* \partial^h (A_{m,l})_- \partial^{-h}$, i.e. $(A_{m,l})^*_- = -\partial^h (A_{m,l})_- \partial^{-h}$ and it is sufficient to let $A^*_{m,l} = -\partial^h \star A_{m,l} \star \partial^{-h}$. Now we will show $A_{m,l} = M^m \star L^l - (-1)^l L^{l-h} \star M^m \star L^h$ satisfy eq.(3.12) as

$$\begin{aligned}
A^*_{m,l} \partial^h &= (M^m \star L^l - (-1)^l L^{l-h} \star M^m \star L^h)^* \partial^h \\
&= ((-1)^l \partial^h L^l \partial^{-h} (L^*)^{-h} \partial^h M^m \partial^{-h} (L^*)^h \\
&\quad -(-1)^l ((-1)^h \partial^h L^h \partial^{-h})(L^*)^{-h} \partial^h M^m \partial^{-h} (L^*)^h (-1)^{l-h} \partial^h L^{l-h} \partial^{-h}) \partial^h \\
&= ((-1)^l \partial^h L^l \partial^{-h} ((-1)^h \partial^h L^h \partial^{-h})^{-h} \partial^h M^m \partial^{-h} ((-1)^h \partial^h L^h \partial^{-h}) - (-1)^l ((-1)^h \partial^h L^h \partial^{-h}) \\
&\quad ((-1)^h \partial^h L^h \partial^{-h})^{-h} \partial^h M^m \partial^{-h} ((-1)^h \partial^h L^h \partial^{-h})(-1)^{l-h} \partial^h L^{l-h} \partial^{-h}) \partial^h \\
&= (-1)^l \partial^h L^{l-h} \star M^m \star L^h - \partial^h M^m \star L^l \\
&= -\partial^h (M^m \star L^l - (-1)^l L^{l-h} \star M^m \star L^h) \\
&= -\partial^h A_{m,l}.
\end{aligned}$$

□

Proposition 3.3. *By acting on the wave operator ϕ or Lax operator L, the additional flows $\partial_{t^*_{m,l}}$ commute with the flows $\partial_{t_{2n+1}}$ of the noncommutative B and C type KP hierarchies, which can be shown as*

$$[\partial_{t^*_{m,l}}, \partial_{t_{2n+1}}]_\star = 0. \tag{3.14}$$

Proof. Firstly we can prove the following identity

$$\frac{\partial A_{n,k}}{\partial t^*_{m,l}} = -[(A_{m,l})_-, A_{n,k}]_\star, \frac{\partial A_{n,k}}{\partial t_{2n+1}} = [B_{2n+1}, A_{n,k}]_\star. \tag{3.15}$$

Let the additional flows $\partial_{t^*_{m,l}}$ and the flows $\partial_{t_{2n+1}}$ act on ϕ, by using eq.(2.9), eq.(3.8) and corollary.(3.1), we can get

$$\begin{aligned}
[\partial_{t^*_{m,l}}, \partial_{t_{2n+1}}]_\star \phi &= \partial_{t^*_{m,l}}(\partial_{t_{2n+1}}\phi) - \partial_{t_{2n+1}}(\partial_{t^*_{m,l}}\phi) \\
&= -\partial_{t^*_{m,l}}(L_-^{2n+1} \star \phi) + \partial_{t_{2n+1}}((A_{m,l})_- \star \phi) \\
&= -(\partial_{t^*_{m,l}} L^{2n+1})_- \star \phi - (L^{2n+1})_- \star (\partial_{t^*_{m,l}}\phi) \\
&\quad +(\partial_{t_{2n+1}} A_{m,l})_- \star \phi + (A_{m,l})_- \star (\partial_{t_{2n+1}}\phi) \\
&= ([(A_{m,l})_-, L^{2n+1}]_\star)_- \star \phi + (L^{2n+1})_- \star (A_{m,l})_- \star \phi \\
&\quad +([(L^{2n+1})_+, A_{m,l}]_\star)_- \star \phi - (A_{m,l})_- \star (L^{2n+1})_- \star \phi \\
&= ([(A_{m,l})_-, L^{2n+1}]_\star)_- \star \phi - ([(A_{m,l})_-, L_+^{2n+1}]_\star)_- \star \phi + [L_-^{2n+1}, (A_{m,l})_-]_\star \star \phi \\
&= ([(A_{m,l})_-, L_-^{2n+1}]_\star)_- \star \phi + [L_-^{2n+1}, (A_{m,l})_-]_\star \star \phi \\
&= 0.
\end{aligned}$$

□

In the process of the above proof, $([L_+^{2n+1}, (A_{m,l})_+]_\star)_- = 0$ and $([L_+^{2n+1}, A_{m,l}]_\star)_- = ([L_+^{2n+1}, (A_{m,l})_-]_\star)_-$ have been used. Therefore, the additional flows $\partial_{t^*_{m,l}}$ are symmetries of the noncommutative B and C type KP hierarchies.

Proposition 3.4. *The additional symmetry flows $\partial_{t^*_{m,l}}$ of the noncommutative B and C type KP hierarchies form the new centerless $W_{1+\infty}^{BC}$, which is a sub-algebra of the centerless $W_{1+\infty}$.*

Proof. By using eq.(3.12), we can easiy get

$$\begin{aligned}
[A_{m,l}, A_{n,k}]_\star^* &= A^*_{n,k} \star A^*_{m,l} - A^*_{m,l} \star A^*_{n,k} \\
&= \partial^h (A_{n,k} \star A_{m,l} - A_{m,l} \star A_{n,k}) \partial^{-h} \\
&= -\partial^h (A_{m,l} \star A_{n,k} - A_{n,k} \star A_{m,l}) \partial^{-h} \\
&= -\partial^h [A_{m,l}, A_{n,k}]_\star \partial^{-h}.
\end{aligned}$$

Thus it has

$$[A_{m,l}, A_{n,k}]_\star = \sum_{p,q} C^{pq}_{nk,ml} A_{p,q}, \tag{3.16}$$

and it implies that

$$([A_{m,l}, A_{n,k}]_\star)_- = -\sum_{p,q} C^{pq}_{nk,ml} (A_{p,q})_-, \tag{3.17}$$

where the coefficient $C^{pq}_{nk,ml}$ is the standard coefficient of the W algebra [24]. Using eq.(3.8) and eq.(3.1), we have

$$[\partial_{t^*_{m,l}}, \partial_{t^*_{n,k}}]_\star \phi = \partial_{t^*_{m,l}}(\partial_{t^*_{n,k}}\phi) - \partial_{t^*_{n,k}}(\partial_{t^*_{m,l}}\phi)$$

$$
\begin{aligned}
&= -\partial_{t^*_{m,l}}((A_{n,k})_- \star \phi) + \partial_{t^*_{n,k}}((A_{m,l})_- \star \phi)\\
&= -(\partial_{t^*_{m,l}}A_{n,k})_- \star \phi - (A_{n,k})_- \star (\partial_{t^*_{m,l}}\phi) + (\partial_{t^*_{n,k}}A_{m,l})_- \star \phi + (A_{m,l})_- \star (\partial_{t^*_{n,k}}\phi)\\
&= ([(A_{m,l})_-, A_{n,k}]_\star)_- \star \phi + (A_{n,k})_- \star (A_{m,l})_- \star \phi\\
&\quad -([(A_{n,k})_-, A_{m,l}]_\star)_- \star \phi - (A_{m,l})_- \star (A_{n,k})_- \star \phi\\
&= ([A_{m,l}, A_{n,k}]_\star)_- \star \phi.
\end{aligned}
$$

Using eq.(3.17), we can get

$$
[\partial_{t^*_{m,l}}, \partial_{t^*_{n,k}}]_\star \phi = -\sum_{p,q} C^{pq}_{nk,ml}(A_{p,q})_- \star \phi = \sum_{p,q} C^{pq}_{nk,ml}(\partial_{t^*_{p,q}}\phi),
$$

which is equal to

$$
[\partial_{t^*_{m,l}}, \partial_{t^*_{n,k}}]_\star = \sum_{p,q} C^{pq}_{nk,ml}\partial_{t^*_{p,q}}.
$$

□

Remark 3.1. When $h = 1$, the additional symmetry flows $\partial_{t^*_{m,l}}$ and the new centerless $W^B_{1+\infty}$ of the noncommutative BKP hierarchy will be derived. When $h = 0$, we will get the additional symmetry flows $\partial_{t^*_{m,l}}$ and the new centerless $W^C_{1+\infty}$ of the noncommutative CKP hierarchy.

To have a better understanding of the additional symmetry flows in noncommutative case, we give a typical example of the noncommutative B and C type KP hierarchies.

Corollary 3.2. *When $l = 1$, we can get $A_{m,1} = M^m \star L + L^{1-h} \star M^m \star L^h$ under the condition of the formula in eq.(3.11), $m \in \mathbb{Z}_+$.*
Let $m = 1$, the corresponding flow on L is

$$
\frac{\partial L}{\partial t^*_{1,1}} = -[(M \star L + L^{1-h} \star M \star L^h)_-, L]_\star = 2L + 2[(L^{1-h} \star M \star L^h)_+, L]_\star, \tag{3.18}
$$

thus

$$
\begin{aligned}
\frac{\partial u_i}{\partial t^*_{1,1}} &= 2\sum_{i=1}^{\infty}(x \star \frac{\partial u_i}{\partial x} + (i+1)u_i) + 2\sum_{j=1}^{\infty}(2j+1)t_{2j+1} \star \frac{\partial u_i}{\partial t_{2j+1}} + 2u_1 \star x - 2x \star u_1\\
&+2\sum_{i=1}^{\infty} x \star (\int u_1 dx) \star u_i - 2\sum_{i=1}^{\infty}(\int u_1 dx) \star x \star u_i\\
&+2\sum_{k+j=i} u_k C^j_{-k}(-1)^j (x \star \frac{\partial^{j-1}u_1}{\partial x} + j\frac{\partial^{j-2}u_1}{\partial x})\\
&-2\sum_{k+j=i} u_k C^j_{-k}(-1)^j (\frac{\partial^{j-1}u_1}{\partial x} \star x + j\frac{\partial^{j-2}u_1}{\partial x}).
\end{aligned}
$$

Proof. Using eq.(2.8) and eq.(3.1), $L^{1-h}\star M\star L^h$ is expressed by

$$L^{1-h}\star M\star L^h=\phi\partial^{1-h}x\partial^h\phi^{-1}+\sum_{i=1}^{\infty}(2i+1)t_{2i+1}\star\phi\partial^{2i+1}\phi^{-1}. \tag{3.19}$$

Furthermore, using $\partial x=x\partial+1$ and $\partial^{-i}x=x\partial^{-i}-i\partial^{-i-1}$, we get

$$(\phi\partial^{1-h}x\partial^h\phi^{-1})_+=\partial^{1-h}x\partial^h+\omega_1\star x-x\star\omega_1, \tag{3.20}$$

with $\phi^{-1}=1-\omega_1\partial^{-1}+...$ being used. Taking eq.(3.20) into eq.(3.19), we have

$$(L^{1-h}\star M\star L^h)_+=\partial^{1-h}x\partial^h+\omega_1\star x-x\star\omega_1+\sum_{i=1}^{\infty}(2i+1)t_{2i+1}\star L_+^{2i+1}. \tag{3.21}$$

Taking eq.(3.21) into eq.(3.18), we have

$$\begin{aligned}\frac{\partial L}{\partial t_{1,1}^*}=&\,2L+2[\partial^{1-h}x\partial^h,L]_\star+2[\sum_{i=1}^{\infty}(2i+1)t_{2i+1}\star L_+^{2i+1},L]_\star\\&+2[\omega_1\star x,L]_\star-2[x\star\omega_1,L]_\star.\end{aligned}$$

We will further get

$$\begin{aligned}\frac{\partial L}{\partial t_{1,1}^*}=&\,2L+2[x\partial,L]_\star+2[\sum_{i=1}^{\infty}(2i+1)t_{2i+1}\star L_+^{2i+1},L]_\star\\&+2[\omega_1\star x,L]_\star-2[x\star\omega_1,L]_\star\\=&\,2\sum_{i=1}^{\infty}(x\star\frac{\partial u_i}{\partial x}+(i+1)u_i)\partial^{-i}+2\sum_{i=1}^{\infty}(2i+1)t_{2i+1}\star\partial_{t_{2i+1}}L\\&+2u_1\star x-2x\star u_1+2\sum_{i=1}^{\infty}x\star(\int u_1dx)\star u_i\partial^{-i}-2\sum_{i=1}^{\infty}(\int u_1dx)\star x\star u_i\partial^{-i}\\&+2\sum_{i=1}^{\infty}u_i(\sum_{j=0}^{\infty}C_{-i}^j(-1)^j(x\star\frac{\partial^{j-1}u_1}{\partial x}+j\frac{\partial^{j-2}u_1}{\partial x})\partial^{-i-j})\\&-2\sum_{i=1}^{\infty}u_i(\sum_{j=0}^{\infty}C_{-i}^j(-1)^j(\frac{\partial^{j-1}u_1}{\partial x}\star x+j\frac{\partial^{j-2}u_1}{\partial x})\partial^{-i-j}).\end{aligned}$$

So we have

$$\begin{aligned}\frac{\partial u_i}{\partial t_{1,1}^*}=&\,2\sum_{i=1}^{\infty}(x\star\frac{\partial u_i}{\partial x}+(i+1)u_i)+2\sum_{j=1}^{\infty}(2j+1)t_{2j+1}\star\frac{\partial u_i}{\partial t_{2j+1}}+2u_1\star x-2x\star u_1\\&+2\sum_{i=1}^{\infty}x\star(\int u_1dx)\star u_i-2\sum_{i=1}^{\infty}(\int u_1dx)\star x\star u_i\\&+2\sum_{k+j=i}u_kC_{-k}^j(-1)^j(x\star\frac{\partial^{j-1}u_1}{\partial x}+j\frac{\partial^{j-2}u_1}{\partial x})\end{aligned}$$

$$-2\sum_{k+j=i} u_k C^j_{-k}(-1)^j(\frac{\partial^{j-1}u_1}{\partial x}\star x + j\frac{\partial^{j-2}u_1}{\partial x}),$$

owing to

$$[x\partial, L]_\star = -\partial + \sum_{i=1}^{\infty}(x\star\frac{\partial u_i}{\partial x} + iu_i)\partial^{-i}.$$

□

Next, we define a generating function of the additional symmetries of the noncommutative B and C type KP hierarchies. Firstly, we give some useful lemmas.

Lemma 3.1. *For two pseudo-differential operators P and Q, we have the following identity*

$$res_z[z^{-1}\star(\partial^j P\star e^{zx})\star(Q\star e^{-zx})^*] = res_\partial[\partial^j P\partial^{-1}Q], j\geq 0, \quad (3.22)$$

in which we define the symbols $res_z(\sum_i a_i\star z^i) = a_{-1}$ *and* $res_\partial(\sum_i b_i\partial^i) = b_{-1}$.

Lemma 3.2. *For two pseudo-differential operators P and Q, we have the following identity*

$$res_z[(P\star e^{xz})\star(Q\star e^{-xz})^*] = res_\partial[P\star Q]. \quad (3.23)$$

Lemma 3.3. *Given an arbitrary pseudo-differential operator* $P = \sum p_i\partial^i$, *we get*

$$P = \sum\partial^i\widetilde{p}_i, P_- = \sum_{i=1}^{\infty}\partial^{-i}res_\partial(\partial^{i-1}P). \quad (3.24)$$

Lemma 3.4. *Let* $f(z) = \sum_{-\infty}^{\infty} a_i z^{-i}$, *then*

$$res_z[\zeta^{-1}(1-z/\zeta)^{-1} + z^{-1}(1-\zeta/z)^{-1}]f(z) = f(\zeta). \quad (3.25)$$

(Here $(1-z/\zeta)^{-1}$ *is understood as a series in* ζ^{-1} *and* $(1-\zeta/z)^{-1}$ *is a series in* z^{-1}.*)*

We define a generating operator $Y_{BC}(\lambda,\mu)$ of the additional symmetries as

$$\begin{aligned} Y_{BC}(\lambda,\mu) &= \sum_{m=0}^{\infty}\frac{(\mu-\lambda)^m}{m!}\sum_{l=-\infty}^{\infty}\lambda^{-l-m-1}(A_{m,m+l})_- \\ &= \sum_{m=0}^{\infty}\frac{(\mu-\lambda)^m}{m!}\sum_{l=-\infty}^{\infty}\lambda^{-l-m-1}(M^m\star L^{m+l} - (-1)^{m+l}L^{m+l-h}\star M^m\star L^h)_-, \end{aligned}$$

which can be expressed by a simple form in the following proposition.

Proposition 3.5. *The following identity holds true*

$$Y_{BC}(\lambda,\mu) = \omega(t,\mu)\partial^{-1}\star\omega^*(t,\lambda) + (\frac{\mu}{\lambda})^h\omega(t,-\lambda)\partial^{-1}\star\omega^*(t,-\mu). \quad (3.26)$$

Proof. Basing on eq.(2.8) and eq.(3.8), and using the above lemma, we get

$$(M^m\star L^{m+l})_- = \sum_{i=1}^{\infty}\partial^{-i}res_z[z^{-1}\star(\partial^{i-1}(M^m\star\phi\partial^{m+l+1}\star e^{\xi}))\star((\phi)^{-1}\star e^{-\xi})^*],$$

$$= \sum_{i=1}^{\infty} res_z[z^{m+l}\partial^{-i}(M^m \star \omega)^{i-1} \star \omega^*(t,z)],$$
$$= res_z[z^{m+l} \star (\partial_z^m \omega)\partial^{-1} \star \omega^*(t,z)],$$

with the help of the identity $f\partial^{-1} = \partial^{-1}f - \partial^{-1}f_x\partial^{-1}$.
Similarly, we can get

$$(L^{m+l-h} \star M^m \star L^h)_- = res_z[z^h \star (\partial_z^m z^{m+l-h} \star \omega(t,z)) \star \partial^{-1}\omega^*(t,z)].$$

Thus, we have a generating function of the noncommutative B and C type KP hierarchies,

$$\begin{aligned} Y_{BC}(\lambda,\mu) = & \sum_{m=0}^{\infty} \frac{(\mu-\lambda)^m}{m!} \star \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} \star res_z[z^{l+m} \star (\partial_z^m \omega(t,z))\partial^{-1}\omega^*(t,z)] \\ & + \sum_{m=0}^{\infty} \frac{(\mu-\lambda)^m}{m!} \star \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} \star res_z[(\partial_z^m z^{l+m} \star \omega(t,z))\partial^{-1} \star \omega^*(t,z)] \\ = & \; res_z[\sum_{n=-\infty}^{+\infty} \frac{z^n}{\lambda^{n+1}} \star \omega(t,z+\mu-\lambda) \star \partial^{-1}\omega^*(t,z)] \\ & + (\frac{\mu}{\lambda})^h \star res_z[\sum_{n=-\infty}^{+\infty} \frac{(z+\mu-\lambda)^n}{(-\lambda)^{n+1}} \star \omega(t,z+\mu-\lambda) \star \partial^{-1}\omega^*(t,z)] \\ = & \; \omega(t,\mu)\partial^{-1} \star \omega^*(t,\lambda) + (\frac{\mu}{\lambda})^h \omega(t,-\lambda)\partial^{-1} \star \omega^*(t,-\mu). \end{aligned}$$

□

Remark 3.2. When $h = 1$, the B type condition admits the following generating symmetries of the noncommutative BKP hierarchy,

$$Y_B(\lambda,\mu) = \frac{1}{\lambda}[\omega(t,-\lambda)\partial^{-1} \star \omega_x(t,\mu) - \omega(t,\mu)\partial^{-1} \star \omega_x(t,-\lambda)].$$

When $h = 0$, the C type condition admits the following generating symmetries of the noncommutative CKP hierarchy,

$$Y_C(\lambda,\mu) = \omega(t,\mu)\partial^{-1} \star \omega(t,-\lambda) + \omega(t,-\lambda)\partial^{-1} \star \omega(t,\mu).$$

4. String equations of the noncommutative B and C type KP hierarchies

In this section, we will consider the string equation of the noncommutative B and C type KP hierarchies. Firstly, the following corollary can be easily derived.

Corollary 4.1. *From eq.(3.11) we get $A_{1,l} = -(l-h)L^{l-1}$ when l is even, the corresponding flows on L are*

$$\partial_{t^*_{1,l}} L = (l-h)[(L^{l-1})_-, L]_\star = \begin{cases} 0, & l = 0,-2,-4,-6,\cdots, \\ -(l-h)(\partial_{t_{1-l}} L), & l = 2,4,6,\cdots. \end{cases} \tag{4.1}$$

We did not mention the case when l is odd in Corollary 4.1. When l is even, we can derive "String equation"([9]) from this case, so it deserve to be discussed with more details. In the next, we set $l = 2k(k = 1,2,3,\cdots)$, we have

$$A_{1,-(l-1)} = 2M \star L^{-(l-1)} - (l-h)L^{-l}, \;\; h = 0,1, \tag{4.2}$$

$$[A_{1,-(l-1)}, L^l]_\star = -2l. \tag{4.3}$$

Hence, we get a special action of the additional flows on L^l

$$\begin{aligned} \partial_{t^*_{1,-(l-1)}} L^l &= [-(A_{1,-(l-1)})_-, L^l]_\star \\ &= [(A_{1,-(l-1)})_+, L^l]_\star + [-(A_{1,-(l-1)}), L^l]_\star \\ &= 2[(M \star L^{-(l-1)})_+, L^l]_\star + 2l. \end{aligned} \tag{4.4}$$

Basing on the above knowledge, we can get the following proposition on the String equation.

Proposition 4.1. *If L^l is a differential operator and it is independent on the additional variables $t^*_{1,-(l-1)}$, then*

$$[L^l, \frac{1}{l}(M \star L^{-(l-1)})_+]_\star = [L^{2k}, \frac{1}{2k} M \star L^{-(2k-1)} - \frac{1}{2}\frac{2k-h}{2k} L^{-2k}]_\star = 1, \; h = 0,1, \tag{4.5}$$

are string equations similar as the standard form $[P,Q]_\star = 1$ (P and Q are two differential operators) of the noncommutative B and C type KP hierarchies.

Proof. Setting l as an even positive number in eq.(4.4) and assuming that L does not depent on $t^*_{1,-(l-1)}$, so we have $\partial_{t^*_{1,-(l-1)}} L^l = 0$, which implies $[L^l, \frac{1}{l}(M \star L^{-(l-1)})_+]_\star = 1$. Furthermore, $(A_{1,-(2k-1)})_- = 0$ implies $(M \star L^{-(2k-1)})_- = \frac{2k-h}{2k} L^{-2k}$, so $(M \star L^{-(2k-1)})_+ = M \star L^{-(2k-1)} - \frac{2k-h}{2} L^{-2k}$, which finishes the proof. □

5. Conclusion and discussion

In this paper, we construct a series of additional flows of the noncommutative B and C type KP hierarchies. The additional symmetries constitute a B and C type infinite dimensional Lie algebra $W_{1+\infty}$. In addition, the String equations of the noncommutative B and C type KP hierarchies were also derived.

As we know, some soliton equations describe real phenomena including shallow water waves in fluid dynamics, optics and so on. By considering noncommutativity in space-time, how to describe soliton dynamics and whether experimental results agree with the strength or the upper bound of the noncommutativity θ^{uv} are interesting questions.

Exact multi-soliton solutions are worth studying from various viewpoints of integrable systems and string theory. The detection of noncommutativity in our universe and the difference of soliton dynamics with moyal products from commutative ones are worth further studying. Because our center of this paper is the additional symmetry of the noncommutative B and C type KP hierarchy, the studies on the soliton solution of these systems might be included in our future work using the similar method as [12].

In some of Peter Olver's papers [21–23], they did some classification of integrable $(1+1)$ dimensional systems on associative algebras. Then how to do the classification of $(2+1)$ dimensional noncommutative integrable systems in our paper such as the $(2+1)$ dimensional noncommutative BKP equation and noncommutative CKP equation might be an interesting subject.

We think there should be a possible model of nonlinear quantum mechanics behind the noncommutative evolution equations we mention. They should be the nonlinear quantization to classical BKP and CKP integrable systems. The effect of such noncommutative evolutions on such basic quantum physical assumptions as the superposition principle remains to be explored.

In addition, the generating function of the additional symmetries can also be proved to have a nice form in terms of wave functions. The nice symmetry can be used to construct a kind of constrained noncommutative B and C type KP hierarchies by doing symmetry reduction. The study on the constrained noncommutative B and C type KP hierarchies is one interesting subject which might be included in our future work. Also the application of the String equations of the noncommutative B and C type KP hierarchies in D-brane dynamics is also an interesting question.

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